The Existential Completion

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Introduction

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- a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is elementary if for every A and C in $\mathcal{C}$, the functor

$$P_{id_C \times \Delta_A}\colon P(C \times A \times A) \longrightarrow P(C \times A)$$

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  has a left adjoint $\exists_{id_C \times \Delta_A}$ and these satisfy Frobenius reciprocity;
- a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential if, for every A_1, A_2 in $\mathcal{C}$, for any projection $pr_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor

$$P_{pr_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$

  has a left adjoint $\exists_{pr_i}$, and these satisfy Beck-Chevalley condition and Frobenius reciprocity.

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{} \arrow[r, phantom, "\Downarrow b"] & \mathbf{InfSL} \\
\mathcal{D}^{op} \arrow[ur, "R"'] &
\end{tikzcd}
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  such that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products, and $b\colon P \longrightarrow R \circ F^{op}$ is a natural transformation.
- a 2-cell is a natural transformation $\theta\colon F \longrightarrow G$ such that for every A in $\mathcal{C}$ and every α in PA, we have

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

Examples

Subobjects. $\mathcal{C}$ has finite limits.

$$Sub_{\mathcal{C}} \colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}\ .$$

The functor assigns to an object A in $\mathcal{C}$ the poset $Sub_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $Sub_{\mathcal{C}}(f) \colon Sub_{\mathcal{C}}(A) \longrightarrow Sub_{\mathcal{C}}(B)$ is given by pulling a subobject back along f.

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Weak Subobjects. $\mathcal{D}$ has finite products and weak pullbacks.

$$\Psi_{\mathcal{D}} \colon \mathcal{D}^{op} \longrightarrow \mathbf{InfSL}\ .$$

$\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the morphism $\Psi_{\mathcal{D}}(f) \colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f.

The Existential Completion

Let $P\colon \mathcal{C}^{op} \longrightarrow \textbf{InfSL}$ be a primary doctrine and let $\mathcal{A} \subset \mathcal{C}_1$ be the class of projections. For every object A of $\mathcal{C}$ consider we define $P^e(A)$ the following poset:

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- the objects are pairs $(B \xrightarrow{g \in \mathcal{A}} A\ ,\ \alpha \in PB)$;
- $(B \xrightarrow{h \in \mathcal{A}} A\ ,\ \alpha \in PB) \leq (D \xrightarrow{f \in \mathcal{A}} A\ ,\ \gamma \in PD)$ if there exists $w\colon B \longrightarrow D$ such that

$$
\begin{tikzcd}
& B \arrow[dl, "w"'] \arrow[d, "h"] \\
D \arrow[r, "f"'] & A
\end{tikzcd}
$$

commutes and $\alpha \leq P_w(\gamma)$.

The Existential Completion

Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, we define

$$P^e_f(\ C \xrightarrow{g\in\mathcal{A}} B\ ,\ \beta \in PC) := (\ D \xrightarrow{g^*f\in\mathcal{A}} A\ ,\ P_{f^*g}(\beta) \in PD)$$

where

$$\begin{tikzcd} D \arrow[r, "g^*f"] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback.

The Existential Completion

Theorem

Given a morphism $f : A \longrightarrow B$ *of* $\mathcal{A}$*, let*

$$\exists^e_f(\; C \xrightarrow{h \in \mathcal{A}} A \;, \alpha \in PC) := (\; C \xrightarrow{fh \in \mathcal{A}} B \;, \alpha \in PC)$$

when $(\; C \xrightarrow{h \in \mathcal{A}} A \;, \alpha \in PC)$ *is in* $P^e(A)$*. Then* $\exists^e_f$ *is left adjoint to* P^e_f*.*

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when $(C \xrightarrow{h \in \mathcal{A}} A , \alpha \in PC)$ *is in* $P^e(A)$. *Then* $\exists^e_f$ *is left adjoint to* P^e_f.

Theorem

Let $P : \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be a primary doctrine, then the doctrine* $P^e : \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is existential.*

The Existential Completion

Theorem

Consider the category $\mathbf{PD}(P,R)$*. We define*

$$\mathrm{E}_{P,R}\colon \mathbf{PD}(P,R) \longrightarrow \mathbf{ED}(P^e,R^e)$$

as follow:

- *for every 1-cell* (F,b), $\mathrm{E}_{P,R}(F,b) := (F,b^e)$*, where* $b^e_A\colon P^eA \longrightarrow R^eFA$ *sends an object* $(\, C \xrightarrow{g} A \,,\ \alpha)$ *in the object* $(\, FC \xrightarrow{Fg} FA \,,\ b_C(\alpha))$*;*
- *for every 2-cell* $\theta\colon (F,b) \Longrightarrow (G,c)$ *,* $\mathrm{E}_{P,R}\theta$ *is essentially the same.*

With the previous assignment E *is a 2-functor and it is 2-adjoint to the forgetful functor.*

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- *The 2-monad* $\mathrm{T}_e \colon \mathbf{PD} \longrightarrow \mathbf{PD}$ *is lax-idempotent;*

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- *The 2-monad* $\mathrm{T}_e \colon \mathbf{PD} \longrightarrow \mathbf{PD}$ *is lax-idempotent;*
- T_e-$\mathbf{Alg} \equiv \mathbf{ED}$.

Exact Completion

Theorem

For every elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *, the doctrine* $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is elementary and existential.*

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Theorem

The 2-functor $\mathbf{Xct} \to \mathbf{PED}$ *that takes an exact category to the elementary doctrine of its subobjects has a left biadjoint which associates the exact category* $\mathcal{T}_{P^e}$ *to an elementary doctrine* $P \colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *.*

Thank you!