The Existential Completion

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Introduction

- Let $\mathcal{C}$ be a category with finite products. A primary doctrine is a functor $P : \mathcal{C}^{\text{op}} \to \text{InfSL}$ from the opposite of the category $\mathcal{C}$ to the category of inf-semilattices;
Let $C$ be a category with finite products. A primary doctrine is a functor $P : C^{op} \to \text{InfSL}$ from the opposite of the category $C$ to the category of inf-semilattices;

a primary doctrine $P : C^{op} \to \text{InfSL}$ is elementary if for every $A$ and $C$ in $C$, the functor

$$P_{id_C \times \Delta_A} : P(C \times A \times A) \to P(C \times A)$$

has a left adjoint $\exists_{id_C \times \Delta_A}$ and these satisfy Frobenius reciprocity;
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a primary doctrine $P : C^{\text{op}} \to \text{InfSL}$ is existential if, for every $A_1$, $A_2$ in $C$, for any projection $pr_i : A_1 \times A_2 \to A_i$, $i = 1, 2$, the functor

$$P_{pr_i} : P(A_i) \to P(A_1 \times A_2)$$

has a left adjoint $\exists_{pr_i}$, and these satisfy Beck-Chevalley condition and Frobenius reciprocity.
Introduction

The category of primary doctrines PD is a 2-category, where:

- A 1-cell is a pair \((F, b)\), where \(F : C \rightarrow D\) is a functor preserving products, and \(b : P \rightarrow R \circ F\) is a natural transformation.

- A 2-cell is a natural transformation \(\theta : F \rightarrow G\) such that for every \(A \in C\) and every \(\alpha \in PA\), we have

  \[
  b_A(\alpha) \leq R \theta_A(c_A(\alpha)).
  \]
Introduction

The category of primary doctrines \( \text{PD} \) is a 2-category, where:

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\[
\begin{array}{c}
\text{C}^{\text{op}} \\
\downarrow \quad P \\
F^{\text{op}} \\
\downarrow b \\
\text{D}^{\text{op}} \\
\downarrow R \\
\text{InfSL}
\end{array}
\]

such that \( F: C \to D \) is a functor preserving products, and \( b: P \to R \circ F^{\text{op}} \) is a natural transformation.
Introduction

The category of primary doctrines $\textbf{PD}$ is a 2-category, where:

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\begin{array}{ccc}
C^{\text{op}} & \xrightarrow{P} & \text{InfSL} \\
\downarrow & & \downarrow \\
F^{\text{op}} & \xrightarrow{b} & D^{\text{op}} \\
\downarrow & & \downarrow R \\
D^{\text{op}} & \xrightarrow{R} & \\
\end{array}
\]

such that $F : C \rightarrow D$ is a functor preserving products, and $b : P \rightarrow R \circ F^{\text{op}}$ is a natural transformation.

- a 2-cell is a natural transformation $\theta : F \rightarrow G$ such that for every $A$ in $C$ and every $\alpha$ in $PA$, we have

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b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).
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Examples

Subobjects. $\mathcal{C}$ has finite limits.

\[
\text{Sub}_{\mathcal{C}} : \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL}.
\]

The functor assigns to an object $A$ in $\mathcal{C}$ the poset $\text{Sub}_{\mathcal{C}}(A)$ of subobjects of $A$ in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $\text{Sub}_{\mathcal{C}}(f) : \text{Sub}_{\mathcal{C}}(A) \longrightarrow \text{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along $f$. 

Weak Subobjects. $\mathcal{D}$ has finite products and weak pullbacks.

$\Psi_{\mathcal{D}} : \mathcal{D}^{\text{op}} \longrightarrow \text{InfSL}.$

$\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the morphism $\Psi_{\mathcal{D}}(f) : \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with $f$. 

Examples

**Subobjects.** \( \mathcal{C} \) has finite limits.

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Sub_\mathcal{C} : \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL}.
\]

The functor assigns to an object \( A \) in \( \mathcal{C} \) the poset \( Sub_\mathcal{C}(A) \) of subobjects of \( A \) in \( \mathcal{C} \) and, for an arrow \( B \xrightarrow{f} A \) the morphism \( Sub_\mathcal{C}(f) : Sub_\mathcal{C}(A) \longrightarrow Sub_\mathcal{C}(B) \) is given by pulling a subobject back along \( f \).

**Weak Subobjects.** \( \mathcal{D} \) has finite products and weak pullbacks.

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\Psi_\mathcal{D} : \mathcal{D}^{\text{op}} \longrightarrow \text{InfSL}.
\]

\( \Psi_\mathcal{D}(A) \) is the poset reflection of the slice category \( \mathcal{D}/A \), and for an arrow \( B \xrightarrow{f} A \), the morphism \( \Psi_\mathcal{D}(f) : \Psi_\mathcal{D}(A) \longrightarrow \Psi_\mathcal{D}(B) \) is given by a weak pullback of an arrow \( X \xrightarrow{g} A \) with \( f \).
The Existential Completion

Let \( P: \mathcal{C}^{\text{op}} \longrightarrow \text{InfSL} \) be a primary doctrine and let \( \mathcal{A} \subseteq \mathcal{C}_1 \) be the class of projections. For every object \( A \) of \( \mathcal{C} \) consider we define \( P^e(A) \) the following poset:
Let $P : C^{\text{op}} \to \text{InfSL}$ be a primary doctrine and let $A \subset C_1$ be the class of projections. For every object $A$ of $C$ consider we define $P^e(A)$ the following poset:

- the objects are pairs $(B \xrightarrow{g \in A} A, \alpha \in PB)$;
The Existential Completion

Let $P : C^{op} \rightarrow \text{InfSL}$ be a primary doctrine and let $A \subset C_1$ be the class of projections. For every object $A$ of $C$ consider we define $P^e(A)$ the following poset:

- the objects are pairs $(B \xrightarrow{g\in A} A, \alpha \in PB)$;
- $(B \xrightarrow{h\in A} A, \alpha \in PB) \leq (D \xrightarrow{f\in A} A, \gamma \in PD)$ if there exists $w : B \rightarrow D$ such that

\[
\begin{array}{ccc}
B & \xrightarrow{w} & D \\
\downarrow{h} & & \downarrow{f} \\
A & \xrightarrow{} & A \\
\end{array}
\]

commutes and $\alpha \leq P_w(\gamma)$. 
The Existential Completion

Given a morphism $f : A \to B$ in $C$, we define

$$P_f^e( \begin{array}{c} C \xrightarrow{g \in A} B, \beta \in PC \end{array} ) := \begin{array}{c} D \xrightarrow{g^*f \in A} A, \quad P_f^*g(\beta) \in PD \end{array}$$

where

\[
\begin{array}{ccc}
D & \xrightarrow{g^*f} & A \\
\downarrow & & \downarrow \\
C & \xrightarrow{f^*g} & B
\end{array}
\]

is a pullback.
The Existential Completion

Theorem

*Given a morphism $f : A \to B$ of $\mathcal{A}$, let*

$$\exists^e_f ( C \xrightarrow{h \in A} A, \alpha \in PC) := ( C \xrightarrow{fh \in A} B, \alpha \in PC)$$

*when $( C \xrightarrow{h \in A} A, \alpha \in PC)$ is in $P^e(A)$. Then $\exists^e_f$ is left adjoint to $P^e_f$.***
The Existential Completion

**Theorem**

Given a morphism $f : A \to B$ of $\mathcal{A}$, let

$$\exists^e_f (C \xrightarrow{h \in \mathcal{A}} A, \alpha \in PC) := (C \xrightarrow{fh \in \mathcal{A}} B, \alpha \in PC)$$

when $(C \xrightarrow{h \in \mathcal{A}} A, \alpha \in PC)$ is in $P^e(A)$. Then $\exists^e_f$ is left adjoint to $P^e_f$.

**Theorem**

Let $P : \mathcal{C}^{op} \to \text{InfSL}$ be a primary doctrine, then the doctrine $P^e : \mathcal{C}^{op} \to \text{InfSL}$ is existential.
The Existential Completion

Theorem
Consider the category $\mathbf{PD}(P, R)$. We define

$$E_{P, R} : \mathbf{PD}(P, R) \to \mathbf{ED}(P^e, R^e)$$

as follow:

$\blacktriangleright$ for every 1-cell $(F, b)$, $E_{P, R}(F, b) := (F, b^e)$, where $b^e_A : P^eA \to R^eFA$ sends an object $(C \xrightarrow{g} A, \alpha)$ in the object $(FC \xrightarrow{Fg} FA, b_C(\alpha))$;

$\blacktriangleright$ for every 2-cell $\theta : (F, b) \to (G, c)$, $E_{P, R}\theta$ is essentially the same.

With the previous assignment $E$ is a 2-functor and it is 2-adjoint to the forgetful functor.
The Existential Completion

Theorem

- The 2-monad $T_e: \text{PD} \to \text{PD}$ is lax-idempotent;
The Existential Completion

Theorem

- The 2-monad \( T_e : \mathsf{PD} \rightarrow \mathsf{PD} \) is lax-idempotent;
- \( T_e \text{-Alg} \equiv \mathsf{ED} \).
Theorem

For every elementary doctrine \( P : C^{\text{op}} \to \text{InfSL} \), the doctrine \( P^e : C^{\text{op}} \to \text{InfSL} \) is elementary and existential.
Exact Completion

Theorem
For every elementary doctrine \( P : C^{op} \to \text{InfSL} \), the doctrine \( P^e : C^{op} \to \text{InfSL} \) is elementary and existential.

Theorem
The 2-functor \( \text{Xct} \to \text{PED} \) that takes an exact category to the elementary doctrine of its subobjects has a left biadjoint which associates the exact category \( T_{P^e} \) to an elementary doctrine \( P : C^{op} \to \text{InfSL} \).
Thank you!