

The Existential Completion

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Introduction

- ▶ Let \mathcal{C} be a category with finite products. A **primary doctrine** is a functor $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ from the opposite of the category \mathcal{C} to the category of inf-semilattices;

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- ▶ a primary doctrine $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is **elementary** if for every A and C in \mathcal{C} , the functor

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- ▶ a primary doctrine $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is **existential** if, for every A_1, A_2 in \mathcal{C} , for any projection $pr_i: A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor

$$P_{pr_i}: P(A_i) \longrightarrow P(A_1 \times A_2)$$

has a left adjoint \exists_{pr_i} , and these satisfy Beck-Chevalley condition and Frobenius reciprocity.

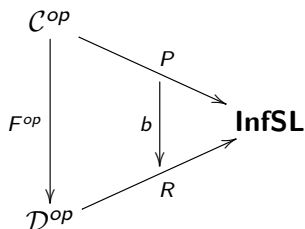
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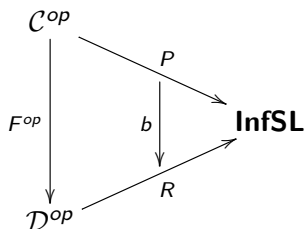


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- ▶ a **2-cell** is a natural transformation $\theta: F \longrightarrow G$ such that for every A in \mathcal{C} and every α in PA , we have

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

Examples

Subobjects. \mathcal{C} has finite limits.

$$Sub_{\mathcal{C}}: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL} .$$

The functor assigns to an object A in \mathcal{C} the poset $Sub_{\mathcal{C}}(A)$ of subobjects of A in \mathcal{C} and, for an arrow $B \xrightarrow{f} A$ the morphism $Sub_{\mathcal{C}}(f): Sub_{\mathcal{C}}(A) \longrightarrow Sub_{\mathcal{C}}(B)$ is given by pulling a subobject back along f .

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Weak Subobjects. \mathcal{D} has finite products and weak pullbacks.

$$\Psi_{\mathcal{D}}: \mathcal{D}^{\text{op}} \longrightarrow \mathbf{InfSL} .$$

$\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category \mathcal{D}/A , and for an arrow $B \xrightarrow{f} A$, the morphism $\Psi_{\mathcal{D}}(f): \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f .

The Existential Completion

Let $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a primary doctrine and let $\mathcal{A} \subset \mathcal{C}_1$ be the class of projections. For every object A of \mathcal{C} consider we define $P^e(A)$ the following poset:

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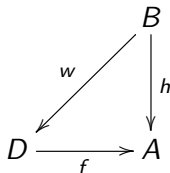
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- ▶ the objects are pairs $(B \xrightarrow{g \in \mathcal{A}} A , \alpha \in PB);$

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- ▶ the objects are pairs $(B \xrightarrow{g \in \mathcal{A}} A , \alpha \in PB)$;
- ▶ $(B \xrightarrow{h \in \mathcal{A}} A , \alpha \in PB) \leq (D \xrightarrow{f \in \mathcal{A}} A , \gamma \in PD)$ if there exists $w: B \longrightarrow D$ such that



commutes and $\alpha \leq P_w(\gamma)$.

The Existential Completion

Given a morphism $f: A \longrightarrow B$ in \mathcal{C} , we define

$$P_f^e(C \xrightarrow{g \in \mathcal{A}} B, \beta \in PC) := (D \xrightarrow{g^* f \in \mathcal{A}} A, P_{f^* g}(\beta) \in PD)$$

where

$$\begin{array}{ccc} D & \xrightarrow{g^* f} & A \\ \downarrow f^* g & \lrcorner & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

is a pullback.

The Existential Completion

Theorem

Given a morphism $f: A \longrightarrow B$ of \mathcal{A} , let

$$\exists_f^e(C \xrightarrow{h \in \mathcal{A}} A, \alpha \in PC) := (C \xrightarrow{fh \in \mathcal{A}} B, \alpha \in PC)$$

when $(C \xrightarrow{h \in \mathcal{A}} A, \alpha \in PC)$ is in $P^e(A)$. Then \exists_f^e is left adjoint to P_f^e .

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Theorem

Let $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a primary doctrine, then the doctrine $P^e: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential.

The Existential Completion

Theorem

Consider the category $\mathbf{PD}(P, R)$. We define

$$\mathbb{E}_{P,R}: \mathbf{PD}(P, R) \longrightarrow \mathbf{ED}(P^e, R^e)$$

as follow:

- ▶ for every 1-cell (F, b) , $\mathbb{E}_{P,R}(F, b) := (F, b^e)$, where $b_A^e: P^e A \longrightarrow R^e F A$ sends an object $(C \xrightarrow{g} A, \alpha)$ in the object $(FC \xrightarrow{Fg} FA, b_C(\alpha))$;
- ▶ for every 2-cell $\theta: (F, b) \Longrightarrow (G, c)$, $\mathbb{E}_{P,R}\theta$ is essentially the same.

With the previous assignment \mathbb{E} is a 2-functor and it is 2-adjoint to the forgetful functor.

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Theorem

- ▶ *The 2-monad $\mathbb{T}_e: \mathbf{PD} \longrightarrow \mathbf{PD}$ is lax-idempotent;*

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- ▶ *The 2-monad $T_e: \mathbf{PD} \longrightarrow \mathbf{PD}$ is lax-idempotent;*
- ▶ $T_e\text{-Alg} \equiv \mathbf{ED}$.

Exact Completion

Theorem

For every elementary doctrine $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, the doctrine $P^e: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is elementary and existential.

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The 2-functor $\mathbf{Xct} \rightarrow \mathbf{PED}$ that takes an exact category to the elementary doctrine of its subobjects has a left biadjoint which associates the exact category \mathcal{T}_{P^e} to an elementary doctrine $P: \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$.

Thank you!