# Continuous complete categories enriched quantales

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# Outline

# Categories as generalize ordered sets

Ordered sets are often viewed as thin categories, and the other way around, categories have also been studied as “generalized ordered structures”.

Illuminating examples include the study of continuous categories and that of completely (totally) distributive categories.


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# Enriched categories as quantitative ordered sets

A bit more generally, categories enriched over a monoidal closed category can be viewed as “ordered sets” with *truth-values* taken in that closed category. This point of view has led to a theory of *quantitative domains*, of which the core objects are categories enriched in a commutative and unital quantale Q.


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- D. Hofmann and P. Waszkiewicz. Approximation in quantale-enriched categories. *Topology and its Applications*, 158: 963–977, 2011.
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# Continuous dcpo

A continuous dcpo (directed complete poset) $P$ is characterized by the relation between $P$ and the poset $\mathrm{Idl}(P)$ of ideals of $P$.

For all $p \in P$, $\downarrow p := \{x \in P : x \leq p\}$ defines an embedding $\downarrow : P \longrightarrow \mathrm{Idl}(P)$. A poset $P$ is directed complete if $\downarrow$ has a left adjoint

$$\sup : \mathrm{Idl}(P) \longrightarrow P$$

and is continuous if there is a string of adjunctions

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# Continuous Category

In a locally small category $\mathcal{E}$, ind-objects, or equivalently, the presheaves generated by ind-objects, play the role of ideals in posets.

Let Ind-$\mathcal{E}$ be the category of all presheaves generated by ind-objects in $\mathcal{E}$ (i.e., filtered colimit of representables). Then, $\mathcal{E}$ has filtered colimits if the Yoneda embedding $\mathrm{y}: \mathcal{E} \longrightarrow \text{Ind-}\mathcal{E}$ has a left adjoint

$$\mathrm{colim}: \text{Ind-}\mathcal{E} \longrightarrow \mathcal{E}$$

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# Continuous Q-category

For categories enriched in a commutative and unital quantale Q, forward Cauchy weights (i.e., presheaves generated by forward Cauchy nets) play the role of ind-objects.

For each Q-category $A$, let $\mathcal{C}A$ be the Q-category of all forward Cauchy weights of $A$. Then, $A$ is called Yoneda complete if the Yoneda embedding $\mathrm{y} : A \longrightarrow \mathcal{C}A$ has a left adjoint

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# Completely distributivity

In the definition of continuous dcpo, if we replace $\mathrm{Idl}(P)$ by the poset of all lower sets of $P$ then we obtain the concept of (constructively) completely distributive lattices.

Similarly, if we replace the category of ind-objects and the Q-category of forward Cauchy weights by the category of all small presheaves and the Q-category of all weights, then we obtain the concepts of completely distributive categories and completely distributive Q-categories, respectively.


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|---|---|---|
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# The question

It is well-known that a completely distributive lattice is necessarily continuous in the sense of Scott. It is natural to ask whether there is an enriched version of this conclusion.

As we shall see in the case of quantale-enriched categories, in contrast to the situation in lattice theory, the answer depends on the structure of the quantale, i.e., the structure of the truth-values.

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# Outline

# Quantales and quantale-enriched categories

$(\mathsf{Q}, \&, k)$ A commutative and unital quantale (a commutative monoid in **Sup**)

$p\&q \leq r \iff p \leq q \to r$

Q-categories a set $X$ with $\hom(x, y) \in \mathsf{Q}$ such that

- $k \leq \hom(x, x)$,
- $\hom(y, z)\& \hom(x, y) \leq \hom(x, z)$.

We often write $\hom(x, y)$ as $X(x, y)$.

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We often write $\hom(x, y)$ as $X(x, y)$.

# Ordered sets and metric spaces are Q-categories

1. If $(\mathsf{Q}, \&, k) = (\{0, 1\}, \wedge, 1)$, then a Q-category is precisely an ordered set.
2. If $(\mathsf{Q}, \&, k) = ([0, \infty]^{\mathrm{op}}, +, 0)$, then a Q-category $X$ is exactly a generalized metric space.

# Q-functor and adjunction

A Q-functor from a Q-category $X$ to a Q-category $Y$ is a map $f : X \longrightarrow Y$ such that for all $x, y \in X$,

$$X(x, y) \leq Y(fx, fy).$$

Given Q-categories $X, Y$, a Q-functor $f : X \longrightarrow Y$ is left adjoint to a Q-functor $g : Y \longrightarrow X$, in symbols $f \dashv g$, if

$$Y(fx, y) = X(x, gy)$$

for all $x \in X$, all $y \in Y$. In this case, we also say that $g$ is right adjoint to $f$.

All Q-categories and Q-functors form a category, written as

$$\text{Q-Cat}.$$

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# The presheaf monad $\mathcal{P}$

A presheaf (or, a weight) $\varphi$ on $A$ is a Q-relation $A \,{\circ\!\!\!\rightarrow}\, *$, or equivalently, a map $\varphi : A \longrightarrow \mathsf{Q}$ such that $\varphi(x) \mathbin{\&} A(y,x) \leq \varphi(y)$ for all $x, y \in A$. Presheaves on $A$ constitute a Q-category $\mathcal{P}A$ with

$$\mathcal{P}A(\varphi,\rho) = \bigwedge_{x \in A} \varphi(x) \to \rho(x).$$

There is a natural way to make $\mathcal{P}$ into a KZ-doctrine $(\mathcal{P}, \mathsf{y}, \mathsf{s})$, the *presheaf monad*, with unit given by the Yoneda embedding

$$\mathsf{y}_A : A \longrightarrow \mathcal{P}A, \quad \mathsf{y}_A(x) = A(-,x)$$

and multiplication given by

$$\mathsf{s}_A : \mathcal{P}\mathcal{P}A \longrightarrow \mathcal{P}A, \quad \mathsf{s}_A(\Lambda) = \Lambda \circ (\mathsf{y}_A)_*.$$

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# The copresheaf monad $\mathcal{P}^\dagger$

Dually, the Q-category $\mathcal{P}^\dagger A$ consists of all *copresheaves* on $A$ with

$$\mathcal{P}^\dagger A(\psi, \sigma) = \bigwedge_{x \in A} \sigma(x) \to \psi(x).$$

The functor $\mathcal{P}^\dagger$ can be made into a co-KZ-doctrine $(\mathcal{P}^\dagger, \mathbf{y}^\dagger, \mathrm{s}^\dagger)$, the *copresheaf monad*, on Q-Cat, where the unit is given by the co-Yoneda embedding

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# Cocomplete Q-categories and $\mathcal{P}$-algebras

Let $A$ be a Q-category.

- $A$ is cocomplete iff $\mathrm{y}_A : A \longrightarrow \mathcal{P}A$ has a left adjoint $\sup_A : \mathcal{P}A \longrightarrow A$.
- $A$ is a $\mathcal{P}$-algebra iff $\mathrm{y}_A : A \longrightarrow \mathcal{P}A$ has a left inverse.

The $\mathcal{P}$-algebras are just the *separated* cocomplete Q-categories.

Dually,

- $A$ is complete iff $\mathrm{y}^\dagger_A : A \longrightarrow \mathcal{P}^\dagger A$ has a right adjoint $\inf_A : \mathcal{P}^\dagger A \longrightarrow A$.
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The $\mathcal{P}^\dagger$-algebras are exactly the separated complete Q-categories.

**Proposition (Stubbe)**

*A Q-category A is complete if and only if it is cocomplete.*

# Outline

# Class of weights

A *saturated class of weights* is a full submonad $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ of the monad $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ on Q-Cat. Explicitly, it is a triple $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ satisfying:

- $\mathcal{T}$ is a subfunctor of $\mathcal{P}$ : Q-Cat $\longrightarrow$ Q-Cat;
- all inclusions $\varepsilon_A : \mathcal{T}A \longrightarrow \mathcal{P}A$ are fully faithful;
- all $\varepsilon_A$ form a natural transformation such that

$$\mathsf{s} \circ (\varepsilon * \varepsilon) = \varepsilon \circ \mathsf{m} \quad \text{and} \quad \varepsilon \circ \mathsf{e} = \mathsf{y}.$$

Since $(\mathcal{P}, \mathsf{s}, \mathsf{y})$ is a KZ-doctrine on Q-Cat, every saturated class of weights $(\mathcal{T}, \mathsf{m}, \mathsf{e})$ is also a KZ-doctrine on Q-Cat.

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# $\mathcal{T}$-continuous $\mathcal{T}$-algebra

Because $(\mathcal{T}, \mathrm{m}, \mathrm{e})$ is also a KZ-doctrine, a *$\mathcal{T}$-algebra $A$* is a Q-category $A$ such that

$$\mathrm{e}_A : A \longrightarrow \mathcal{T}A$$

has a left inverse (which is necessarily a left adjoint of $\mathrm{e}_A$):

$$\sup_A : \mathcal{T}A \longrightarrow A.$$

A $\mathcal{T}$-algebra $A$ is $\mathcal{T}$-continuous if there is a string of adjunctions

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# Examples of $\mathcal{T}$-continuous $\mathcal{T}$-algebras

1. Let $\mathcal{T} = \mathcal{P}$. Then a Q-category $A$ is a $\mathcal{T}$-continuous $\mathcal{T}$-algebra if $A$ is separated and there is a string of adjunctions

$$t_A \dashv \sup_A \dashv y_A : A \longrightarrow \mathcal{P}A.$$

In this case, $A$ is called a completely distributive Q-category.
Particularly, if $(Q, \&, k) = (\{0, 1\}, \wedge, 1)$, then completely distributive Q-categories degenerate to (constructively) completely distributive lattices.

2. Let $(Q, \&, k) = (\{0, 1\}, \wedge, 1)$ and $\mathcal{T} = \mathrm{Idl}$. Then a $\mathcal{T}$-continuous $\mathcal{T}$-algebra is precisely a continuous dcpo.

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# Examples of $\mathcal{T}$-continuous $\mathcal{T}$-algebras

1. Let $\mathcal{T} = \mathcal{P}$. Then a Q-category $A$ is a $\mathcal{T}$-continuous $\mathcal{T}$-algebra if $A$ is separated and there is a string of adjunctions

$$t_A \dashv \sup_A \dashv \mathrm{y}_A : A \longrightarrow \mathcal{P}A.$$

   In this case, $A$ is called a completely distributive Q-category. Particularly, if $(\mathsf{Q}, \&, k) = (\{0,1\}, \wedge, 1)$, then completely distributive Q-categories degenerate to (constructively) completely distributive lattices.

2. Let $(\mathsf{Q}, \&, k) = (\{0,1\}, \wedge, 1)$ and $\mathcal{T} = \mathrm{Idl}$. Then a $\mathcal{T}$-continuous $\mathcal{T}$-algebra is precisely a continuous dcpo.

# Relation to distributive law

Let $\mathcal{T}$ be a saturated class of weights, considered as a submonad of the presheaf monad $\mathcal{P}$ on Q-Cat.

A lifting of $\mathcal{T}$ through the forgetful functor $U$ : Q-Inf $\longrightarrow$ Q-Cat is a monad $\widetilde{\mathcal{T}}$ on Q-Inf that makes

$$\begin{tikzcd}
\text{Q-Inf} \arrow[r, "\widetilde{\mathcal{T}}"] \arrow[d, "U"'] & \text{Q-Inf} \arrow[d, "U"] \\
\text{Q-Cat} \arrow[r, "\mathcal{T}"'] & \text{Q-Cat}
\end{tikzcd}$$

commutative. Since the forgetful functor $U$ : Q-Inf $\longrightarrow$ Q-Cat is injective on objects, the monad $\mathcal{T}$ has at most one lifting through $U$.

A distributive law of the monad $\mathcal{P}^\dagger$ over $\mathcal{T}$ is a natural transformation $\delta : \mathcal{P}^\dagger\mathcal{T} \longrightarrow \mathcal{T}\mathcal{P}^\dagger$ satisfying certain conditions.

Since Q-Inf is the category of Eilenberg-Moore algebras of the monad $\mathcal{P}^\dagger$, it follows that the distributive laws of $\mathcal{P}^\dagger$ over $\mathcal{T}$ correspond bijectively to the liftings of $\mathcal{T}$ through $U$.

Therefore, distributive laws of $\mathcal{P}^\dagger$ over $\mathcal{T}$, when exist, are unique. So, in this case, we simply say that $\mathcal{P}^\dagger$ distributes over $\mathcal{T}$.

D. Hofmann, G. J. Seal, and W. Tholen, editors. *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, volume 153 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2014.

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## Theorem

*For a saturated class of weights $\mathcal{T}$ on* Q-Cat, *the following statements are equivalent:*

1. *Every completely distributive* Q*-category is $\mathcal{T}$-continuous.*
2. *The copresheaf monad $\mathcal{P}^{\dagger}$ distributes over $\mathcal{T}$.*

# Outline

# Presheaves generated by forward Cauchy nets

Let $A$ be a Q-category. A net $\{x_\lambda\}$ in $A$ is called forward Cauchy if

$$\bigvee_{\lambda}\bigwedge_{\gamma\geq\mu\geq\lambda} A(x_\mu, x_\gamma) \geq k.$$

A presheaf $\varphi : A \mathrel{\circ\!\!\rightarrow} \star$ is called forward Cauchy if

$$\varphi(x) = \bigvee_{\lambda}\bigwedge_{\lambda\leq\mu} A(x, x_\mu)$$

for some forward Cauchy net $\{x_\lambda\}$ in $A$.

Forward Cauchy weights in a Q-category are analogue of ideals in a partially ordered set and ind-objects in a locally small category.

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# Continuous Q-categories

Let Q be a quantale whose underlying lattice is continuous. Then, assigning each Q-category $A$ to the Q-category

$$\mathcal{C}A := \{\varphi \in \mathcal{P}A \mid \varphi \text{ is forward Cauchy}\}$$

defines a saturated class of weights on Q-Cat, which is denoted by $\mathcal{C}$.

A Q-category $A$ is continuous if it is a $\mathcal{C}$-continuous $\mathcal{C}$-algebra, that is, $\sup : \mathcal{C}A \longrightarrow A$ has a left adjoint.

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# Continuous t-norm

A continuous t-norm is a continuous map $\&\colon [0,1]^2 \longrightarrow [0,1]$ that makes $([0,1], \&, 1)$ into a commutative quantale. Basic continuous t-norms include:

- The Gödel t-norm $\&_M$: $p \&_M q = \min\{p, q\}$.
- The Łukasiewicz t-norm $\&_Ł$: $p \&_Ł q = \max\{p + q - 1, 0\}$.
- The product $\&_P$: $p \&_P q = p \cdot q$.

The quantale $([0,1], \&_P, 1)$ is isomorphic to Lawvere's quantale $([0,\infty]^{\mathrm{op}}, +, 0)$.

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# Structure of a continuous t-norm

Let & be a continuous t-norm. An element $a \in [0,1]$ is called idempotent if $a \mathbin{\&} a = a$. For any idempotent elements $a, b$ with $a < b$, the restriction of & to $[a, b]$ makes $[a, b]$ into a commutative quantale with $b$ being the unit element.

**Theorem (Mostert and Shields, 1957)**

*Let & be a continuous t-norm. If $a \in [0,1]$ is non-idempotent, then there exist idempotent elements $a^-, a^+ \in [0,1]$ such that $a^- < a < a^+$ and the quantale $([a^-, a^+], \&, a^+)$ is isomorphic either to $([0,1], \&_Ł, 1)$ or to $([0,1], \&_P, 1)$.*

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# The main result

## Theorem

*Let* $\mathsf{Q} = ([0,1], \&, 1)$ *with* $\&$ *being a continuous t-norm. Then the following statements are equivalent:*

1. *Every completely distributive* $\mathsf{Q}$*-category is continuous.*
2. *The* $\mathsf{Q}$*-category* $([0,1], \to)$ *is continuous.*
3. *For each non-idempotent element* $a \in [0,1]$*, the quantale* $([a^-, a^+], \&, a^+)$ *is isomorphic to* $([0,1], \&_P, 1)$ *whenever* $a^- > 0$.
4. *For each* $p \in (0,1]$*, the map* $p \to - : [0,1] \longrightarrow [0,1]$ *is continuous on the interval* $[0,p)$.
5. *For every complete* $\mathsf{Q}$*-category* $A$*, the inclusion* $\mathcal{C}A \hookrightarrow \mathcal{P}A$ *has a left adjoint.*
6. *The copresheaf monad* $\mathcal{P}^\dagger$ *distributes over* $\mathcal{C}$.

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Thanks for your attention