Dagger limits

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Structure of the talk

1. Dagger categories
2. Dagger limits
3. Polar decomposition
4. Further topics?

Dagger $=$ a functorial way of reversing arrows:

$$A \xrightarrow{f = f^{\dagger\dagger}} B$$

$$A \xleftarrow[f^{\dagger}]{} B$$

Category	Objects	Morphisms	Dagger
Rel	Sets	Relations	inverse
PInj	Sets	Partial injections	inverse
FHilb	F.d. Hilbert spaces	linear maps	adjoint
Hilb	Hilbert spaces	bounded linear maps	adjoint
Groupoid **G**	ob(**G**)	mor(**G**)	inverse

Dictionary

Ordinary notion	Dagger counterpart	Added condition
Isomorphism	Unitary	$f^{-1} = f^\dagger$
Mono	Dagger mono	$f^\dagger f = \mathrm{id}$
Epi	Dagger epi	$ff^\dagger = \mathrm{id}$
	Partial isometry	$f = ff^\dagger f$
Idempotent $p = p^2$	Projection	$p = p^\dagger$
Functor	Dagger Functor	$F(f^\dagger) = F(f)^\dagger$
Natural transformation	Natural transformation	-
Adjunction $F \dashv G$	Dagger adjunction	F and G dagger
Monad (T, μ, η)	Dagger monad	T dagger and $\mu_T \circ T\mu^\dagger = T\mu \circ \mu_T^\dagger$

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What should dagger limits be?

- Unique up to unique *unitary*
- Defined (canonically) for arbitrary diagrams
- Definition shouldn't depend on additional structure (e.g. enrichment)
- Generalizes dagger biproducts and dagger equalizers
- Connections to dagger adjunctions etc.

Why is this not (trivially) trivial?

- Unitaries rather than mere isos
- **DagCat** is not just a 2-category, it is a *dagger* 2-category.
- I.e. 2-cells have a dagger, so one should require unitary 2-cells etc.
- The forgetful functor **DagCat** $\to$ **Cat** has both 1-adjoints but no 2-adjoints.
- Previously in CT 2016: only dagger limits of dagger functors.

Biproducts

A biproduct is a product + coproduct

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} A \oplus B \underset{p_B}{\overset{i_B}{\leftrightarrows}} B$$

such that

$$p_A i_A = \mathrm{id}_A \qquad p_B i_B = \mathrm{id}_B$$
$$p_B i_A = 0_{A,B} \qquad p_A i_B = 0_{B,A}$$

Known examples of dagger limits

- Dagger biproduct of A and B is a biproduct of the form $(A \oplus B, p_A, p_B, p_A^\dagger, p_B^\dagger)$
- Dagger equalizer is an equalizer e that is dagger monic
- Given a diagram from an indiscrete category $\mathbf{J}$ to $\mathbf{C}$: one dagger limit for each choice of $A \in \mathbf{J}$

How to generalize?

1. Maps $A \oplus B \to A, B$ are dagger epic, whereas dagger equalizers $E \to A$ are dagger monic.

2. Requiring the structure maps to be partial isometries generalizes both.

3. Based on equalizers and indiscrete diagrams, one can only require this on a weakly initial set.

4. One also needs to generalize from $A \to A \oplus B \to B = 0_{A,B}$

5. This can be done by saying that the induced projections on the limit commute.

Defining dagger limits

Definition

Let $D\colon \mathbf{J} \to \mathbf{C}$ be a diagram and let $\Omega \subseteq J$ be weakly initial. A *dagger limit of* (D, Ω) is a limit L of D whose cone $l_A\colon L \to D(A)$ satisfies the following two properties:

normalization l_A is a partial isometry for every $A \in \Omega$;

independence the projections on L induced by these partial isometries commute, *i.e.* $l_A^\dagger l_A l_B^\dagger l_B = l_B^\dagger l_B l_A^\dagger l_A$ for all $A, B \in \Omega$.

Uniqueness

Theorem

Let L be a dagger limit of (D, Ω) and M a limit of D. The canonical isomorphism $L \to M$ is unitary iff M is a dagger limit of (D, Ω).

Often Ω is forced on us:

- Products $\bullet \quad \bullet$
- Equalizers $\bullet \rightrightarrows \bullet$
- Pullbacks $\bullet \to \bullet \leftarrow \bullet$

But not always: $\bullet \leftrightarrows \bullet$ or $\bullet \leftrightarrows \bullet$

Definition

A dagger-shaped dagger limit is the dagger limit of a dagger functor.

E.g. products, limits of projections, unitary representations of groupoids.

Definition

A set $\Omega \subset \mathbf{J}$ is a *basis* when every object B allows a unique $A \in \Omega$ making $\mathbf{J}(A, B)$ non-empty.
(Finitely) based dagger limit: Ω is a (finite) basis

- Products: $\bullet \qquad \bullet$
- Equalizers: $\bullet \rightrightarrows \bullet$
- Indiscrete categories $\bullet \leftrightarrows \bullet$
- Nonexample: $\bullet \to \bullet \leftarrow \bullet$

- If **C** has zero morphisms, L is a dagger-shaped limit iff
  - each $L \to D(A)$ is a partial isometry
  - $D(A) \to L \to D(B) = 0$ whenever $\hom(A, B)$ is empty.
- If **C** is enriched in commutative monoids, then finitely based dagger limits can be equivalently defined by

$$\mathrm{id}_L = \sum_{A \in \Omega} L \to D(A) \to L$$

Theorem

A dagger category has dagger-shaped limits iff it has dagger split infima of projections, dagger stabilizers, and dagger products.

Theorem

A dagger category has all finitely based dagger limits iff it has dagger equalizers, dagger intersections and finite dagger products.

Interlude: Biproducts without zero morphisms

A biproduct is a product + coproduct

$$\begin{tikzcd} A \arrow[r, shift right, "i_A"'] & A \oplus B \arrow[l, shift right, "p_A"'] \arrow[r, shift right, "p_B"'] & B \arrow[l, shift right, "i_B"'] \end{tikzcd}$$

such that

$$p_A i_A = \mathrm{id}_A \qquad p_B i_B = \mathrm{id}_B$$

$$i_A p_A i_B p_B = i_B p_B i_A p_A$$

This defines biproducts up to iso, requires no enrichment and is equivalent to the usual definitions when enrichment is available. Can be generalized for other limit-colimit coincidences.

Polar Decomposition

Definition

Let $f\colon A \to B$ be a morphism in a dagger category. A *polar decomposition* of f consists of two factorizations of f as $f = pi = jp$,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, "f"] & A \arrow[d, "p"] \\ B \arrow[r, "j"'] & B \end{tikzcd}$$

where p is a partial isometry and i and j are self-adjoint bimorphisms.

A category *admits polar decomposition* when every morphism has a polar decomposition.

Polar Decomposition

Fact: **Hilb** has polar decomposition.

Let f have a polar decomposition $f = pi = jp$.

- If f is an iso, then p is unitary
- If f splits a dagger idempotent e, then p is a dagger splitting of it and $e = pp^\dagger$.

Polar Decomposition

If $E \xrightarrow{e} A \rightrightarrows B$ is an equalizer and

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "p"'] \arrow[dr, "e"] & E \arrow[d, "p"] \\ A \arrow[r, "j"'] & A \end{tikzcd}$$

is a polar decomposition, then $E \xrightarrow{p} A \rightrightarrows B$ is a dagger equalizer.

Theorem
This works for all $\mathbf{J}$ *with a basis (mod independence)*

Theorem
If $\mathbf{C}$ *is balanced, one can build from a limit of* D *a dagger limit of* $D' \cong D$ *(mod independence).*

Commuting limits with colimits

Naively, dagger limits should always commute with dagger colimits: given $D\colon \mathbf{J} \times \mathbf{K} \to \mathbf{C}$, one would like to define $\hat{D}\colon \mathbf{J} \times \mathbf{K}^{\mathrm{op}} \to \mathbf{C}$ by "applying the dagger to the second variable" and then calculate as follows:

$$\begin{aligned} &\operatorname{dcolim}_k \operatorname{dlim}_j D(j,k) = \operatorname{dlim}_k \operatorname{dlim}_j \hat{D}(j,k) \\ &\cong_\dagger \operatorname{dlim}_j \operatorname{dlim}_k \hat{D}(j,k) = \operatorname{dlim}_j \operatorname{dcolim}_k D(j,k) \end{aligned}$$

However, $\hat{D}$ is not guaranteed to be a bifunctor, and when it isn't, $\operatorname{dcolim}_k \operatorname{dlim}_j D(j,k)$ can differ from $\operatorname{dlim}_j \operatorname{dcolim}_k D(j,k)$.

Theorem

If $\hat{D}$ is a bifunctor, then dagger limits commute with dagger colimits up to unitary iso.

Further topics

- Can be formalized as adjoints to the diagonal such that...
- Oddly completions don't seem to work: dagger equalizers and infinite dagger products imply that the category is indiscrete.
- Can be generalized to an enrichment-free viewpoint on limit-colimit coincidences

Conclusion

- Daglims unique up to unique unitary iso
- Defined for arbitrary diagrams
- Definition doesn't need enrichment
- Generalizes dagger biproducts and dagger equalizers
- Polar decomposition turns limits into dagger limits
- Connections to dagger adjunctions etc.