### Dagger limits

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### Structure of the talk

1. Dagger categories

2. Dagger limits

3. Polar decomposition

4. Further topics?

Dagger = a functorial way of reversing arrows:

$$A \xrightarrow{f = f^{\dagger \dagger}} B$$

$$A \xleftarrow{f^{\dagger}} B$$

Category	Objects	Morphisms	Dagger
Rel	Sets	Relations	inverse
Plnj	Sets	Partial injections	inverse
FHilb	F.d. Hilbert spaces	linear maps	adjoint
Hilb	Hilbert spaces	bounded linear maps	adjoint
Groupoid <b>G</b>	ob( <b>G</b> )	$mor(\mathbf{G})$	inverse

# Dictionary

Ordinary notion	Dagger counterpart	Added condition
Isomorphism	Unitary	$f^{-1} = f^{\dagger}$
Mono	Dagger mono	$f^{\dagger}f = \mathrm{id}$
Epi	Dagger epi	$ff^{\dagger}=\mathrm{id}$
	Partial isometry	$f = ff^{\dagger}f$
Idempotent $p = p^2$	Projection	$ ho= ho^\dagger$
Functor	Dagger Functor	$F(f^{\dagger}) = F(f)^{\dagger}$
Natural transformation	Natural transformation	_
Adjunction $F \dashv G$	Dagger adjunction	F and $G$ dagger
		T dagger and
Monad $(T, \mu, \eta)$	Dagger monad	$\mu_{\mathcal{T}}\circ \mathcal{T}\mu^{\dagger}$
		$= T\mu \circ \mu_T^{\dagger}$

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### What should dagger limits be?

- Unique up to unique unitary
- Defined (canonically) for arbitrary diagrams
- Definition shouldn't depend on additional structure (e.g. enrichment)
- Generalizes dagger biproducts and dagger equalizers
- Connections to dagger adjunctions etc.

## Why is this not (trivially) trivial?

- Unitaries rather than mere isos
- ▶ **DagCat** is not just a 2-category, it is a *dagger* 2-category.
- ▶ I.e. 2-cells have a dagger, so one should require unitary 2-cells etc.
- The forgetful functor DagCat → Cat has both 1-adjoints but no 2-adjoints.
- Previously in CT 2016: only dagger limits of dagger functors.

### **Biproducts**

A biproduct is a product + coproduct

$$A \stackrel{p_A}{\longleftrightarrow} A \oplus B \stackrel{i_B}{\longleftrightarrow} B$$

such that

$$p_A i_A = \mathrm{id}_A$$
  $p_B i_B = \mathrm{id}_B$   
 $p_B i_A = 0_{A,B}$   $p_A i_B = 0_{B,A}$ 

### Known examples of dagger limits

▶ Dagger biproduct of A and B is a biproduct of the form  $(A \oplus B, p_A, p_B, p_A^{\dagger}, p_B^{\dagger})$ 

Dagger equalizer is an equalizer e that is dagger monic

▶ Given a diagram from an indiscrete category J to C: one dagger limit for each choice of  $A \in J$ 

### How to generalize?

- 1. Maps  $A \oplus B \to A$ , B are dagger epic, whereas dagger equalizers  $E \to A$  are dagger monic.
- 2. Requiring the structure maps to be partial isometries generalizes both.
- 3. Based on equalizers and indiscrete diagrams, one can only require this on a weakly initial set.
- 4. One also needs to generalize from  $A \to A \oplus B \to B = 0_{A,B}$
- 5. This can be done by saying that the induced projections on the limit commute.

### Defining dagger limits

#### Definition

Let  $D: \mathbf{J} \to \mathbf{C}$  be a diagram and let  $\Omega \subseteq J$  be weakly initial. A dagger limit of  $(D, \Omega)$  is a limit L of D whose cone  $I_A: L \to D(A)$  satisfies the following two properties:

normalization  $I_A$  is a partial isometry for every  $A \in \Omega$ ; independence the projections on L induced by these partial isometries commute, i.e.  $I_A^\dagger I_A I_B^\dagger I_B = I_B^\dagger I_B I_A^\dagger I_A$  for all  $A, B \in \Omega$ .

### Uniqueness

#### **Theorem**

Let L be a dagger limit of  $(D,\Omega)$  and M a limit of D. The canonical isomorphism  $L \to M$  is unitary iff M is a dagger limit of  $(D,\Omega)$ .

Often  $\Omega$  is forced on us:

- ► Products •
- ▶ Equalizers ⇒ •
- Pullbacks → ← •

But not always:  $\bullet \leftrightarrows \bullet$  or  $\bullet \leftrightarrows \bullet$ 

#### **Definition**

A dagger-shaped dagger limit is the dagger limit of a dagger functor.

E.g. products, limits of projections, unitary representations of groupoids.

#### **Definition**

A set  $\Omega \subset \mathbf{J}$  is a *basis* when every object B allows a unique  $A \in \Omega$  making  $\mathbf{J}(A,B)$  non-empty.

(Finitely) based dagger limit:  $\Omega$  is a (finite) basis

- ▶ Products: •
- ▶ Indiscrete categories ≒ •
- ▶ Nonexample:  $\bullet \rightarrow \bullet \leftarrow \bullet$

- ▶ If **C** has zero morphisms, *L* is a dagger-shaped limit iff
  - ightharpoonup each L o D(A) is a partial isometry
  - ▶  $D(A) \rightarrow L \rightarrow D(B) = 0$  whenever hom(A, B) is empty.
- ▶ If **C** is enriched in commutative monoids, then finitely based dagger limits can be equivalently defined by

$$\mathrm{id}_L = \sum_{A=0}^{\infty} L \to D(A) \to L$$

#### **Theorem**

A dagger category has dagger-shaped limits iff it has dagger split infima of projections, dagger stabilizers, and dagger products.

#### Theorem

A dagger category has all finitely based dagger limits iff it has dagger equalizers, dagger intersections and finite dagger products.

### Interlude: Biproducts without zero morphisms

A biproduct is a product + coproduct

$$A \stackrel{p_A}{\longleftrightarrow} A \oplus B \stackrel{i_B}{\longleftrightarrow} B$$

such that

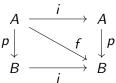
$$\begin{aligned} p_A i_A &= \mathrm{id}_A & p_B i_B &= \mathrm{id}_B \\ i_A p_A i_B p_B &= i_B p_B i_A p_A \end{aligned}$$

This defines biproducts up to iso, requires no enrichment and is equivalent to the usual definitions when enrichment is available. Can be generalized for other limit-colimit coincidences.

### Polar Decomposition

#### Definition

Let  $f: A \rightarrow B$  be a morphism in a dagger category. A *polar decomposition* of f consists of two factorizations of f as f = pi = jp,



where p is a partial isometry and i and j are self-adjoint bimorphisms.

A category *admits polar decomposition* when every morphism has a polar decomposition.

## Polar Decomposition

Fact: Hilb has polar decomposition.

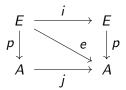
Let f have a polar decomposition f = pi = jp.

If f is an iso, then p is unitary

▶ If f splits a dagger idempotent e, then p is a dagger splitting of it and  $e = pp^{\dagger}$ .

## Polar Decomposition

If  $E \xrightarrow{e} A \Rightarrow B$  is an equalizer and



is a polar decomposition, then  $E \xrightarrow{p} A \Rightarrow B$  is a dagger equalizer.

#### **Theorem**

This works for all **J** with a basis (mod independence)

#### **Theorem**

If  ${\bf C}$  is balanced, one can build from a limit of D a dagger limit of  $D'\cong D$  (mod independence).

### Commuting limits with colimits

Naively, dagger limits should always commute with dagger colimits: given  $D \colon \mathbf{J} \times \mathbf{K} \to \mathbf{C}$ , one would like to define  $\hat{D} \colon \mathbf{J} \times \mathbf{K}^{\mathrm{op}} \to \mathbf{C}$  by "applying the dagger to the second variable" and then calculate as follows:

$$dcolim_k dlim_j D(j, k) = dlim_k dlim_j \hat{D}(j, k)$$
  

$$\cong_{\dagger} dlim_j dlim_k \hat{D}(j, k) = dlim_j dcolim_k D(j, k)$$

However,  $\hat{D}$  is not guaranteed to be a bifunctor, and when it isn't,  $dcolim_k dlim_j D(j, k)$  can differ from  $dlim_j dcolim_k D(j, k)$ .

#### **Theorem**

If  $\hat{D}$  is a bifunctor, then dagger limits commute with dagger colimits up to unitary iso.

### Further topics

► Can be formalized as adjoints to the diagonal such that...

Oddly completions don't seem to work: dagger equalizers and infinite dagger products imply that the category is indiscrete.

 Can be generalized to an enrichment-free viewpoint on limit-colimit coincidences

#### Conclusion

- Daglims unique up to unique unitary iso
- Defined for arbitrary diagrams
- ▶ Definition doesn't need enrichment
- Generalizes dagger biproducts and dagger equalizers
- Polar decomposition turns limits into dagger limits
- Connections to dagger adjunctions etc.