Dagger limits

Martti Karvonen (joint work with Chris Heunen)
Structure of the talk

1. Dagger categories

2. Dagger limits

3. Polar decomposition

4. Further topics?
Dagger = a functorial way of reversing arrows:

\[ A \xrightarrow{f} B \]
\[ A \xleftarrow{f^\dagger} B \]

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<tr>
<th>Category</th>
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<td>Groupoid G</td>
<td>( \text{ob}(G) )</td>
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<td>Isomorphism</td>
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<td>$f^{-1} = f^\dagger$</td>
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<td>Mono</td>
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<td>$f^\dagger f = \text{id}$</td>
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<td>Epi</td>
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<td>Idempotent $p = p^2$</td>
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### Dictionary

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What should dagger limits be?

- Unique up to unique *unitary*

- Defined (canonically) for arbitrary diagrams

- Definition shouldn’t depend on additional structure (e.g. enrichment)

- Generalizes dagger biproducts and dagger equalizers

- Connections to dagger adjunctions etc.
Why is this not (trivially) trivial?

- Unitaries rather than mere isos

- **DagCat** is not just a 2-category, it is a *dagger* 2-category.

- I.e. 2-cells have a dagger, so one should require unitary 2-cells etc.

- The forgetful functor **DagCat** → **Cat** has both 1-adjoints but no 2-adjoints.

- Previously in CT 2016: only dagger limits of dagger functors.
A biproduct is a product + coproduct

\[ A \leftarrow_{p_A} i_A \quad A \oplus B \leftarrow_{i_B} \rightarrow_{p_B} B \]

such that

\[ p_A i_A = \text{id}_A \quad \quad p_B i_B = \text{id}_B \]
\[ p_B i_A = 0_{A,B} \quad \quad p_A i_B = 0_{B,A} \]
Known examples of dagger limits

- Dagger biproduct of $A$ and $B$ is a biproduct of the form
  \[(A \oplus B, p_A, p_B, p_A^\dagger, p_B^\dagger)\]

- Dagger equalizer is an equalizer $e$ that is dagger monic

- Given a diagram from an indiscrete category $J$ to $C$: one
dagger limit for each choice of $A \in J$
How to generalize?

1. Maps $A \oplus B \to A, B$ are dagger epic, whereas dagger equalizers $E \to A$ are dagger monic.

2. Requiring the structure maps to be partial isometries generalizes both.

3. Based on equalizers and indiscrete diagrams, one can only require this on a weakly initial set.

4. One also needs to generalize from $A \to A \oplus B \to B = 0_{A,B}$

5. This can be done by saying that the induced projections on the limit commute.
Defining dagger limits

Definition
Let \( D : J \rightarrow \mathbf{C} \) be a diagram and let \( \Omega \subseteq J \) be weakly initial. A \textit{dagger limit of} \((D, \Omega)\) is a limit \( L \) of \( D \) whose cone \( l_A : L \rightarrow D(A) \) satisfies the following two properties:

- \textbf{normalization} \( l_A \) is a partial isometry for every \( A \in \Omega \);
- \textbf{independence} the projections on \( L \) induced by these partial isometries commute, \( i.e. \ l_A^\dagger l_A l_B^\dagger l_B = l_B^\dagger l_B l_A^\dagger l_A \) for all \( A, B \in \Omega \).
Uniqueness

**Theorem**

Let $L$ be a dagger limit of $(D, \Omega)$ and $M$ a limit of $D$. The canonical isomorphism $L \to M$ is unitary iff $M$ is a dagger limit of $(D, \Omega)$.

Often $\Omega$ is forced on us:

- Products
- Equalizers
- Pullbacks

But not always: $\bullet \iff \bullet$ or $\bullet \iff \bullet$
Definition
A dagger-shaped dagger limit is the dagger limit of a dagger functor.
E.g. products, limits of projections, unitary representations of groupoids.

Definition
A set $\Omega \subset J$ is a basis when every object $B$ allows a unique $A \in \Omega$ making $J(A, B)$ non-empty.

(Finitely) based dagger limit: $\Omega$ is a (finite) basis

- Products: $\bullet \quad \bullet$
- Equalizers: $\bullet \iff \bullet$
- Indiscrete categories $\bullet \leftrightarrow \bullet$
- Nonexample: $\bullet \to \bullet \leftarrow \bullet$
If $C$ has zero morphisms, $L$ is a dagger-shaped limit iff

- each $L \to D(A)$ is a partial isometry
- $D(A) \to L \to D(B) = 0$ whenever $\text{hom}(A, B)$ is empty.

If $C$ is enriched in commutative monoids, then finitely based dagger limits can be equivalently defined by

$$\text{id}_L = \sum_{A \in \Omega} L \to D(A) \to L$$
Theorem
A dagger category has dagger-shaped limits iff it has dagger split infima of projections, dagger stabilizers, and dagger products.

Theorem
A dagger category has all finitely based dagger limits iff it has dagger equalizers, dagger intersections and finite dagger products.
Interlude: Biproducts without zero morphisms

A biproduct is a product + coproduct

\[
A \overset{p_A}{\leftarrow} A \oplus B \overset{i_B}{\leftarrow} B
\]

such that

\[
p_A i_A = \text{id}_A \quad p_B i_B = \text{id}_B
\]

\[
i_A p_A i_B p_B = i_B p_B i_A p_A
\]

This defines biproducts up to iso, requires no enrichment and is equivalent to the usual definitions when enrichment is available. Can be generalized for other limit-colimit coincidences.
Polar Decomposition

Definition
Let \( f : A \to B \) be a morphism in a dagger category. A polar decomposition of \( f \) consists of two factorizations of \( f \) as \( f = pi = jp \),

\[
\begin{array}{ccc}
A & \xrightarrow{i} & A \\
\downarrow{p} & & \downarrow{p} \\
B & \xrightarrow{j} & B \\
\end{array}
\]

where \( p \) is a partial isometry and \( i \) and \( j \) are self-adjoint bimorphisms.
A category admits polar decomposition when every morphism has a polar decomposition.
Polar Decomposition

Fact: **Hilb** has polar decomposition.

Let $f$ have a polar decomposition $f = pi = jp$.

- If $f$ is an iso, then $p$ is unitary

- If $f$ splits a dagger idempotent $e$, then $p$ is a dagger splitting of it and $e = pp^\dagger$. 
Polar Decomposition

If $E \xrightarrow{e} A \supseteq B$ is an equalizer and

$$
\begin{array}{c}
E \xrightarrow{i} E \\
\downarrow p \\
A \xrightarrow{j} A
\end{array}
\quad \begin{array}{c}
E \xleftarrow{e} E \\
\downarrow p
\end{array}
$$

is a polar decomposition, then $E \xrightarrow{p} A \supseteq B$ is a dagger equalizer.

**Theorem**

*This works for all $J$ with a basis (mod independence)*

**Theorem**

*If $C$ is balanced, one can build from a limit of $D$ a dagger limit of $D' \cong D$ (mod independence).*
Commuting limits with colimits

Naively, dagger limits should always commute with dagger colimits: given $D : J \times K \to C$, one would like to define $\hat{D} : J \times K^{\text{op}} \to C$ by “applying the dagger to the second variable” and then calculate as follows:

$$d\text{colim}_k d\text{lim}_j D(j, k) = d\text{lim}_k d\text{lim}_j \hat{D}(j, k)$$

$$\cong^{\dagger} d\text{lim}_j d\text{lim}_k \hat{D}(j, k) = d\text{lim}_j d\text{colim}_k D(j, k)$$

However, $\hat{D}$ is not guaranteed to be a bifunctor, and when it isn’t, $d\text{colim}_k d\text{lim}_j D(j, k)$ can differ from $d\text{lim}_j d\text{colim}_k D(j, k)$.

**Theorem**

*If $\hat{D}$ is a bifunctor, then dagger limits commute with dagger colimits up to unitary iso.*
Further topics

- Can be formalized as adjoints to the diagonal such that...

- Oddly completions don’t seem to work: dagger equalizers and infinite dagger products imply that the category is indiscrete.

- Can be generalized to an enrichment-free viewpoint on limit-colimit coincidences
Conclusion

- Daglims unique up to unique unitary iso
- Defined for arbitrary diagrams
- Definition doesn’t need enrichment
- Generalizes dagger biproducts and dagger equalizers
- Polar decomposition turns limits into dagger limits
- Connections to dagger adjunctions etc.