Artin glueings of frames and toposes as semidirect products

Peter Faul
peter@faul.io
University of Cambridge

CT 2019, Edinburgh
Artin glueings of topological spaces

Let $N = (|N|, \mathcal{O}(N))$ and $H = (|H|, \mathcal{O}(H))$ be topological spaces.

What topological spaces $G = (|G|, \mathcal{O}(G))$ satisfy that $H$ is an open subspace and $N$ its closed complement?

Such a space $G$ we call an Artin glueing of $H$ by $N$.

It must be that $|G| = |N| \sqcup |H|$.

Each open $U \in \mathcal{O}(G)$ then corresponds to a pair $(U_N, U_H)$ where $U_N = U \cap N$ and $U_H = U \cap H$.

Thus $\mathcal{O}(G)$ is isomorphic to the frame $L_G$ of certain pairs $(U_N, U_H)$ with componentwise union and intersection.
The associated meet-preserving map

For each $U \in \mathcal{O}(H)$ there is a largest open $V \in \mathcal{O}(N)$ such that $(V, U)$ occurs in $L_G$.

Let $f_G: \mathcal{O}(H) \to \mathcal{O}(N)$ be a function which assigns to each $U \in \mathcal{O}(H)$ the largest $V$.

This function preserves finite meets.

We have that $(V, U) \in L_G$ if and only if $V \subseteq f(U)$.

Given any finite-meet preserving map $f: \mathcal{O}(H) \to \mathcal{O}(N)$ we can construct a frame $\text{Gl}(f)$ as above.

This frame $\text{Gl}(f)$ will satisfy the required properties and we call it the Artin glueing of $f$. 
Semidirect products of groups are analogous

Let $N$ and $H$ be groups.

A semidirect product of $H$ by $N$ is a group $G$ satisfying the following conditions.

1. $H$ is a subgroup and $N$ a normal subgroup.
2. $N \cap H = \{e\}$.
3. $N \vee H = NH = \{nh : n \in N, h \in H\} = G$.

Conditions 2 and 3 together say that $N$ and $H$ are complements.

Every such group $G$ is determined by a unique group homomorphism $\alpha : H \to \text{Aut}(N)$. 
Split extensions of groups

A diagram $N \xrightarrow{k} G \xleftarrow{e} H$ is a split extension when the following hold:

1. $k$ is the kernel of $e$,
2. $e$ is the cokernel of $k$,
3. $se = id$.

Every semidirect product of $N$ and $H$ yields a split extension.

Every split extension is of this form.

Every element $g \in G$ can be written $g = k(n)s(h)$ for $n \in N$ and $h \in H$. Thus $G$ is generated by the images of $k$ and $s$. 

5
Artin glueings of frames as extensions

We want a pointed category in which there is a class of extensions of $H$ by $N$ which are precisely the Artin glueings of $N$ and $H$.

The usual category of locales is no good as it does not have zero morphisms.

Instead consider the category $\mathbf{RFrm}$ whose objects are frames and whose morphisms are finite-meet preserving maps.

The maps $\top_{X,Y} : X \to Y$ sending each element of $X$ to the top element 1 of $Y$ form a class of zero morphisms.
Cokernels

The cokernel of a morphism \( f : N \to G \) in RFrm exists and is \( e : G \to \downarrow f(0) \), where \( e(g) = g \land f(0) \).

We call such a map a normal epimorphism.

Furthermore \( e \) has a right adjoint section \( e_*(x) = (f(0) \Rightarrow x) \).

- Let \( t : G \to X \) be such that \( tf = \top \).

\[
\begin{array}{ccc}
N & \xrightarrow{f} & G \\
\downarrow & & \downarrow e \\
H & \xleftarrow{e_*} & X \\
\downarrow t & & \downarrow te_* \\
X & & \\
\end{array}
\]

- If \( e(x) = e(y) \) (i.e. \( x \land f(0) = y \land f(0) \)) then

\[
t(x) = t(x) \land 1 = t(x) \land t(f(0)) = t(x \land f(0)) = t(y \land f(0)) = t(y)
\]

- Because \( e(e_*e(x)) = e(x) \) we have that \( t(e_*e(x)) = t(x) \).

- Thus \( t = te_*e \) and \( te_* \) is the unique such map as \( e \) is epic.
Kernels do not always exist in RFrm, but kernels of normal epis do. The kernel of a normal epi $e : G \rightarrow \downarrow u$ is the inclusion $k : \uparrow u \hookrightarrow G$. It has a left adjoint $k^*(x) = x \lor u$.

Split extensions in RFrm are of the form $\uparrow u \xrightarrow{k} G \xleftarrow{s} \downarrow u$.

Notice that $\uparrow u$ is the closed sublocale corresponding to $u$ and that $\downarrow u$ is isomorphic to the open sublocale corresponding to $u$.

Only the splitting remains mysterious.
Schreier-type extensions and the splitting

Recall that if \( N \xrightarrow{k} G \xleftarrow{e} H \) is a split extension of groups, then each element of \( G \) can be written \( g = k(n)s(h) \).

For split extensions of monoids this does always hold. Split extensions where this holds are called weakly Schreier.

In \( \text{RFrm} \) a split extension will be weakly Schreier if and only if \( s \) is the right adjoint of \( e \).

Thus a weakly Schreier extension is always of the form
\[
\uparrow u \xrightarrow{k} G \xleftarrow{e} \downarrow u. \quad \text{It is entirely determined by the normal epi } e.
\]
Artin glueings are weakly Schreier

Recall that the Artin glueing $\text{Gl}(f)$ of a finite-meet preserving map $f : H \to N$ is the frame of pairs $(n, h)$ where $n \leq f(h)$.

We have projections $\pi_1 : \text{Gl}(f) \to N$ and $\pi_2 : \text{Gl}(f) \to H$ which preserve finite meets.

These projections have right adjoints which lie in $\text{RFrm}$.

1. $\pi_1^*(n) = (n, 1)$.
2. $\pi_2^*(h) = (f(h), h)$.

We have that $N \xrightarrow{\pi_1^*} \text{Gl}(f) \xleftarrow{\pi_2^*} H$ is a weakly Schreier extension.

Think of $\pi_2$ as the map $(-) \wedge (0, 1) : \text{Gl}(f) \to \downarrow(0, 1)$.

Notice that $f$ can be recovered via $f = \pi_1 \pi_2^*$. 
All extensions are Artin glueings

From a weakly Schreier extension $N \xrightarrow{k} G \xleftarrow{e} H$ we can form

\[ N \xrightarrow{\pi_1^*} \text{Gl}(k^*e_*) \xrightarrow{\pi_2} H. \]

The frames $G$ and $\text{Gl}(k^*e_*)$ are isomorphic

1. $f : G \to \text{Gl}(k^*e_*)$ sends $g$ to $(k^*(g), e(g))$,
2. $f^{-1} : \text{Gl}(k^*e_*) \to G$ sends $(n, h)$ to $k(n) \land e_*(h)$.

Furthermore $f$ and $f^{-1}$ make the three squares below commute.

\[
\begin{array}{c}
N \xrightarrow{k} G \xleftarrow{e} H \\
\| \quad \frown \quad \frown \\
N \xrightarrow{\pi_1^*} \text{Gl}(k^*e_*) \xleftarrow{\pi_1} H
\end{array}
\]
Further thoughts

Since Artin glueings correspond to finite-meet preserving maps, we have that $\text{Hom}(−,−)$ is the bifunctor of weakly Schreier extensions.

The homsets of $\mathbb{RFrm}$ are actually meet-semilattices. This operation gives a Baer sum on our extensions.


The Artin glueing construction works on toposes and there is an analogue on that level.
Artin glueings of toposes

Given toposes $\mathcal{N}$ and $\mathcal{H}$ we can ask for which toposes $\mathcal{G}$ is $\mathcal{H}$ an open subtopos and $\mathcal{N}$ its closed complement.

Such toposes will always be determined by a left exact functor $F: \mathcal{H} \to \mathcal{N}$ in the following way.

- The objects are triples $(N, H, \ell)$ such that $\ell: N \to F(H)$.
- The morphisms are pairs $(f, g): (N, H, \ell) \to (N', H', \ell')$ such that

$$
\begin{array}{ccc}
N & \xrightarrow{f} & N' \\
\downarrow{\ell} & & \downarrow{\ell'} \\
F(H) & \xrightarrow{F(g)} & F(H')
\end{array}
$$

commutes.

We write $\mathcal{G}_1(F)$ and call this topos the Artin glueing along $F$. 
We can consider the 2-category $\mathbf{RTopos}$ of toposes with left exact functors and natural transformations.

Between any two toposes there are left exact functors that send every object in the domain to a terminal object in the codomain. These behave like zero morphisms.

We can define the 2-cokernel of a left exact functor $F$ as the 2-coequaliser of $F$ with a zero morphism. Similarly we can define the 2-kernel.
2-Cokernels and 2-kernels

2-Cokernels always exist.

Given a left exact functor $F$ the 2-cokernel is the open subtopos $\mathcal{O}_{F(0)}$ corresponding to the subterminal $F(0)$.

2-Kernels in general do not exist but 2-kernels of 2-cokernels $E : \mathcal{G} \to \mathcal{O}_U$ do exist.

It is given by the inclusion of the full subcategory of objects sent to a terminal object by $E$.

This turns out to be the inclusion $K : \mathcal{C}_U \to \mathcal{G}$ of the closed subtopos corresponding to the subterminal $U$. 
Adjoints extensions

An adjoint extension is a diagram \( \mathbf{N} \xrightarrow{K} \mathbf{G} \xleftarrow{E} \mathbf{H} \) where

1. \( K \) is the 2-kernel of \( E \) and
2. \( E \) is the 2-cokernel of \( K \).

We see that the adjoint extensions in \( \mathbf{RTopos} \) are all of the form \( \mathfrak{C}_U \xrightarrow{K} \mathbf{G} \xleftarrow{E} \mathfrak{D}_U \).

But this is the inclusion of complemented subtoposes, so this is an Artin glueings.

\( \mathfrak{G} \) is given by \( \text{Gl}(K^*E^*) \).
Further thoughts

Similarly we can think of $\text{Hom}$ as giving an extension bifunctor.

Since $\text{Hom}(-,-)$ returns finitely complete categories we should be able to compute limits of extensions.

There seems to be a relationship to recollements, the additive (or triangulated) version of Artin glueings.