A framework for formal higher category theory

- Virtual Double Categories
- Modules
- Globular Multicategories
- Higher Modules
- Weakening
Formal Category Theory

- Abstract setting for studying “category-like” structures
- Key notions of category theory can be defined once and for all
Virtual Double Categories

A virtual double category consists of a collection of:

- **objects** or 0-types

  \[ A : 0 \text{-Type} \]

- **0-terms**

  \[ x : A \vdash fx : B \]
Virtual Double Categories

- 1-types

\[ \frac{M}{A \xrightarrow{M} B} \]

\[ x : A, y : B \vdash M(x, y) : 1 \text{-Type}(A, B) \]
Virtual Double Categories

1-terms

\[ \begin{array}{c}
A \xrightarrow{M} B \xrightarrow{N} C \\
f \downarrow \phi \downarrow g \\
D \xrightarrow{O} E
\end{array} \]

\[ m : M(x, y), \; n : N(y, z) \vdash \phi(m, n) : O(fx, gz) \]

\[ \begin{array}{c}
A \xrightarrow{=} A \\
f \downarrow \psi \downarrow g \\
D \xrightarrow{O} D
\end{array} \]

\[ a : A \vdash \psi(a) : O(fa, ga) \]
Terms have an associative and unital notion of composition
Example: Virtual Double Category of Categories

- 0-types are categories
  - $\bullet$

- 0-terms are functors
  - $\bullet$
    - $\Downarrow$
    - $\bullet$

- 1-types are profunctors

- 1-terms are transformations between profunctors

Example: Virtual Double Category of Spans

For any category $C$ with pullbacks, there is a virtual double category $\text{Span}(C)$ whose:

- 0-types are objects of $C$
- 0-terms are arrows of $C$
- 1-types are spans
- 1-terms are transformations between spans.
Example: Virtual Double Category of Spans

1-terms are transformations between spans. A term

\[
\begin{array}{c}
A \xrightarrow{M} B \xrightarrow{N} C \\
\downarrow \quad \quad \quad \downarrow \\
D \quad \quad \quad O \\
\downarrow \quad \quad \quad \downarrow \\
E
\end{array}
\]

corresponds to a diagram

\[
\begin{array}{c}
A \xrightarrow{M \times B \ N} C \\
\downarrow \quad \quad \quad \downarrow \\
D \quad \quad \quad E
\end{array}
\]
Identity Types

Typically for any 0-type $A$, there is a 1-type

$$A \xrightarrow{\mathcal{H}_A} A$$

which can be thought of as the **Hom-type** of $A$. This comes with a canonical **reflexivity term**

$$a : A \vdash r_A : \mathcal{H}_A(a, a)$$
Identity Types

Composition with

\[
\begin{array}{c}
A \\ r_A \\ \mathcal{H}_A \\
A \rightarrow A
\end{array}
\]

gives a bijection between terms of the following forms:

\[
\begin{array}{c}
A \\ r_A \\ M \\
A \rightarrow A
\end{array}
\]

\[
\begin{array}{c}
A \\ r_A \\ M \\
A \rightarrow A
\end{array}
\]

This is an abstract form of the Yoneda Lemma.
Identity Types

Composition with

\[
\begin{array}{c}
A \\ \downarrow \quad r_A \\ A \\
\end{array}
\]

\[
A \xrightarrow{\mathcal{H}_A} A
\]

gives a bijection between terms of the following forms:

\[
\begin{array}{c}
A \\ \downarrow \\ B \\
\end{array}
\xrightarrow{\mathcal{H}_A}
\begin{array}{c}
A \\ \downarrow \\ C \\
\end{array}
\]

\[
\begin{array}{c}
A \\ \downarrow \\ C \\
\end{array}
\]

\[
p : \mathcal{H}_A(x, y) \vdash \phi(p) : M(x, y)
\]

\[
a : A \vdash \phi(r_a) : M(a, a)
\]

This is an abstract form of the **Yoneda Lemma**.
Identity Types

Composition with

\[
\begin{array}{ccc}
A & \xrightarrow{\mathcal{H}_A} & A \\
\downarrow r_A & & \downarrow \text{id}_M \\
A & \xrightarrow{M} & B
\end{array}
\]

gives a bijection between terms of the following forms:

\[
\begin{array}{ccc}
A & \xrightarrow{\mathcal{H}_A} & A & \xrightarrow{M} & B \\
& \downarrow f & & \downarrow g & \\
C & \xrightarrow{N} & D
\end{array}
\quad
\begin{array}{ccc}
A & \xrightarrow{M} & B \\
& \downarrow f & & \downarrow g \\
C & \xrightarrow{N} & D
\end{array}
\]

\[
p : \mathcal{H}_A(x, y), m : M(y, z) \vdash \phi(x, y, z, p, m) : N(fx, gz)
\]

\[
y : A, m : M(y, z) \vdash \phi(y, y, z, r_y, m) : N(fy, gz)
\]
Identity Types

In fact $\mathcal{H}_A$ and $r_A$ are characterised by such properties. We say that a virtual double category with this data has **identity types**.
Identity Types

- Let $\text{VDbl}$ be the category of virtual double categories.
- Let $\overline{\text{VDbl}}$ be the category of virtual double categories with identity types.
- The forgetful functor $U : \overline{\text{VDbl}} \to \text{VDbl}$ has both a left and a right adjoint.
- The right adjoint $\text{Mod}$ is the monoids and modules construction.
Monoids and Modules

Given any virtual double category $X$, there is a virtual double
Mod($X$) such that:

- 0-types are monoids in $X$
  A monoid consists of a 0-type $A$, a 1-type $\mathcal{H}_A$ together with a unit
  \[
  \begin{array}{c}
  A \xrightarrow{=} A \\
  \| \quad \downarrow r_A \quad \| \\
  A \xrightarrow{\mathcal{H}_A} A
  \end{array}
  \]
  and a multiplication
  \[
  \begin{array}{c}
  A \xrightarrow{\mathcal{H}_A} A \xrightarrow{\mathcal{H}_A} A \\
  \| \quad \downarrow m_A \quad \|
  \end{array}
  \]
satisfying unit and associativity axioms.
0-terms are monoid homomorphisms in $X$. A monoid homomorphism $f : A \rightarrow B$ is a term:

\[
\begin{array}{ccc}
A & \xrightarrow{\mathcal{H}_A} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{\mathcal{H}_B} & B
\end{array}
\]

compatible with the multiplication and unit terms of $A$ and $B$. 
1-types are modules in \( X \). A module \( M : A \to B \) consists of a 1-type \( M \) together with left and right multiplication terms

\[
\begin{align*}
A & \overset{\mathcal{H}_A}{\longrightarrow} A & A & \overset{M}{\longrightarrow} B \\
A & \overset{\lambda_M}{\downarrow} & A & \overset{\rho_M}{\downarrow} \\
A & \overset{\mathcal{H}_A}{\longrightarrow} A & A & \overset{\mathcal{H}_A}{\longrightarrow} A
\end{align*}
\]

compatible with the multiplication of \( A \) and \( B \) and each other.
1-terms are module homomorphisms in $X$. A typical module homomorphism $f$ is a term

$$
\begin{array}{c}
A \xrightarrow{M} B \xrightarrow{N} C \\
\downarrow f \\
D \xrightarrow{O} E
\end{array}
$$

satisfying equivariance laws.
Equivariance Laws

For example

\[ A \xrightarrow{M} B \xrightarrow{\mathcal{H}_B} B \xrightarrow{N} C \]
\[ A \xrightarrow{M} B \xrightarrow{\mathcal{H}_B} B \xrightarrow{N} C \]

\[ \Downarrow \rho_M \quad \Downarrow f \quad \Downarrow f \]

\[ \Downarrow \lambda_N \]

\[ D \xrightarrow{O} E \]

\[ D \xrightarrow{O} E \]
Monoids and Modules

Many familiar types of “category-like” object are the result of applying the monoids and modules construction. For example:

- The virtual double category of categories internal to $\mathcal{C}$ is $\text{Mod}(\text{Span}(\mathcal{C}))$.
See

- T. Leinster. Higher Operads, Higher Categories
- G.S.H. Cruttwell and Michael A. Shulman. A unified framework for generalized multicategories
Virtual double categories are $T$-multicategories where $T$ is the free category monad on 1-globular sets.

- Shapes of pasting diagrams of arrows in a category are parametrised by $T1$.
- The terms of a virtual double category are arrows sending such pasting diagrams of types to types.

\[
\begin{array}{c}
X_1 \\
\downarrow \\
TX_0 \\
\downarrow \\
X_0
\end{array}
\]
Virtual double categories are \( T \)-multicategories where \( T \) is the free category monad on 1-globular sets.

- Shapes of pasting diagrams of arrows in a category are parametrised by \( T1 \).
- The terms of a virtual double category are arrows sending such pasting diagrams of types to types.

What about other \( T \)? In particular the free strict \( \omega \)-category monad on globular sets
A **globular multicategory** consists of a collection of:

- 0-types
- For each $n \geq 1$, $n$-types

Suppose that we have parallel $(n-1)$-types $A$ and $B$. Given $M(u, v) : n$-Type$(A, B)$ and $N(u, v) : n$-Type$(A, B)$, we have

$$x : M(u, v), y : N(u, v) \vdash O(x, y) : (n + 1)$-Type$(M, N)$
Globular Multicategories

- $n$-terms sending a pasting diagram of types to an $n$-type.
Globular Multicategories

\[ \Gamma = A \xrightarrow{M} B \xrightarrow{L} A \]

\[ \Gamma(0) = [a : A, b : B, a' : A, b' : B] \]
\[ \Gamma(1) = [m : M(a, b), m' : M(a, b), n : N(a, b), l : L(b, a'), m' : M(a', b'), n' : N(a', b')] \]
\[ \Gamma(2) = [o : O(m, n), p : P(m, n'), q : Q(m', n')] \]

We have

\[ \Gamma \vdash \phi(l, o, p, q) : O(a, b') \]
Example: Globular Multicategory of Spans

For any category $C$ with pullbacks, there is a globular multicategory $\text{Span}(C)$ whose:

- 0-types are objects of $C$
- 1-types are spans
- 2-types are spans between spans (That is 2-spans.)
- 3-types are spans between 2-spans (That is 3-spans.)
Example: Globular Multicategory of Spans

For any category $C$ with pullbacks, there is a globular multicategory $\text{Span}(C)$ whose:

- 0-types are objects of $C$
- 1-types are spans
- 2-types are spans between spans (or 2-spans)
- 3-types are spans between 2-spans (That is 3-spans). That is a diagram
Example: Globular Multicategories of Spans

For any category $C$ with pullbacks, there is a globular multicategory $\text{Span}(C)$ whose:

- 0-types are sets
- 0-terms are functions
- 1-types are spans
- 2-types are spans between spans (or 2-spans)
- 3-types are spans between 2-spans (That is 3-spans).
- etc.
- Terms are transformations from a pullback of spans to a span.
Globular Multicategories associated to Type Theories

- There is a globular multicategory associated to any model of dependent type theory
- Types, contexts and terms correspond to the obvious things in the type theory.
- See Benno van den Berg and Richard Garner. Types are weak $\omega$-groupoids
There is a globular multicategory associated to any model of dependent type theory.

Types, contexts and terms correspond to the obvious things in the type theory.

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When we have identity types, what structure does this globular multicategory have?
Globular Multicategories with Strict Identity Types

- For each $n$-type $M$, we require an identity $(n + 1)$ type $\mathcal{H}_M$ with a reflexivity term $r : M \rightarrow \mathcal{H}_M$.

\[ A \xrightarrow{M} B \xleftarrow{r_M} A \]

- Composition with reflexivity terms gives bijective correspondences which “add and remove identity” types.
Globular Multicategories with Strict Identity Types

- The forgetful functor

\[ U : \text{GlobMult} \rightarrow \text{GlobMult} \]

has both a left and a right adjoint.

- The right adjoint Mod is the strict higher modules construction.
Higher Modules

In general, $n$-modules can be acted on by their $k$-dimensional source and target modules for any $k < n$. 
Higher Modules

Given a 2-module $O$, depicted

there are actions whose sources are
Higher Module Homomorphisms

Given a homomorphism $f$ with source $\Gamma$, there is an equivariance law for each place in $\Gamma$ that an identity type can be added.
Higher Module Homomorphisms

Given a homomorphism $f$ with source

there are two ways of building terms with source

using either left or right actions.
Globular multicategory of strict $\omega$-categories

Applying this construction to $\text{Span}(\text{Set})$ we obtain a globular multicategory whose

- 0-types are strict $\omega$-categories,
- 1-types are profunctors
- 2-types are profunctors between profunctors
- etc.
- 0-terms are strict $\omega$-functors,
- Higher terms are transformations between profunctors
Let

\[ U : \overline{T}\text{-Mult} \to \overline{T}\text{-Mult} \]

be the functor which forgets strict identity types. Let

\[ F : T\text{-Mult} \to \overline{T}\text{-Mult} \]

be its left adjoint. Let \( u \) be a generic type (or term). We have

\[ u \to U \text{Mod}(X) \]
\[ Fu \to \text{Mod}(X) \]
\[ UFu \to X \]
Weakening

- The boundary inclusions of the shapes of globular multicategory cells, induce a weak factorization system.
- A weak map of globular multicategories is a strict map from a cofibrant replacement

\[ QX \rightarrow Y \]

- Thus, we define a weak $n$-module (or homomorphism) to be a map

\[ QUFu \rightarrow X \]

- Weak 0-modules are precisely Batanin-Leinster $\omega$-categories. See Richard Garner. A homotopy-theoretic universal property of Leinster’s operad for weak $\omega$-categories
Composition of weak higher module homomorphisms

A pair of composable terms in a globular multicategory is the same as a diagram
Composition of weak higher module homomorphisms

Let $\Gamma$ be a context in $X$ with shape $\pi$ and let $u : \Delta \to \Gamma, v : \Gamma \to A$ be a composable pair in $X$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\pi & \xrightarrow{\Gamma} & X \\
\downarrow \pi & & \downarrow g \\
\Delta & \xrightarrow{\Gamma} & \Gamma \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi & \xrightarrow{\Gamma} & X \\
\downarrow \pi & & \downarrow g \\
\Delta & \xrightarrow{\Gamma} & \Gamma \\
\end{array}
\]

Hence, we have a diagram

\[
\begin{array}{ccc}
\pi & \xrightarrow{\Gamma} & X \\
\downarrow \pi & & \downarrow g \\
\Delta & \xrightarrow{\Gamma} & \Gamma \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi & \xrightarrow{\Gamma} & X \\
\downarrow \pi & & \downarrow g \\
\Delta & \xrightarrow{\Gamma} & \Gamma \\
\end{array}
\]

\[
\begin{array}{ccc}
\pi & \xrightarrow{\Gamma} & X \\
\downarrow \pi & & \downarrow g \\
\Delta & \xrightarrow{\Gamma} & \Gamma \\
\end{array}
\]
Composition of weak higher module homomorphisms

- Let $w$ be the shape of $u, v$. Then $f; g$ is defined by the following commutative diagram:

\[
\begin{array}{c}
w \xrightarrow{\text{composite}} u +_\pi v \xrightarrow{f+\gamma g} X \\
\end{array}
\]

- Since $UF$ is cocontinuous, composition of strict homomorphisms defined by the following commutative diagram:

\[
\begin{array}{c}
UFw \xrightarrow{UF(\text{composite})} UFu +_{UF\pi} UFv \xrightarrow{f+\gamma g} X \\
\end{array}
\]
Composition of weak higher module homomorphisms

- We would like a diagram

\[
\begin{array}{c}
QUF_w \
\downarrow \quad QUF_{(\text{composite})} \\
QUF_u +_{QUF_\pi} QUF_v \\f +_g \quad \downarrow \\
X \\
\end{array}
\]

but \( Q \) is not cocontinuous.

- However \( QUF_u +_{QUF_\pi} QUF_v \) is still cofibrant. This allows us to construct a well-behaved composition map

\[
\begin{array}{c}
QUF_w \\
\rightarrow \\
QUF_u +_{QUF_\pi} QUF_v \\
\end{array}
\]
Weak Modules

Applying this construction to Span(Set) we obtain notions of

- Weak $\omega$-categories, profunctors, profunctors between profunctors, etc.
- Weak transformations between profunctors
- Composition of these terms
Weak Modules

Applying this construction to $\text{Span}(\text{Set})$ we obtain notions of

- Weak $\omega$-categories, profunctors, profunctors between profunctors, etc.
- Weak transformations between profunctors
- Composition of these terms

We can use data to construct an $\omega$-category of $\omega$-categories
Future Work

- Semi-strictness results and comparison to dependent type theory.
- Develop higher category theory and higher categorical logic.
Thanks

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