Uniform Kan fibrations in simplicial sets
(jww Eric Faber)

Benno van den Berg
ILLC, University of Amsterdam

Category Theory 2019, Edinburgh, 8 July 2019

Warning

This is work in progress and not as thoroughly checked as I would have liked!

Terminology might still change!

I'm in a hurry, so I will not have time to properly discuss related work!

Section 1

The main question

Kan fibrations

Definition

A map $p : Y \to X$ is a *Kan fibration* if for any horn $\Lambda^n_k \to \Delta^n$ and any commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

there exists a dotted arrow making both triangles commute.

First question

In a constructive setting, should we demand the existence of such fillers (property) or should we say that a Kan fibration is a map equipped with a choice of fillers (structure)?

Kan fibrations

Definition

A map $p : Y \to X$ is a *Kan fibration* if for any horn $\Lambda^n_k \to \Delta^n$ and any commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

there exists a dotted arrow making both triangles commute.

First question

In a constructive setting, should we demand the existence of such fillers (property) or should we say that a Kan fibration is a map equipped with a choice of fillers (structure)?

Our answer

It should be structure!

Uniform Kan fibrations: the very idea

Definition

A map $p : Y \to X$ is a *algebraic Kan fibration* if for any commutative diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

it comes equipped with a choice of filler (the dotted arrow).

Second question

In a constructive setting, should these fillers satisfy some compatibility conditions or can they be completely unrelated?

Uniform Kan fibrations: the very idea

Definition

A map $p : Y \to X$ is a *algebraic Kan fibration* if for any commutative diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

it comes equipped with a choice of filler (the dotted arrow).

Second question

In a constructive setting, should these fillers satisfy some compatibility conditions or can they be completely unrelated?

Our answer

Some compatibility (uniformity) conditions should be satisfied!

Goal

But what should the compatibility/uniformity conditions be?

Purpose of the talk

Propose a (new) definition of a *uniform Kan fibration*.

Section 2

Algebraic weak factorisation systems

Algebraic weak factorisation systems

Functorial factorisation

A functor $\mathcal{C}^{\to} \to \mathcal{C}^{\to\to}$ is a *functorial factorisation* on a category $\mathcal{C}$ if it is a section of the composition functor $\circ : \mathcal{C}^{\to\to} \to \mathcal{C}^{\to}$.

So a functorial factorisation writes every map f in $\mathcal{C}$ as a composition:

$$
\begin{tikzcd}
X \arrow[rr, "f"'] \arrow[d, dotted, no head] & & Y \arrow[d, dotted, no head] \\
X \arrow[r, "Lf"] & Ef \arrow[r, "Rf"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\
Ef \arrow[r, "Rf"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\
Y \arrow[r, "1"'] & Y
\end{tikzcd}
$$

This turns the functors L and R into (co)pointed endofunctors $\mathcal{C}^{\to} \to \mathcal{C}^{\to}$.

Algebraic weak factorisation system (Grandis-Tholen-Garner)

A functorial factorisation is an algebraic weak factorisation system (AWFS) if L and R can be extended to a comonad and a monad on $\mathcal{C}^{\to}$, respectively, and a distributive law holds (for the comonad over the monad).

Left and right maps

Given an AWFS:

- a *left map* is a coalgebra for the comonad.
- a *right map* is an algebra for the monad.

Both are closed under composition and the left maps have the LLP wrt to the right maps.

But the classes are *not* closed under retracts. Their retract closures give one an ordinary weak factorisation system.

Cofibrations

Cofibrations, constructively

A map $f : Y \to X$ in simplicial sets is a *cofibration* if it is a monomorphism, and given any $x \in X_n$, we can decide whether x lies in the image of f, and if so, we can effectively find the $y \in Y_n$ such that $f_n(y) = x$.

These cofibrations form the left class in an AFWS. The associated right class we will call *uniform trivial Kan fibrations*.

Simplicial Moore path object

In Van den Berg & Garner, we defined a simplicial Moore path functor. The idea is that there is an endofunctor M on simplicial sets together with natural transformations $r : X \to MX$, $s, t : MX \to X$ and $\circ : MX \times_X MX \to MX$ equipping X with the structure of an internal category. In addition, there is a contraction $\Gamma : MX \to MMX$.

Two new results:

Theorem

This functor M is polynomial.

Theorem

The functorial factorisation sending $f : Y \to X$ to

$$Y \xrightarrow{(1, r.f)} Y \times_X MX \xrightarrow{s.p_2} X$$

is part of an algebraic weak factorisation system.

HDRs and naive fibrations

Hyperdeformation retracts (HDRs)

A *hyperdeformation retract* is a left map for this AWFS: that is, a map $i : Y \to X$ for which there is a retraction $j : X \to Y$ and a homotopy $H : X \to MX$ with $H : 1 \simeq i.j$ such that $\Gamma.H = PH.H$.

Naive fibrations

A *naive fibration* is a right map for this AWFS: that is, a map $p : Y \to X$ which comes equipped with a transport operation

$$T : Y \times_X MX \to Y$$

with $p.T = s.p_2, \; T.(1, r.p) = 1$ and
$T.(p_1, \mu.(p_2, p_3)) = T.(T.(p_1, p_2), p_3)$.

Kan fibrations are naive fibrations, but the converse is false. Indeed, every map $X \to 1$ is a naive fibration.

Section 3

Uniform Kan fibrations

Mould square

Mould square

A square of the form
$$\begin{tikzcd} A_0 \arrow[r, "i_0"] \arrow[d, "a"'] & B_0 \arrow[d, "b"] \\ A_1 \arrow[r, "i_1"'] & B_1 \end{tikzcd}$$
is a *mould square* if:

- the maps a and b are cofibrations and the square is a morphism of cofibrations when read from left to right (which only means that it is a pullback).
- the maps i_0 and i_1 are HDRs and the square is a morphism of HDRs when read from top to bottom.
- the square for the retracts
$$\begin{tikzcd} B_0 \arrow[r, "j_0"] \arrow[d, "b"'] & A_0 \arrow[d, "a"] \\ B_1 \arrow[r, "j_1"'] & A_1 \end{tikzcd}$$
is a pullback as well.

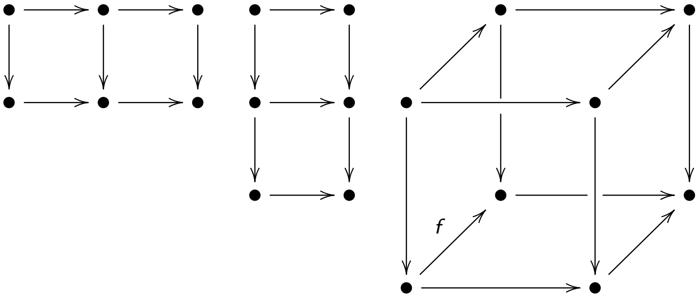
Properties of mould squares

Lemma

Any pair consisting of an HDR $i_1 : A_1 \to B_1$ and a cofibration $a : A_0 \to A_1$ can be extended to a mould square in an (up to iso) unique way.

Properties of mould squares

Mould squares can be composed horizontally and vertically, and they can be pulled back.



Uniform Kan fibration

Definition

To equip a map $p : Y \to X$ with the structure of a *uniform Kan fibration* means that one should specify for any solid commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ C \arrow[r] \arrow[urr, dotted] & D \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

in which the left square is a mould square and for any map $C \to Y$, a particular morphism $D \to Y$ making everything commute, in a way which respects horizontal and vertical composition, as well as base change of mould squares.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[urrr, dotted] & \bullet \arrow[r] \arrow[urr, dotted] & \bullet \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

Horn squares

Proposition

Uniform Kan fibration have the RLP wrt Horn inclusions.

Proof.

There is a special class of mould squares, which we call *Horn squares*:

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & s_i^*(\partial\Delta^n) \arrow[d] \\
\Delta^n \arrow[r, bend left, "d_i/d_{i+1}"] & \Delta^{n+1} \arrow[l, bend left, "s_i"]
\end{tikzcd}$$

The induced map from the pushout to the bottom-right object is the horn inclusion $\Lambda^{n+1}_{i/i+1} \to \Delta^{n+1}$. Therefore uniform Kan fibration have the RLP wrt Horn inclusions. $\square$

Classically OK

In fact, one can show (with quite some effort!) that the lifts against the mould squares determine the lifts against all the mould squares, and that the uniformity conditions can be expressed purely as conditions on the lifts against horn squares. This can be used to show:

Theorem

Classically (in **ZFC**) every Kan fibration can be equipped with the structure of a uniform Kan fibration.

Towards an algebraic model structure

The main motivation for our work was to give constructive proofs of:

- the existence of an algebraic model structure on simplicial sets.
- the existence of a model of univalent type theory in simplicial sets.

Currently we have constructive proofs/proof sketches for:

- the existence of a model structure on the simplicial sets, when restricted to those that are uniformly Kan.
- the existence of a model of type theory with $\Pi, \Sigma, \mathbb{N}, 0, 1, +, \times$.

Future work

What remains to be proven (constructively!):

- We can show that universal uniform Kan fibration exist, but we haven't shown they are univalent.
- We haven't shown that universes are uniformly Kan.
- And we haven't shown that there exists an algebraic model structure on the entire category of simplicial sets based on our notion of a uniform Kan fibration.

THANK YOU!

Section 4

Comparison with Gambino & Sattler

Gambino & Sattler

Gambino and Sattler (in their paper "The Frobenius condition, right properness, and uniform fibrations") also propose a definition of a uniform Kan fibration.

Proposition

Uniform Kan fibrations in our sense are also uniform Kan fibrations in the sense of Gambino and Sattler.

I expect the converse to be false (constructively!).

One key difference is that our definition can be shown to be *local*.

Local class

Definition

Let us say that a structure on morphisms is *local* if: to equip a morphism $f : Y \to X$ with this structure it is necessary and sufficient to do this for every pullback of f along a map $x : \Delta^n \to X$, in such a way our choices are stable under pulling back along maps $\alpha : \Delta^m \to \Delta^n$.

$$\begin{tikzcd}
Y_{x\cdot\alpha} \arrow[r] \arrow[d] & Y_x \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\
\Delta^m \arrow[r, "\alpha"'] & \Delta^n \arrow[r, "x"'] & X
\end{tikzcd}$$

Our notion is local (this follows from the fact that our uniformity conditions can be expressed purely terms of fillers against horn squares), but it is unclear whether the uniform Kan fibrations of Gambino & Sattler are as well (constructively).

This prevents them from showing (constructively) that universal Kan fibrations exist.

Section 5

The horn square definition

The horn square definition

We introduce some notation.

First of all, let us write for each $n \in \mathbb{N}$:

$$\begin{aligned} \mathcal{A}_n &= \big\{(i,j,i+1,j) : i,j \leq n, j < i\big\} \\ &\cup \big\{(i,j,i,j+1) : i,j \leq n, j > i\big\} \\ &\cup \big\{(i,i,i,i+1) : i \leq n\big\} \\ &\cup \big\{(i,i,i+1,i) : i \leq n\big\} \end{aligned}$$

Secondly, let us write

$$S_j^{n+1} = s_j^* \partial\Delta[n] = \Lambda^{n+1}_{j,j+1} \cup (d_j \cap d_{j+1}).$$

The horn square definition

To equip a map $p : Y \to X$ with the structure of a uniform Kan fibration it is necessary and sufficient to equip p with chosen fillers against horn squares, in such a way that for any $n \in \mathbb{N}$, $(i, j, i^*, j^*) \in \mathcal{A}_n$ and $\pm \in \{+, -\}$: if f is our chosen solution to the lifting problem

$$\begin{tikzcd}
(\partial\Delta[n], \langle\rangle) \arrow[r] \arrow[d] & (\partial\Delta[n], \langle i, \pm\rangle) \arrow[r, "y"] \arrow[d] & Y \arrow[d, "p"] \\
(\Delta[n], \langle\rangle) \arrow[r] \arrow[urr, dotted, "g"] & (\Delta[n], \langle i, \pm\rangle) \arrow[r, "x"'] \arrow[ur, dotted, "f"'] & X,
\end{tikzcd}$$

then our chosen solution to the lifting problem

$$\begin{tikzcd}
(\partial\Delta[n+1], \langle\rangle) \arrow[r] \arrow[d] & (\partial\Delta[n+1], \langle i^*, \pm\rangle) \arrow[r, "y'"] \arrow[d] & Y \arrow[d, "p"] \\
(\Delta[n+1], \langle\rangle) \arrow[r] \arrow[urr, dotted, "g.s_j"] & (\Delta[n+1], \langle i^*, \pm\rangle) \arrow[r, "x.s_{j^*}"'] \arrow[ur, dotted] & X
\end{tikzcd}$$

should be $f.s_{j^*}$, where y' is the map which is $y \cdot s_{j^*}$ on S_j^{n+1} and f on the faces d_k with $k \in \{j, j+1\} - \{i^*\}$.