Codensity monads and $D$-ultrafilters

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joint work with Jiří Adámek

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$E : \mathcal{A} \to \mathcal{K}$ is **codense** if, for every $X \in \mathcal{K}$,

$$X \xrightarrow{a} EA,$$  

$a \in X/E,$

is a limit cone of $X/E \to \mathcal{K}$. 
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Terminology for $E$ the inclusion of a full subcategory $\mathcal{A}$ into $\mathcal{K}$: $\mathcal{A}$ is codense in $\mathcal{K}$. 

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**Terminology for $E$ the inclusion of a full subcategory $\mathcal{A}$ into $\mathcal{K}$:**

$\mathcal{A}$ is codense in $\mathcal{K}$.

**Examples:**

- FinSet dense in Set
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Terminology for $E$ the inclusion of a full subcategory $\mathcal{A}$ into $\mathcal{K}$: $\mathcal{A}$ is codense in $\mathcal{K}$.

**Examples:**

- $	ext{FinSet}$ dense in $\text{Set}$
- $\mathcal{K}_{fp}$ dense in $\mathcal{K}$, for $\mathcal{K}$ locally finitely presentable
The codensity monad shows how far the subcategory $A$ is from being condense in $\mathcal{K}$.

An inspiring paper:

[Tom Leinster, Codensity and the ultrafilter monad, TAC, 2013]
The **codensity monad** shows how far the subcategory $\mathcal{A}$ is from being condense in $\mathcal{K}$. 

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Codensity monad of $E : \mathcal{A} \to \mathcal{K}$:

$$T = \text{Ran}_E E$$
Codensity monad of $E : \mathcal{A} \to \mathcal{K}$:

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For $\mathcal{A}$ small full subcategory of complete $\mathcal{K}$,

$$TX = \lim \left( \frac{X/\mathcal{A}}{\mathcal{K}} \right)$$

$$(X \xrightarrow{a} A) \mapsto A$$
For $\text{FinSet} \leftrightarrow \text{Set}$, 

$$T = \text{ultrafilter monad}$$

[Kennison, Gildenhuys, JPPA, 1971]

For $\text{FD}(K\text{-Vec}) \leftrightarrow K\text{-Vec}$, 

$$T = \text{double-dualization monad}$$

[Tom Leinster, 2013]
General setting:

\[ \mathcal{A} \hookrightarrow \mathcal{K} \]

(essentially) small \quad complete with a cogenerator \quad \mathcal{D} \quad \text{belonging to} \quad \mathcal{A}
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(essentially) small complete with a cogenerator \( D \) belonging to \( \mathcal{A} \)

1. \( \mathcal{K} \) symmetric monoidal closed and complete, \( \mathcal{A} = \mathcal{K}_{fp} \)

- commutative varieties
- cartesian closed, complete categories
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1. \( \mathcal{K} \) symmetric monoidal closed and complete, \( \mathcal{A} = \mathcal{K}_{\text{fp}} \)
   - commutative varieties
   - cartesian closed, complete categories

2. More general
Commutative varieties:

varieties where every operation is a homomorphism

They are precisely the varieties \( \mathcal{K} \) which are symmetric monoidal closed for the usual tensor product,

\[
A \otimes B = \text{free algebra in } \mathcal{K} \text{ on } |A| \times |B|
\]

with

\[
I = \text{free algebra in } \mathcal{K} \text{ on one generator}
\]

[Banaschewski, Nelson, CMB, 1976]
$D$ finitely presentable cogenerator of $\mathcal{K}$

$$- \otimes D \dashv [D, -] : \mathcal{K} \to \mathcal{K}$$

$(-)^* = [-, D] \dashv [-, D] : \mathcal{K}^{\text{op}} \to \mathcal{K}$

$$(-)^{**} : \mathcal{K} \to \mathcal{K}$$

double-dualization monad
Examples.

<table>
<thead>
<tr>
<th>Category</th>
<th>$D$</th>
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<tbody>
<tr>
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<td>${0}$</td>
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<td>$K$-Vector spaces</td>
<td>$K$</td>
</tr>
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</table>
| Join-semilattices              | $\begin{array}{c}
0 \\
1
\end{array}$ |
| $M$-Sets                       | $\mathcal{P}M$ |
| Posets                         | $\begin{array}{c}
0 \\
1
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| $\Sigma$-structures            | $\{0, 1\}$ complete |
| Undirected graphs              | $\begin{array}{c}
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### Examples.

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$(mR = \{x \in M \mid mx \in R\})$
An object $D$ is a $\ast$-object if, for $(-)^\ast = [-, D]$,

$$A^\ast \xrightarrow{a^\ast} X^\ast, \quad a \in X/K_{fp},$$

is a cocone colimit.
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is a cocone colimit.

For commutative varieties $\mathcal{K}$, and for posets, $\Sigma$-structures ($\Sigma$ finite), undirected graphs, ...

... every $D \in \mathcal{K}_{fp}$ is a $\ast$-object.

And conversely, $D \ast$-object $\Rightarrow$ $D \in \mathcal{K}_{fp}$, when $I \in \mathcal{K}_{fp}$. 
An object $D$ is a $\ast$-object if, for $(-)^* = [-, D]$,

$$A^* \overset{a^*}{\longrightarrow} X^*, \quad a \in X/\mathcal{K}_{fp},$$

is a cocone colimit.

For commutative varieties $\mathcal{K}$, and for posets, $\Sigma$-structures ($\Sigma$ finite), undirected graphs, ...

every $D \in \mathcal{K}_{fp}$ is a $\ast$-object.

And conversely, $D$ $\ast$-object $\Rightarrow$ $D \in \mathcal{K}_{fp}$, when $I \in \mathcal{K}_{fp}$.

$\ast$-cogenerator $:= \text{cogenerator} + \ast$-object
Let $D$ be a $\ast$-cogenerator.
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Then, $(-)^{**}$ has the limit property:

\[ X^{**} \xrightarrow{a^{**}} A^{**}, \quad a \in X/\mathcal{K}_{fp}, \quad \text{is a limit cone.} \]
Let $D$ be a $\ast$-cogenerator.

Then, $(-)^{\ast\ast}$ has the limit property:

$$X^{\ast\ast} \xrightarrow{a^{\ast\ast}} A^{\ast\ast}, \quad a \in X/K_{fp},$$

is a limit cone.

Moreover, $(-)^{\ast\ast}$ has monic units.
Construction of the $D$-ultrafilter monad $\mathbb{T}$:

\[
\begin{array}{ccc}
A' & \xrightarrow{a'} & X^{**} \\
\downarrow & & \downarrow \\
A & \xrightarrow{\eta_A} & A^{**}
\end{array}
\]

$p(a) \quad a^{**}$

$a \in X/\mathcal{K}_{fp}$
Construction of the $D$-ultrafilter monad $\mathbb{T}$:

\[
\begin{array}{c}
TX \\
\downarrow q(a) \\
A' \\
\downarrow p(a) \\
A
\end{array}
\quad
\begin{array}{c}
i_X \\
\downarrow a' \\
X^{**} \\
\downarrow a^{**} \\
A^{**}
\end{array}
\quad
\begin{array}{c}
a \in X/K_{fp}
\end{array}
\]
**D-ultrafilters**: external elements $f$ of $X^{**}$ which factorize through every $a'$, $a \in X/K_{fp}$:

\[
\begin{array}{ccc}
I & \longrightarrow & X^{**} \\
\downarrow & & \downarrow \\
A' & \rightarrow & X^{**} \\
\end{array}
\]

In all the above examples, they are just the elements of the underlying set of $TX$. A different approach to the generalization of the notion of ultrafilter on one object in the study of codensity monads: [Barry-Patrick Devlin, PhD Thesis, Univ. Edinburgh, 2015]
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A different approach to the generalization of the notion of ultrafilter on one object in the study of codensity monads: [Barry-Patrick Devlin, PhD Thesis, Univ. Edinburgh, 2015]
Let $\mathcal{K}$ be a complete, symmetric monoidal closed category with a $\ast$-cogenerator $D$.

Then,

the $D$-ultrafilter monad is a submonad of $(\dash)\ast\ast$ and it is the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$. 


The $D$-ultrafilter monad $T$

Let $\mathcal{K}$ be a complete, symmetric monoidal closed category with a $\ast$-cogenerator $D$.

Then,

the $D$-ultrafilter monad is a submonad of $(-)^{**}$ and it is the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$.

Corollary. The codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ is the largest submonad of $(-)^{**}$ whose unit has invertible components at all finitely presentable objects.
## Examples.

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<td>$X^{**} = \text{prime, upwards closed collections of prime } \uparrow\text{-sets of } X$</td>
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<td>nonempty, prime collections of $\uparrow\text{-sets of } X$ closed under upper sets and finite intersections</td>
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General setting:

\[ \mathcal{A} \hookrightarrow \mathcal{K} \]

(essentially) small \quad complete with a cogenerator

\( D \) lying in \( \mathcal{A} \)
General setting:

\[ \mathcal{A} \xrightarrow{\text{(essentially) small}} \mathcal{K} \]

\[ \text{complete with a cogenerator } \{ D_i \}_{i \in I} \text{ lying in } \mathcal{A} \]

\[ L \dashv R : (\text{Set}^I)^{\text{op}} \to \mathcal{K} \]

with \[ LX = (\mathcal{K}(X, D_i))_{i \in I} \] and \[ R(M_i)_{i \in I} = \prod_{i \in I} D_i^{M_i}. \]
General setting:

\[ \mathcal{A} \rightarrow \mathcal{K} \]

(essentially) small complete with a cogenerator \( \{D_i\}_{i \in I} \) lying in \( \mathcal{A} \)

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with \( LX = (\mathcal{K}(X, D_i))_{i \in I} \) and \( R(M_i)_{i \in I} = \prod_{i \in I} D_i^{M_i} \).

\[ S := \text{the monad induced on } \mathcal{K} \text{ by } L \dashv R \]
The unit $\eta^S$ of $S$ is pointwise monic.
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$S$ has the limit property:

$$SX \xrightarrow{Sa} SA, \quad a \in X/\mathcal{A}, \quad \text{is a limit cone.}$$
The “pullback-intersection” construction with $S$ in lieu of $(-)^{**}$:

\[
\begin{array}{ccc}
TX & \xrightarrow{i_X} & SX \\
q(a) & \downarrow & \downarrow \\
A' & \xrightarrow{a'} & SX \\
p(a) & \downarrow & \downarrow \\
A & \xrightarrow{\eta^S_A} & SA \\
\end{array}
\]

Also gives rise to the codensity monad.
The “pullback-intersection” construction with \( S \) in lieu of \((-)^{**}\):

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A' \quad \downarrow a' \quad SX \\
\downarrow p(a) \\
A \quad \downarrow \eta^S_A \quad SA \\
\downarrow \\
a \in X/\mathcal{A}
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Also gives rise to the codensity monad.
The codensity monad of $\mathcal{A} \hookrightarrow \mathcal{K}$ is the smallest submonad of $\mathcal{S}$ with the limit property.
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**Question.** When is $(-)^{**}$ a submonad of $\mathcal{S}$?

It is,

- in the previous examples;
- whenever $D$ has the property that $\eta_D : D \to D^{**}$ is invertible.
Example.

- Codensity monad of $\{K\} \rightarrow K\text{-Vec}$:

  $TX = \text{all homogeneous functions in } SX = K^{X^*}$
  
  (homogeneous = preserving the scalar multiplication)
Example.

- Codensity monad of \( \{ K \} \xrightarrow{\cdot} K\text{-Vec} \):

  \[
  TX = \text{all homogeneous functions in } SX = K^{X^*}
  \]
  (homogeneous = preserving the scalar multiplication)

- Codensity monad of \( \{ K, K^2 \} \xrightarrow{\cdot} K\text{-Vec} \):

  double-dualization monad
Example.

- Codensity monad of \( \{ \{0, 1\} \} \to \text{Set} : \)

  \[
  TX = \text{collections of nonempty sets of } X \text{ including either } Y \text{ or } \overline{Y} \text{ for every } Y \subseteq X
  \]

- Codensity monad of \( \{ \{0, 1, 2\} \} \to \text{Set} : \)

  ultrafilter monad

  [Leinster, 2013]

- Codensity monad of

  \[
  \{ \text{sets of cardinality less than } \lambda \} \to \text{Set} : \\
  (\lambda \text{ infinite})
  \]

  \[
  TX = \lambda\text{-complete ultrafilters on } X
  \]

  [Adámek, Brooke-Taylor, Campion, Positselski, Rosický, 2019]
Examples.

- The codensity monad of \( \text{FinTop} \xrightarrow{\triangleleft} \text{Top} \)
  is given by

\[
TX = \text{all ultrafilters on the set } X
\]

with topology basis \( \triangle G = \{ U \in TX \mid G \in U \} \), \( G \) open in \( X \).

- The codensity monad of \( \text{FinTop}_0 \xrightarrow{\square} \text{Top}_0 \)
  is the prime open filter monad,

\[
TX = \text{prime filters of } (\Omega X, \subseteq)
\]

with topology basis \( \square G = \{ U \in TX \mid G \in U \} \), \( G \) open in \( X \).