Codensity monads and D-ultrafilters

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$E : \mathcal{A} \to \mathcal{K}$ is codense if, for every $X \in \mathcal{K}$,

$$X \xrightarrow{\quad a \quad} EA \ , \qquad a \in X/E,$$

is a limit cone of $X/E \to \mathcal{K}$.

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Examples:

- FinSet dense in Set
- $\mathcal{K}_{\mathsf{fp}}$ dense in $\mathcal{K}$, for $\mathcal{K}$ locally finitely presentable

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An inspiring paper:

[Tom Leinster, *Codensity and the ultrafilter monad*, TAC, 2013]

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For $\mathcal{A}$ small full subcategory of complete $\mathcal{K}$,

$$TX = \lim \left(X/\mathcal{A} \xrightarrow{\quad C_X \quad} \mathcal{K} \right)$$

$$(X \xrightarrow{a} A) \quad \mapsto \quad A$$

For FinSet $\hookrightarrow$ Set ,

T = ultrafilter monad

[Kennison, Gildenhuys, JPPA, 1971]

For FD(K-Vec) $\hookrightarrow$ K-Vec ,

T = double-dualization monad

[Tom Leinster, 2013]

General setting:

$$\mathcal{A} \hookrightarrow \mathcal{K}$$

(essentially) small

complete with a cogenerator D belonging to $\mathcal{A}$

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   - commutative varieties
   - cartesian closed, complete categories

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   - cartesian closed, complete categories
2. More general

Commutative varieties:

varieties where every operation is a homomorphism

They are precisely the varieties $\mathcal{K}$ which are symmetric monoidal closed for the usual tensor product,

$$A \otimes B = \text{free algebra in } \mathcal{K} \text{ on } |A| \times |B|$$

with

$$I = \text{free algebra in } \mathcal{K} \text{ on one generator}$$

[Banaschewski, Nelson, CMB, 1976]

D finitely presentable cogenerator of $\mathcal{K}$

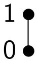
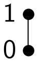
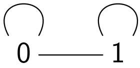
$$- \otimes D \dashv [D, -] : \mathcal{K} \to \mathcal{K}$$

$$(-)^* = [-, D] \dashv [-, D] : \mathcal{K}^{\mathsf{op}} \to \mathcal{K}$$

$$\boxed{(-)^{**} : \mathcal{K} \to \mathcal{K}}$$

double-dualization monad

Examples.

Category	D
Sets	$\{0,1\}$
Sets and partial functs.	$\{0\}$
K-Vector spaces	K
Join-semilattices	
M-Sets	$\mathcal{P}M$
Posets	
Σ-structures	$\{0,1\}$ complete
Undirected graphs	

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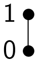
Join-semilattices	
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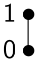
Posets	
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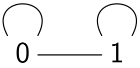
Undirected graphs	

$\big(mR = \{x \in M \mid mx \in R\}\big)$

An object D is a $*$-object if, for $(-)^* = [-, D]$,

$$A^* \xrightarrow{a^*} X^*, \qquad a \in X/\mathcal{K}_{\mathsf{fp}},$$

is a cocone colimit.

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For commutative varieties $\mathcal{K}$,

and for posets, Σ-structures (Σ finite), undirected graphs, ...

every $D \in \mathcal{K}_{\mathsf{fp}}$ is a $*$-object.

And conversely, D $*$-object $\Rightarrow$ $D \in \mathcal{K}_{\mathsf{fp}}$,
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$\star$-cogenerator := cogenerator + $\star$-object

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Then, $(-)^{**}$ has the limit property:

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Moreover, $\boxed{(-)^{**} \text{ has monic units.}}$

Construction of the D-ultrafilter monad $\mathbb{T}$:

$$
\begin{tikzcd}
A' \arrow[r, tail, "a'"] \arrow[d, "p(a)"'] \arrow[dr, phantom, "\lrcorner", very near start] & X^{**} \arrow[d, "a^{**}"] \\
A \arrow[r, tail, "\eta_A"'] & A^{**}
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TX \arrow[d, "q(a)"'] \arrow[dr, "i_X"] & \\
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D-ultrafilters: external elements f of X^{**} which factorize through every a', $a \in X/\mathcal{K}_{\mathsf{fp}}$:

$$\begin{tikzcd} & I \arrow[dl, dashed] \arrow[d, "f"] \\ A' \arrow[r, "a'"'] & X^{**} \end{tikzcd}$$

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A different approach to the generalization of the notion of ultrafilter on one object in the study of codensity monads: [Barry-Patrick Devlin, PhD Thesis, Univ. Edinburgh, 2015]

The D-ultrafilter monad $\mathbb{T}$

Let $\mathcal{K}$ be a complete, symmetric monoidal closed category with a $*$-cogenerator D.

Then,

the D-ultrafilter monad is a submonad of $(-)^{**}$ and it is the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$.

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Corollary. The codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ is the largest submonad of $(-)^{**}$ whose unit has invertible components at all finitely presentable objects.

Examples.

Category	D	TX
Sets	$\{0,1\}$	ultrafilters on X
Sets and partial functs.	$\{0\}$	ultrafilters on X
K-Vector spaces	K	X^{**} = double dual space
Join-semilattices	$\begin{tikzcd} 1 \arrow[d, dash] \\ 0 \end{tikzcd}$	X^{**} = prime, upwards closed collections of prime $\uparrow$-sets of X
M-Sets	$\mathcal{P}M$	ultrafilters on the underlying set
Posets	$\begin{tikzcd} 1 \arrow[d, dash] \\ 0 \end{tikzcd}$	nonnempty, prime collections of $\uparrow$-sets of X closed under upper sets and finite intersections
Σ-structures	$\{0,1\}$ complete	ultrafilters on the underlying set
Undirected graphs	$\begin{tikzcd} 0 \arrow[loop above, dash] \arrow[r, dash] & 1 \arrow[loop above, dash] \end{tikzcd}$	ultrafilters on the set of vertices

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$$L \dashv R : \left(\mathsf{Set}^I\right)^{\mathrm{op}} \to \mathcal{K}$$

with $\quad LX = \big(\mathcal{K}(X, D_i)\big)_{i \in I} \quad$ and $\quad R(M_i)_{i \in I} = \prod_{i \in I} D_i^{M_i}.$

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$\mathbb{S} :=$ the monad induced on $\mathcal{K}$ by $L \dashv R$

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S has the limit property:

$$SX \xrightarrow{Sa} SA, \quad a \in X/\mathcal{A}, \qquad \text{is a limit cone.}$$

The "pullback-intersection" construction with S in lieu of $(-)^{**}$:

$$\begin{tikzcd}
TX \arrow[d, "q(a)"'] \arrow[dr, "i_X"] & \\
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$$a \in X/\mathcal{A}$$

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> Also gives rise to the codensity monad.

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Question. When is $(-)^{**}$ a submonad of $\mathbb{S}$?

It is,

- in the previous examples;
- whenever D has the property that $\eta_D : D \to D^{**}$ is invertible.

Example.

- Codensity monad of $\{K\} \hookrightarrow K\text{-}Vec$:

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- Codensity monad of $\{K,\, K^2\} \hookrightarrow K\text{-}Vec$:

double-dualization monad

Example.

- Codensity monad of $\{\,\{0,1\}\,\} \hookrightarrow \mathsf{Set}$:

$TX =$ collections of nonempty sets of X including either Y or $\overline{Y}$ for every $Y \subseteq X$

- Codensity monad of $\{\,\{0,1,2\}\,\} \hookrightarrow \mathsf{Set}$:

ultrafilter monad

[Leinster,2013]

- Codensity monad of

$\{$ sets of cardinality less than $\lambda\,\} \hookrightarrow \mathsf{Set}$:

(λ infinite)

$TX =$ λ-complete ultrafilters on X

[Adámek, Brooke-Taylor, Campion, Positselski, Rosický, 2019]

Examples.

- The codensity monad of $\mathsf{FinTop} \hookrightarrow \mathsf{Top}$

  is given by

  $$TX = \text{ all ultrafilters on the set } X$$

  with topology basis $\triangle G = \{\mathcal{U} \in TX \mid G \in \mathcal{U}\}$, G open in X.

- The codensity monad of $\mathsf{FinTop}_0 \hookrightarrow \mathsf{Top}_0$

  is the prime open filter monad,

  $$TX = \text{ prime filters of } (\Omega X, \subseteq)$$

  with topology basis $\Box G = \{\mathcal{U} \in TX \mid G \in \mathcal{U}\}$, G open in X.