Hopf-Frobenius Algebras

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Hopf-Frobenius Algebras


\[
\text{Hopf Algebra} \quad \begin{array}{c|c}
\text{Frobenius Algebra} & \\
\quad \text{Antipodes} & \\
\quad \text{Frobenius Algebra} & \\
\quad \text{Hopf Algebra} & \\
\text{Hopf Algebra} & \\
\end{array}
\]
Preliminaries
Definition

In a symmetric monoidal category, an object $A$ has a dual $A^*$ if there exists morphisms $d : I \to A \otimes A^*$ and $e : A^* \otimes A \to I$, which are depicted by assigning an orientation to the wire and bending it.

$d := \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]
  
  \node (a) at (0,0) {$A$};
  \node (b) at (1,0) {$A^*$};

  \draw[->] (a) to [bend left] (b);
  \end{tikzpicture}$

$e := \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]

  \node (a) at (1,0) {$A$};
  \node (b) at (0,0) {$A^*$};

  \draw[->] (a) to [bend left] (b);
\end{tikzpicture}$

such that

$\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]

  \node (a) at (0,0) {$A$};
  \node (b) at (1,0) {$A$};

  \draw[->] (a) to [bend left] (b);
\end{tikzpicture} = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]

  \node (a) at (1,0) {$A$};
  \node (b) at (0,0) {$A$};

  \draw[->] (a) to [bend left] (b);
\end{tikzpicture}$

and

$\begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]

  \node (a) at (0,0) {$A^*$};
  \node (b) at (1,0) {$A^*$};

  \draw[->] (a) to [bend left] (b);
\end{tikzpicture} = \begin{tikzpicture}[baseline={([yshift=-.5ex]current bounding box.center)}]

  \node (a) at (1,0) {$A^*$};
  \node (b) at (0,0) {$A^*$};

  \draw[->] (a) to [bend left] (b);
\end{tikzpicture}$
Monoids

Definition

A monoid in a symmetric monoidal category $\mathcal{C}$ consists of an object $M$ in $\mathcal{C}$ equipped with two structure maps $\mu : M \otimes M \to M$, $\eta : I \to M$ which are associative and unital, depicted graphically below.
Comonoids

Definition

A comonoid in a symmetric monoidal category \( \mathcal{C} \) consists of an object \( C \) in \( \mathcal{C} \) equipped with two structure maps \( \Delta : C \to C \otimes C \), \( \varepsilon : M \to I \) which are coassociative and counital, depicted graphically below.
Definition

A bialgebra in symmetric monoidal category $C$ consists of a monoid and a comonoid $(F, \delta, \epsilon, \eta)$, which jointly obey the copy, cocopy, bialgebra, and scalar laws depicted below.
**Definition**

A *Hopf algebra* consists of a bialgebra \((H, \bigcirc, \bullet, \checkmark, \blacklozenge, \blacklozenge)\) and an endomorphism \(s : H \to H\) called the *antipode* which satisfies the *Hopf law*:

\[
s := \quad = \quad =
\]

Where unambiguous, we abuse notation slightly and use \(H\) to refer the whole Hopf algebra.
Definition

A *Frobenius algebra* in a symmetric monoidal category $C$ consists of a monoid and a comonoid $(F, \bullet, \triangleleft, \triangleleft, \triangleright, \triangleright)$ obeying the Frobenius law:
A Frobenius algebra in a symmetric monoidal category $\mathcal{C}$ consists of a monoid $(F, \cdot, \circ)$ and a Frobenius form $\triangleleft : F \otimes F \to I$, which admits an inverse, $\triangleright : I \to F \otimes F$, satisfying:

\[
\begin{align*}
\triangleleft \circ \triangleright &= 1 \\
\triangleright \circ \triangleleft &= 1 \\
\end{align*}
\]
Definition

A Frobenius algebra in a symmetric monoidal category $C$ consists of a monoid and a comonoid $(F, \bullet, \circ, \triangleright, \triangleleft)$ obeying the Frobenius law:
**Definition**

A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid $(F, \alpha, \beta, \delta, \epsilon)$ obeying the Frobenius law: 

\[
\begin{align*}
\begin{array}{c}
\quad = \\
\quad = \\
\end{array}
\end{align*}
\]
Definition

A Frobenius algebra in a symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid $(F, \triangleright, \triangleleft, \bigtriangledown, \vartriangleleft)$ obeying the Frobenius law:
Hopf-Frobenius Algebra

Definition

A Hopf-Frobenius algebra or HF algebra consists of an object $H$ bearing a green monoid $(\bigotimes, \bigodot)$, a green comonoid $(\bigoplus, \boxminus)$, a red monoid $(\bigcirc, \bigcirc)$, a red comonoid $(\bigtriangleup, \bigcirc)$ and endomorphisms $\Box, \Box$ such that

- $(\bigotimes, \bigodot, \bigotimes, \bigodot)$ and $(\bigoplus, \boxminus, \bigoplus, \boxminus)$ are Frobenius algebras,
- $(\bigotimes, \bigodot, \bigcirc, \bigcirc, \bigtriangleup)$ and $(\bigoplus, \boxminus, \bigcirc, \bigcirc, \bigtriangleup)$ are Hopf algebras
- $\Box$ and $\Box$ satisfy the left and right equations below respectively

\[
\begin{align*}
\Box &= \bigotimes, & \Box &= \bigoplus
\end{align*}
\]
А Hopf-Frobenius algebra or HF algebra consists of an object $H$ bearing a green monoid $(\bigcirc, \bigcirc)$, a green comonoid $(\bigcirc, \bigcirc)$, a red monoid $(\bigcirc, \bigcirc)$, a red comonoid $(\bigcirc, \bigcirc)$ and endomorphisms □, □ that give us the following structures.
Definition

A *left (co)integral* on $H$ is a copoint $\nabla : H \to I$ (resp. a point $\delta : I \to H$), satisfying the equations:

A *right (co)integral* is defined similarly.
Definition

A left (co)integral on $H$ is a copoint $\nabla : H \to I$ (resp. a point $\Delta : I \to H$), satisfying the equations:

\[=\]

A right (co)integral is defined similarly.

Definition

An integral Hopf algebra $(H, \Delta, \nabla)$ is a Hopf algebra $H$ equipped with a choice of left cointegral $\Delta$, and right integral $\nabla$, such that $\nabla \circ \Delta = \text{id}_I$. 
Definition

An integral Hopf algebra \((H, \uparrow, \downarrow)\) is a Hopf algebra \(H\) equipped with a choice of left cointegral \(\uparrow\), and right integral \(\downarrow\), such that \(\downarrow \circ \uparrow = \text{id}_H\).
Lemma

Let \((H, \triangledown, \nabla)\) be an integral Hopf algebra. Then the following map is the inverse of the antipode.

\[
\begin{align*}
\begin{array}{c}
\quad = \quad \quad = \\
\end{array}
\end{align*}
\]

In particular, the following identities are satisfied

\[
\begin{align*}
\begin{array}{c}
\quad = \quad \quad = \\
\end{array}
\end{align*}
\]
Lemma

Let \((H, \uparrow, \downarrow)\) be an integral Hopf algebra, and define

\[
\beta := \quad \gamma := \quad
\]

then \(\beta\) is a Frobenius form for \((H, \uparrow, \downarrow)\) iff \(\beta\) and \(\gamma\) are a cup and a cap.

If the following identity holds

\[
= \quad =
\]

then \((H, \uparrow, \downarrow)\) is a Hopf-Frobenius algebra
Definition

Let the object $H$ have a dual $H^*$. The integral morphism $\mathcal{I} : H \to H$ is defined as shown below.
Definition

We say that a Hopf algebra satisfies the *Frobenius condition* if there exists maps $\uparrow$ and $\downarrow$ such that

\[
\text{[Diagram with arrows and nodes]}
\]

and

\[
\text{[Another diagram with arrows and nodes]}
\]
**Definition**

We say that a Hopf algebra satisfies the *Frobenius condition* if there exists maps $\uparrow$ and $\downarrow$ such that

$$(H, \uparrow, \downarrow) \text{ is an integral Hopf algebra}$$
Theorem

\[ H \text{ satisfies the Frobenius condition if and only if } H \text{ is a Hopf-Frobenius algebra with the Frobenius forms and their inverses as shown below.}\]

Every Hopf algebra in the category of finite dimensional vector spaces satisfies the Frobenius condition.
The explicit definitions of the green comonoid and red monoid structures are shown below.
Hopf-Frobenius Algebra

The explicit definitions of the green comonoid and red monoid structures are shown below.

Lemma

*If* $H$ *is a Hopf-Frobenius algebra, then every left cointegral (right integral) is a scalar multiple of* $\bullet$ *(resp. $\circ$)*.
Corollary

If $H$ is a Hopf-Frobenius algebra, then it is unique up to an invertible scalar.

Explicitly, let $(H, \begin{tikzpicture}[baseline=-0.5ex]
    
    
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture})$ be a Hopf algebra. Suppose that $H$ has two Hopf-Frobenius algebra structures

- $(\begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture})$
- $(\begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture}, \begin{tikzpicture}[baseline=-0.5ex]
    
    \end{tikzpicture})$

Then for some invertible scalar $k : I \to I$, $\begin{tikzpicture}[baseline=-0.5ex]
    \end{tikzpicture} = k \otimes \begin{tikzpicture}[baseline=-0.5ex]
    \end{tikzpicture}$, and $\begin{tikzpicture}[baseline=-0.5ex]
    \end{tikzpicture} = k^{-1} \otimes \begin{tikzpicture}[baseline=-0.5ex]
    \end{tikzpicture}$. 
Corollary

If $H$ is a Hopf-Frobenius algebra, then it is unique up to an invertible scalar.
Drinfeld Double
Definition
A bialgebra $H$ is quasi-triangular if there exists a universal $R$-matrix $R : I \to H \otimes H$ such that

- $R$ is invertible with respect to $\gamma$
- $R = R$
- $R = R$
- $R = R$
Theorem

The category of modules over a bialgebra is braided if and only if the bialgebra is quasi-triangular
Definition

Let \((H, \circ, \otimes, \otimes', \ell, \mu, \Delta)\) be a Hopf algebra, and suppose that the object \(H\) has a dual \(H^*\). We define the dual Hopf algebra \((H^*, \circ^*, \otimes^*, \otimes'^*, \ell^*, \mu^*, \Delta^*)\) as:

[Diagrams of algebraic structures involving \(*\) operations]
Definition

Let $H$ be a Hopf algebra with an invertible antipode, and dual $H^*$. The *Drinfeld double of $H$, denoted $D(H) = (H \otimes H^*, \mu, 1, \Delta, \epsilon, s)$, is a Hopf algebra defined in the following manner:
Drinfeld Double

Definition

Let $H$ be a Hopf algebra with an invertible antipode, and dual $H^*$. The Drinfeld double of $H$, denoted $D(H) = (H \otimes H^*, \mu, 1, \Delta, \epsilon, s)$, is a Hopf algebra defined in the following manner:
Drinfeld Double

Definition
Let $H$ be a HF algebra. The red Drinfeld double, denoted $D_{\text{red}}(H) = (H \otimes H, \mu, 1, \Delta, \epsilon, s)$, is a Hopf algebra on the object $H \otimes H$ with structure maps

\[
\Delta := \begin{array}{c}\begin{array}{c}\text{Diagram for } \Delta \end{array}\end{array},
\epsilon := \begin{array}{c}\begin{array}{c}\text{Diagram for } \epsilon \end{array}\end{array},
1 := \begin{array}{c}\begin{array}{c}\text{Diagram for } 1 \end{array}\end{array},
\text{ and } s := \begin{array}{c}\begin{array}{c}\text{Diagram for } s \end{array}\end{array}.
\]
Conclusions

What’s next?

- Category of representations

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- Red Drinfeld double may be useful in the context of Kitaev double

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- Useful whenever the dual Hopf algebra is encountered

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What’s next?

- Category of representations
- Red Drinfeld double may be useful in the context of Kitaev double
- Useful whenever the dual Hopf algebra is encountered
- More interesting examples of Hopf-Frobenius algebras?

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