Hopf-Frobenius Algebras

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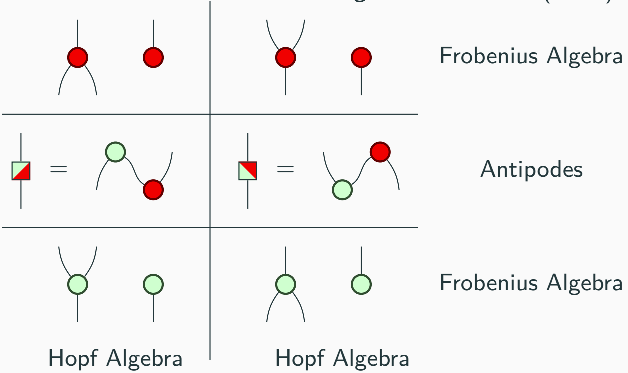
Hopf-Frobenius Algebras

- Ross Duncan and Kevin Dunne. *Interacting Frobenius Algebras are Hopf.* (2016).
- Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. *Interacting Hopf Algebras.* (2014).
- John Baez, and Jason Erbele. *Categories in control* (2014)

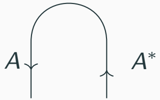


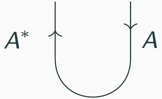
Preliminaries

Duals

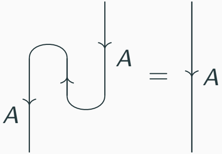
Definition

In a symmetric monoidal category, an object A has a dual A^* if there exists morphisms $d : I \to A \otimes A^*$ and $e : A^* \otimes A \to I$, which are depicted by assigning an orientation to the wire and bending it

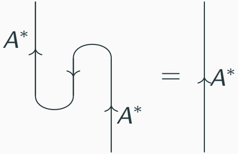
$d :=$ 

$e :=$ 

such that

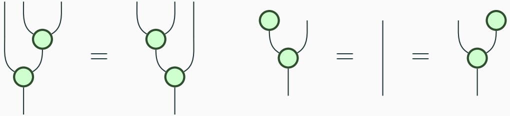


and



Definition

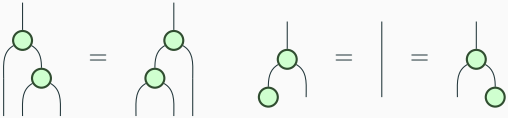
A *monoid* in a symmetric monoidal category $\mathcal{C}$ consists of an object M in $\mathcal{C}$ equipped with two structure maps  $: M \otimes M \to M$,  $: I \to M$ which are *associative* and *unital*, depicted graphically below



Comonoids

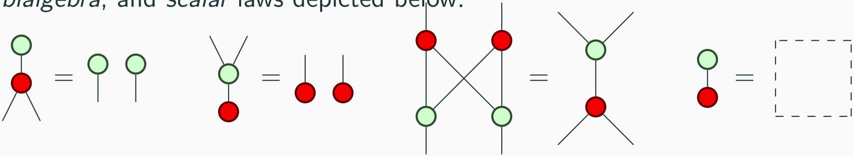
Definition

A *comonoid* in a symmetric monoidal category $\mathcal{C}$ consists of an object C in $\mathcal{C}$ equipped with two structure maps  $: C \to C \otimes C$,  $: M \to I$ which are *coassociative* and *counital*, depicted graphically below



Definition

A *bialgebra* in symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid (F, , , , ), which jointly obey the *copy*, *cocopy*, *bialgebra*, and *scalar* laws depicted below.

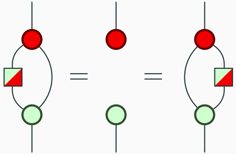


Hopf Algebra

Definition

A *Hopf algebra* consists of a bialgebra (H,  ,  ,  , ) and an endomorphism $s : H \to H$ called the *antipode* which satisfies the *Hopf law*:

$s \;:=$ 

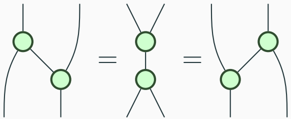


Where unambiguous, we abuse notation slightly and use H to refer the whole Hopf algebra.

Frobenius Algebra

Definition

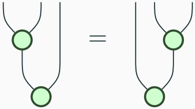
A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid (F, , , ,) obeying the Frobenius law:

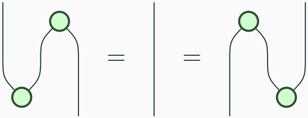


Frobenius Algebra

Definition

A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid (F,  , ) and a *Frobenius form*  $: F \otimes F \to I$, which admits an inverse,  $: I \to F \otimes F$, satisfying:





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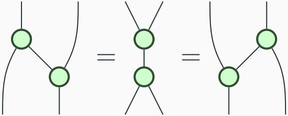

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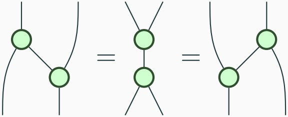

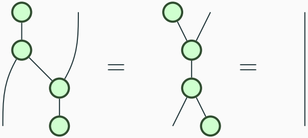

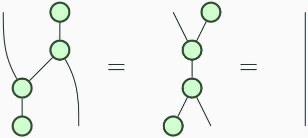

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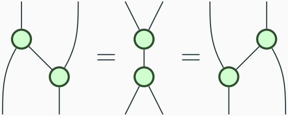
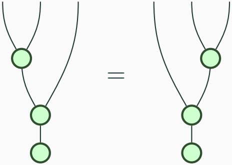




Hopf-Frobenius Algebra

Definition

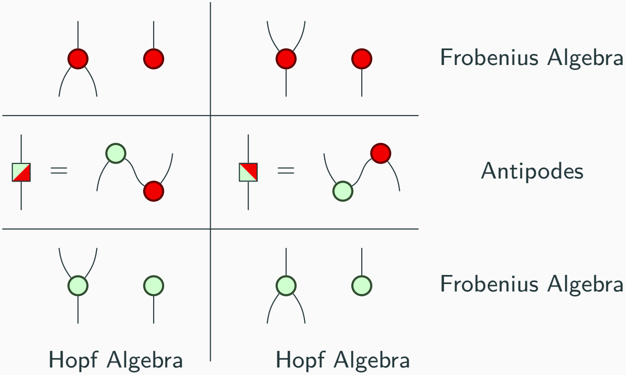
A *Hopf-Frobenius algebra* or *HF algebra* consists of an object H bearing a green monoid (,), a green comonoid (,), a red monoid (,), a red comonoid (,) and endomorphisms , such that

- (, , ,) and (, , ,) are Frobenius algebras,
- (, , , ,) and (, , , ,) are Hopf algebras
- and satify the left and right equations below respectively

Hopf-Frobenius Algebra

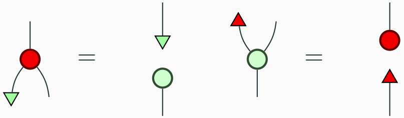
Definition

A *Hopf-Frobenius algebra* or *HF algebra* consists of an object H bearing a green monoid (,), a green comonoid (,), a red monoid (,), a red comonoid (,) and endomorphisms , that give us the following structures



Definition

A *left (co)integral* on H is a copoint  : $H \to I$ (resp. a point  : $I \to H$), satisfying the equations:



A *right (co)integral* is defined similarly.

Integrals

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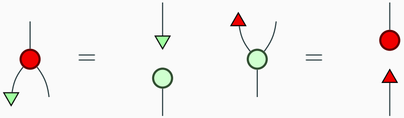


A *right (co)integral* is defined similarly.

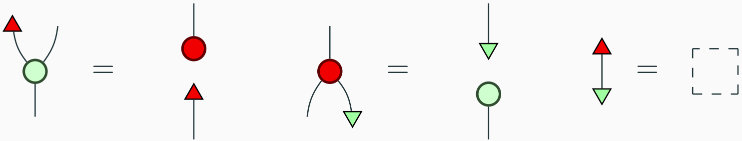
Definition

An *integral Hopf algebra* $(H, ,)$ is a Hopf algebra H equipped with a choice of left cointegral , and right integral , such that $\circ = \mathrm{id}_I$.

Integrals

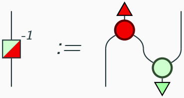
Definition

An *integral Hopf algebra* $(H, \uparrow, \downarrow)$ is a Hopf algebra H equipped with a choice of left cointegral $\uparrow$, and right integral $\downarrow$, such that $\downarrow \circ \uparrow = \mathrm{id}_I$.



Lemma

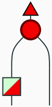
Let $(H,$ , $)$ *be an integral Hopf algebra. Then the following map is the inverse of the antipode.*



In particular, the following identities are satified

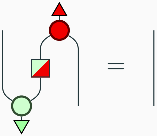
Lemma

Let $(H, ,)$ be an integral Hopf algebra, and define

$\beta :=$  $\gamma :=$ 

then β is a Frobenius form for $(H, ,)$ iff β and γ are a cup and a cap.

If the following identity holds

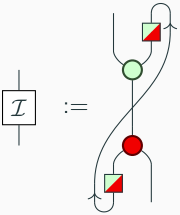


then $(H, ,)$ is a Hopf-Frobenius algebra

Frobenius Condition

Definition

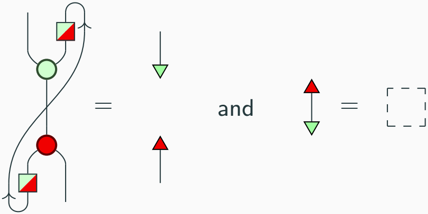
Let the object H have a dual H^*. The *integral morphsim* $\mathcal{I} : H \to H$ is defined as shown below.



Frobenius Condition

Definition

We say that a Hopf algebra satisfies the *Frobenius condition* if there exists maps and such that



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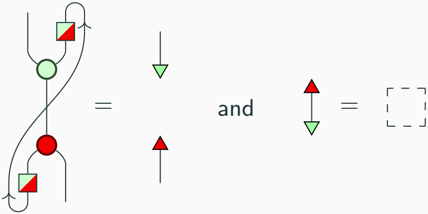
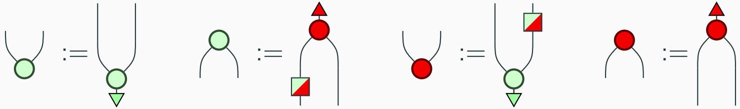


$(H, ,)$ is an integral Hopf algebra

Hopf-Frobenius Algebra

Theorem

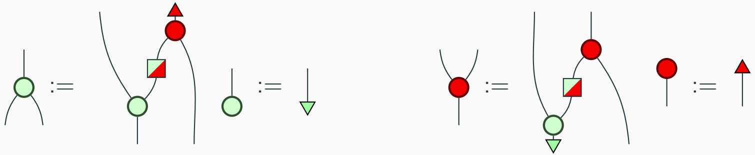
H satisfies the Frobenius condition if and only if H is a Hopf-Frobenius algebra with the Frobenius forms and their inverses as shown below.



Every Hopf algebra in the category of finite dimensional vector spaces satisfies the Frobenius condition.

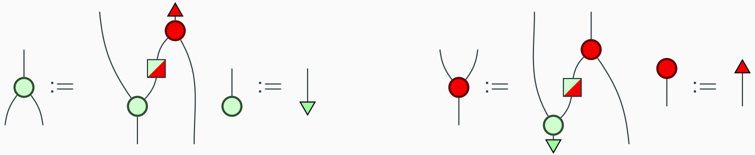
Hopf-Frobenius Algebra

The explicit definitions of the green comonoid and red monoid structures are shown below.



Hopf-Frobenius Algebra

The explicit definitions of the green comonoid and red monoid structures are shown below.



Lemma

If H is a Hopf-Frobenius algebra, then every left cointegral (right integral) is a scalar multiple of  *(resp.*  *)*

Hopf-Frobenius Algebra

Corollary

If H is a Hopf-Frobenius algebra, then it is unique up to an invertible scalar

Explicitly, let (H, , , , ,) be a Hopf algebra. Suppose that H has two Hopf-Frobenius algebra structures

- (, , , ,)
- ($'$, $'$, $'$, $'$, $'$)

Then for some invertible scalar $k : I \to I$, $' = k \otimes$, and $' = k^{-1} \otimes$.

Hopf-Frobenius Algebra

Corollary

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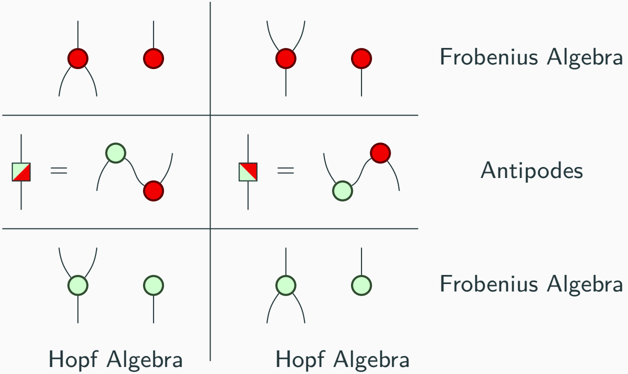
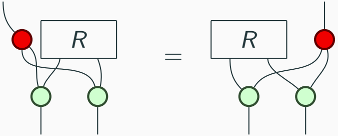


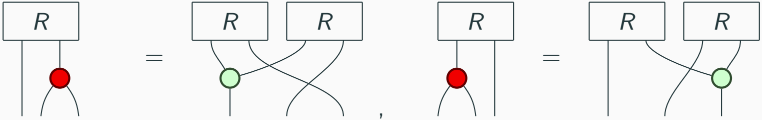
Drinfeld Double

Drinfeld Double

Definition

A bialgebra H is *quasi-triangular* if there exists a *universal R-matrix* $R : I \to H \otimes H$ such that

- R is invertible with respect to 
- 

- 

Drinfeld Double

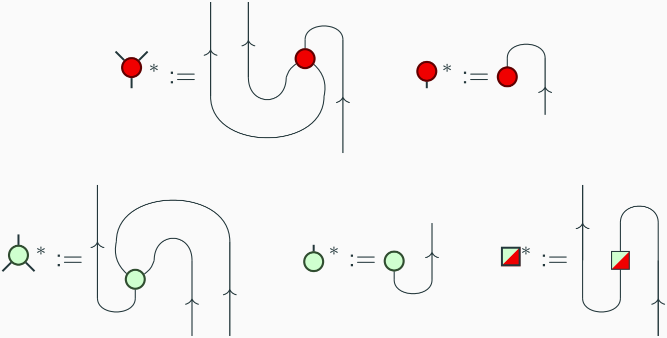
Theorem

The category of modules over a bialgebra is braided if and only if the bialgebra is quasi-triangular

Dual Hopf Algebra

Definition

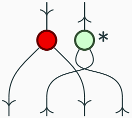
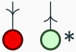
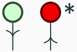
Let (H, , , , ,) be a Hopf algebra, and suppose that the object H has a dual H^*. We define the *dual Hopf algebra* (H^*, *, *, *, *, *) as :



Drinfeld Double

Definition

Let H be a Hopf algebra with an invertible antipode, and dual H^*. The *Drinfeld double* of H, denoted $D(H) = (H \otimes H^*, \mu, 1, \Delta, \epsilon, s)$, is a Hopf algebra defined in the following manner:

$\Delta :=$  $\epsilon :=$  $1 :=$ 

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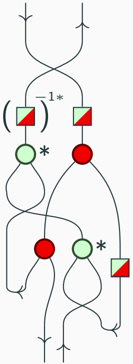
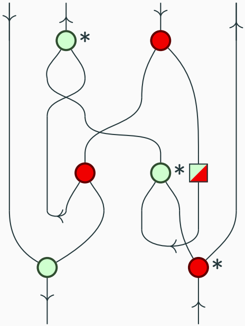
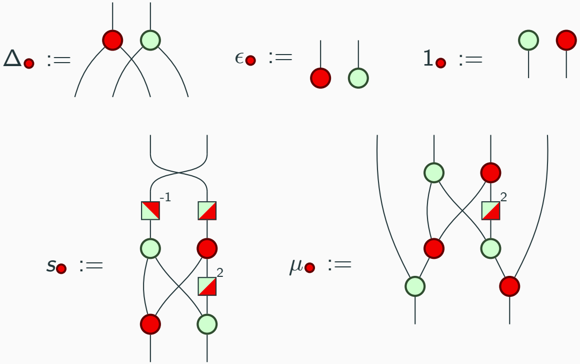
$s :=$ 

$\mu :=$ 

Definition

Let H be a HF algebra. The *red Drinfeld double*, denoted $D_{\bullet}(H) = (H \otimes H, \mu_{\bullet}, 1_{\bullet}, \Delta_{\bullet}, \epsilon_{\bullet}, s_{\bullet})$, is a Hopf algebra on the object $H \otimes H$ with structure maps



Conclusions

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What's next?

- Category of representations
- Red Drinfeld double may be useful in the context of Kitaev double
- Useful whenever the dual Hopf algebra is encountered
- More interesting examples of Hopf-Frobenius algebras?

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