A language for closed cartesian bicategories

—work in progress—

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Overview

- Context:
  - formal category theory
  - ~~directed type theory~~ directed first order logic
- motivation:
  - discussions with Colin Zwanziger about ‘2-toposes’
  - past discussions with Paul-André Melliès
- idea: logical properties of the 2-topos **Cat** are best understood by embedding into the **cartesian bicategory**[1][2] **Prof**.

$$\mathbf{Cat} \quad \hookrightarrow \quad \mathbf{Prof}$$

[1] A. Carboni and R.F.C. Walters. “Cartesian bicategories I”. In: *Journal of pure and applied algebra* 49.1-2 (1987), pp. 11–32.

[2] A. Carboni, G.M. Kelly, R.F.C. Walters, and R.J. Wood. “Cartesian bicategories II”. In: *Theory and Applications of Categories* 19.6 (2008), pp. 93–124.

Cartesian Bicategories

For $\mathcal{B}$ a bicategory, $\mathrm{Map}(\mathcal{B})$ is the sub-bicategory of left adjoints.

Definition

$\mathcal{B}$ is called **pre-cartesian**, if

1. $\mathrm{Map}(\mathcal{B})$ has finite bicategorical products;
2. all $\mathcal{B}(A, B)$ have finite products.

For pre-cartesian $\mathcal{B}$, the nullary and binary product functors

$$1 : 1 \to \mathrm{Map}(\mathcal{B}) \qquad \times : \mathrm{Map}(\mathcal{B}) \times \mathrm{Map}(\mathcal{B}) \to \mathrm{Map}(\mathcal{B})$$

canonically extend to *lax* functors

$$1 : 1 \to \mathcal{B} \qquad \otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}. \tag{$\dagger$}$$

Definition

A **cartesian bicategory** is a pre-cartesian bicategory $\mathcal{B}$ such that the lax functors ($\dagger$) are pseudo-functors.

CKWW show that the pseudofunctors ($\dagger$) define a symmetric monoidal structure on a cartesian bicategory.

The cartesian bicategory **Prof**

- A **profunctor** $\Phi : \mathbb{A} \nrightarrow \mathbb{B}$ between small categories is by definition a set-valued functor $\Phi : \mathbb{B}^{\mathrm{op}} \times \mathbb{A} \to \mathbf{Set}$.
- Profunctors can be composed via coends:

$$(\psi \circ \phi)(C, A) \;=\; \int^{B \in \mathbb{B}} \psi(C, B) \times \phi(B, A)$$

  for $\phi : \mathbb{A} \to \mathbb{B}$,$\psi : \mathbb{B} \to \mathbb{C}$.
- The bicategory **Prof** has small categories as objects, profunctors as 1-cells, and natural transformations as 2-cells.

Functors as maps in **Prof**

- Every functor $F : \mathbb{A} \to \mathbb{B}$ induces an adjunction of profunctors

  $$F_* \dashv F^* : \mathbb{B} \nrightarrow \mathbb{A}$$

  where $F_*(B, A) = \mathbb{B}(B, FA)$ and $F^*(A, B) = \mathbb{B}(FA, B)$.
- Conversely, every adjunction $\psi \dashv \phi : \mathbb{B} \nrightarrow \mathbb{A}$ is induced by an essentially unique functor, *provided* $\mathbb{B}$ *is Cauchy-complete*.
- Using this one can show that **Prof** is a cartesian bicategory, and $\mathrm{Map}(\mathbf{Prof})$ is equivalent to the 2-category $\mathbf{Cat}_{\mathrm{cc}}$ of Cauchy-complete categories.

$$\mathrm{Map}(\mathbf{Prof}) \simeq \mathbf{Cat}_{\mathrm{cc}} \quad \hookrightarrow \quad \mathbf{Prof}$$

Autonomous structure on **Prof**

Prof is **autonomous**, in the sense that every object has a monoidal dual.
The dual object to $\mathbb{C}$ is its **dual category** $\mathbb{C}^{\mathsf{op}}$, with unit and counit profunctors given by

$$\eta : 1 \nrightarrow \mathbb{C} \times \mathbb{C}^{\mathsf{op}} \qquad \eta(C, D) = \mathrm{hom}(D, C)$$

$$\varepsilon : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \nrightarrow 1 \qquad \varepsilon(C, D) = \mathrm{hom}(C, D)$$

Closed structure of **Prof**

Prof is a **closed bicategory**: for profunctors $\phi : \mathbb{B} \nrightarrow \mathbb{C}$ and small categories $\mathbb{A}, \mathbb{D}$, the pre- and postcomposition functors

$$(\phi \circ -) : \mathbf{Prof}(\mathbb{A}, \mathbb{B}) \to \mathbf{Prof}(\mathbb{A}, \mathbb{C}) \quad \text{and}$$
$$(- \circ \phi) : \mathbf{Prof}(\mathbb{C}, \mathbb{D}) \to \mathbf{Prof}(\mathbb{B}, \mathbb{D})$$

have right adjoints

$$(\phi \circ -) \dashv (\phi \multimap -) : \mathbf{Prof}(\mathbb{A}, \mathbb{C}) \to \mathbf{Prof}(\mathbb{A}, \mathbb{B}) \quad \text{and}$$
$$(- \circ \phi) \dashv (- \multimapinv \phi) : \mathbf{Prof}(\mathbb{B}, \mathbb{D}) \to \mathbf{Prof}(\mathbb{C}, \mathbb{D})$$

given by **ends**:

$$(\phi \multimap \psi)(B, A) = \int_C \psi(C, A)^{\phi(C,B)} \quad \text{and}$$
$$(\gamma \multimapinv \phi)(D, C) = \int_B \gamma(D, B)^{\phi(C,B)}$$

More closed cartesian bicategories

Examples of **closed cartesian bicategories with duals** are:

- the locally ordered category **Rel** of sets and relations
- the bicategory **Span** of sets and spans
- the bicategory $\mathbb{C}$-**Prof** of ~~profunctors~~ **discrete 2-sided fibrations** between presheaves of categories on a small category $\mathbb{C}$

Taking presheaves of categories on a **2-category** instead of a category in the last example gives closed cartesian bicategories that do in general **not** have duals (as pointed out by Shulman).

I will present a formal calculus for this generality – for cartesian bicategories that are closed but do not necessarily have duals.

Ends and coends vs quantification

The bijection characterizing ends

$$\frac{\gamma(C^+) \longrightarrow \phi(C^+, D^-, D^+) \text{ extranatural}}{\gamma(C^+) \longrightarrow \int_D \phi(C^+, D^-, D^+) \text{ natural}}$$

(where $\varphi : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \times \mathbb{D} \to \mathbf{Set}$ and $\psi : \mathbb{D} \to \mathbf{Set}$) resembles the adjunction

$$\frac{\Gamma \;\vdash\; \phi[x]}{\Gamma \;\vdash\; \forall x\,.\,\phi[x]}\; x \notin \mathrm{FV}(\Gamma)$$

for universal quantification in first order logic.

Similarly, the bijection characterizing coends $\int^D$ resembles the adjunction for $\exists$.
However, 'naive' application of first order logic rules is not possible because of mixed variances, and **dinatural transformations don't compose**.
In the following I introduce a system of constructions for 1- and 2-cells in closed cartesian bicategories that can be seen as refinement of this naive analogy between (co)ends and quantification.

Variances of variables controlled by a particular **form of judgment**.

The language

Signature:

- **Sort symbols** $\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{U}, \boldsymbol{V}, \ldots$
- **Functor symbols** with arities

$$\begin{aligned} F &: \boldsymbol{S}_1 \ldots \boldsymbol{S}_n \to \boldsymbol{T} \\ G &: \boldsymbol{U}_1 \ldots \boldsymbol{U}_m \to \boldsymbol{T} \\ &\ldots \end{aligned}$$

- **Relator symbols** with arities

$$\begin{aligned} R &: \boldsymbol{S}_1 \ldots \boldsymbol{S}_n \nleftarrow \boldsymbol{T}_1 \ldots \boldsymbol{T}_k \\ S &: \boldsymbol{U}_1 \ldots \boldsymbol{U}_m \nleftarrow \boldsymbol{V}_1 \ldots \boldsymbol{V}_l \\ &\ldots \end{aligned}$$

Contexts, terms, formulas

- **Contexts** are lists $\vec{X} = X_1, \ldots, X_n$ of sorted variables
- **Terms-in-context** $\langle\vec{X}\rangle\, T : \boldsymbol{S}$ are built up from sorted variables and functor symbols
- **Formulas** $\langle\vec{X}\rangle\, \varphi\, \langle\vec{Y}\rangle$ – with a context $\langle\vec{X}\rangle$ of negative variables on the left, and a context $\langle\vec{Y}\rangle$ of positive variables on the right – are generated by the rules

$$\frac{\begin{array}{ll}\langle\vec{X}\rangle\, S_i : \boldsymbol{S}_i & (1 \le i \le n)\\ \langle\vec{Y}\rangle\, T_j : \boldsymbol{T}_j & (1 \le j \le k)\end{array}}{\langle\vec{X}\rangle\, R(\vec{S}; \vec{T})\, \langle\vec{Y}\rangle} \qquad \text{if } R : \boldsymbol{S}_1 \ldots \boldsymbol{S}_n \nrightarrow \boldsymbol{T}_1 \ldots \boldsymbol{T}_k$$

$$\frac{\langle\vec{X}\rangle\, S : \boldsymbol{T} \qquad \langle\vec{Y}\rangle\, T : \boldsymbol{T}}{\langle\vec{X}\rangle\, \mathrm{hom}(S, T)\, \langle\vec{Y}\rangle} \qquad \frac{\langle\vec{X}\rangle\, \varphi\, \langle\vec{Y}, \vec{Z}\rangle \qquad \langle\vec{X}, \vec{Y}\rangle\, \psi\, \langle\vec{Z}\rangle}{\langle\vec{X}\rangle\, \varphi \otimes_{\vec{Y}} \psi\, \langle\vec{Z}\rangle}$$

$$\frac{\langle\vec{X}\rangle\, \varphi\, \langle\vec{Z}\rangle \qquad \langle\vec{Y}\rangle\, \psi\, \langle\vec{Z}\rangle}{\langle\vec{X}\rangle\, \varphi \overset{\vec{Z}}{\multimapinv}\, \psi\, \langle\vec{Y}\rangle} \qquad \frac{\langle\vec{Z}\rangle\, \varphi\, \langle\vec{X}\rangle \qquad \langle\vec{Z}\rangle\, \psi\, \langle\vec{Y}\rangle}{\langle\vec{X}\rangle\, \varphi \overset{\vec{Z}}{\multimap}\, \psi\, \langle\vec{Y}\rangle}$$

The connectives

Intuitions:

- $\mathsf{hom}(S, T)$: 'directed equality predicate' (identity arrow in CCBC)
- $\varphi \otimes_{\vec{X}} \psi$: 'combined existential quantification & conjunction' (composition in CCBC)
- $\varphi \overset{\vec{X}}{\multimap} \psi$: 'combined universal quantification & implication' (closed structure in CCBC)

Judgments

Judgments (sequents) are of the form

$$\langle \vec{X}_0 \rangle\, \varphi_1\, \langle \vec{X}_1 \rangle\, \ldots \varphi_{n-1}\, \langle \vec{X}_{n-1} \rangle\, \varphi_n\, \langle \vec{X}_n \rangle \;\vdash\; \psi$$

such that

$$\langle \vec{X}_0 \ldots \vec{X}_{j-1} \rangle\, \varphi_j\, \langle \vec{X}_j \ldots \vec{X}_n \rangle$$

(for $1 \leq j \leq n$), and

$$\langle \vec{X}_0 \rangle\, \psi\, \langle \vec{X}_n \rangle$$

are well-formed formulas.

In other words, variables declared on the left of φ_i may appear negatively in φ_i, and variables declared on the right may appear positively (and only so).

Variables may appear in different variances in a judgment, but never in the same formula.

Structural Rules

$$\frac{}{\langle \vec{X} \rangle\, \varphi\, \langle \vec{Y} \rangle \;\vdash\; \varphi}$$ axiom

$$\frac{\langle \vec{X}_0, \ldots, \vec{X}_{i-1} \rangle\, \Gamma\, \langle \vec{X}_i, \ldots, \vec{X}_n \rangle \vdash \varphi_i \qquad \langle \vec{X}_0 \rangle\, \varphi_1 \ldots \varphi_i \ldots \varphi_n\, \langle \vec{X}_n \rangle \;\vdash\; \psi}{\langle \vec{X}_0 \rangle\, \varphi_1 \ldots \Gamma \ldots \varphi_n\, \langle \vec{X}_n \rangle \;\vdash\; \psi}$$ cut

$$\frac{\Gamma \varphi, \varphi, \Delta \;\vdash\; \psi}{\Gamma, \varphi, \Delta \;\vdash\; \psi}$$ contraction

$$\frac{\Gamma, \Delta \;\vdash\; \psi}{\Gamma, \varphi, \Delta \;\vdash\; \psi}$$ weakening

$$\frac{\Gamma\, \langle \vec{X}, Y \rangle\, \Delta \;\vdash\; \varphi \qquad \langle \vec{X} \rangle\, T : \boldsymbol{T}}{\Gamma[T/Y]\, \langle \vec{X} \rangle\, \Delta[T/Y] \;\vdash\; \varphi}$$ substitution for inner variable

$$\frac{\langle \vec{X}, Y \rangle\, \Delta \;\vdash\; \varphi \qquad \langle \vec{X} \rangle\, T : \boldsymbol{T}}{\langle \vec{X} \rangle\, \Delta[T/Y] \;\vdash\; \varphi[T/Y]}$$ substitution for left variable

$$\frac{\Gamma\, \langle \vec{X}, Y \rangle \;\vdash\; \varphi \qquad \langle \vec{X} \rangle\, T : \boldsymbol{T}}{\Gamma[T/Y]\, \langle \vec{X} \rangle \;\vdash\; \varphi[T/Y]}$$ substitution for right variable

Logical rules – in 'adjunction style'

$$\frac{\Gamma\ \varphi\ \langle\vec{X}\rangle\ \psi\ \Delta\ \vdash\ \theta}{\Gamma\ \varphi \otimes_{\vec{X}} \psi\ \Delta\ \vdash\ \theta}$$

if $\{\vec{X}\} \cap \mathrm{FV}(\Gamma, \Delta) = \varnothing$

$$\frac{\Gamma, \varphi\ \langle\vec{X}\rangle\ \psi\ \langle\vec{Y}\rangle\ \vdash\ \theta}{\Gamma, \varphi\ \langle\vec{X}\rangle\ \vdash\ \theta \overset{\vec{Y}}{\multimapinv} \psi}$$

if $\{\vec{Y}\} \cap \mathrm{FV}(\Gamma, \varphi) = \varnothing$
and $\mathrm{FV}_{-}(\psi) \subseteq \{\vec{X}\}$

$$\frac{\langle\vec{X}\rangle\ \varphi\ \langle\vec{Y}\rangle\ \psi, \Gamma\ \vdash\ \theta}{\langle\vec{Y}\rangle\ \psi, \Gamma\ \vdash\ \varphi \overset{\vec{X}}{\multimap} \theta}$$

if $\{\vec{X}\} \cap \mathrm{FV}(\psi, \Gamma) = \varnothing$
and $\mathrm{FV}_{+}(\varphi) \subseteq \{\vec{Y}\}$

$$\frac{\Gamma\ \langle X\rangle\ \Delta, \Theta\ \vdash\ \psi}{\Gamma\ \langle X\rangle\ \Delta, \mathsf{hom}(X, Y)\ \langle Y\rangle\ \Theta[Y/X]\ \vdash\ \psi}$$

$$\frac{\Gamma, \Delta\ \langle Y\rangle\ \Theta\ \vdash\ \psi}{\Gamma[X/Y]\ \langle X\rangle\ \mathsf{hom}(X, Y), \Delta\ \langle Y\rangle\ \Theta\ \vdash\ \psi}$$

Example derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\langle X\rangle\, \varphi[X] \otimes \psi[X] \;\vdash\; \theta[X]}{\langle X\rangle\, \varphi[X], \psi[X] \;\vdash\; \theta[X]}}{\langle X\rangle\, \varphi[X], \mathsf{hom}(X, Y)\, \langle Y\rangle\, \psi[Y] \;\vdash\; \theta[X]}}{\langle X\rangle\, \varphi[X] \otimes \mathsf{hom}(X, Y)\, \langle Y\rangle\, \psi[Y] \;\vdash\; \theta[X]}}{\langle Y\rangle\, \psi[Y] \;\vdash\; \varphi[X] \otimes \mathsf{hom}(X, Y) \overset{X}{\multimap} \theta[X]}$$

'Proof' that $\mathcal{B}(1, A)$ is cartesian closed in any closed cartesian bicategory $\mathcal{B}$.

Soundness and completeness, future work

- For now, I think I can show soundness and completeness for **posetal** closed cartesian bicategories.
- To extend to non-posetal case need equational theory on derivations, ideally via **term calculus**
- However: 'natural deduction style' system problematic, since the type of reflexivity $\mathsf{hom}(X, X)$ is not well formed.
- Same problem for left/right sequent calculus system

Thanks for your attention!