COALGEBRAS FOR ENRICHEd HAUSDORFF (AND VIETORIS) FUNCTORS

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July 12, 2019

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INTRODUCTION
For a functor $F : \mathbf{C} \to \mathbf{C}$, one defines coalgebra homomorphism:

$$\begin{array}{cc}
FX & FY \\
\downarrow c & \downarrow d \\
X & Y
\end{array}$$

The corresponding category we denote as $\text{CoAlg}(F)$. 

Theorem
The forgetful functor $\text{CoAlg}(F) \to \mathbf{C}$ creates all colimits and those limits which are preserved by $F$. 

A QUICK REMINDER

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The forgetful functor $\text{CoAlg}(F) \to \mathbf{C}$ creates all colimits and those limits which are preserved by $F$. 
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• The power-set functor $P: \text{Set} \to \text{Set}$ does not have a fix-point; hence $P$ does not admit a final coalgebra. \(^a\)

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\(^a\)Georg Cantor. “Über eine elementare Frage der Mannigfaltigkeitslehre”. In: Jahresbericht der Deutschen Mathematiker-Vereinigung 1 (1891), pp. 75–78.
Recall

- The final coalgebra for $F: C \to C$ is a fix-point of $F$.
- The power-set functor $P: \text{Set} \to \text{Set}$ does not have a fix-point; hence $P$ does not admit a final coalgebra.
- The finite power-set functor $P_{\text{fin}}: \text{Set} \to \text{Set}$ admits a final coalgebra (for instance, because $P_{\text{fin}}$ is finitary).

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- *Somehow more general*: the Vietoris functor $V: \text{CompHaus} \to \text{CompHaus}$ admits a final coalgebra$^a$

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$^a$Here: $V$ preserves codirected limits. This result appears as an exercise in Ryszard Engelking.

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- *Somehow more general*: the Vietoris functor $V: \text{CompHaus} \to \text{CompHaus}$ admits a final coalgebra (and the same is true for $V: \text{PosComp} \to \text{PosComp}$).
- *A bit more general*: the compact Vietoris functor $V_c: \text{Top} \to \text{Top}$ admits a final coalgebra.
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- The finite power-set functor $P_{\text{fin}}: \text{Set} \rightarrow \text{Set}$ admits a final coalgebra (for instance, because $P_{\text{fin}}$ is finitary).
- *Somehow more general*: the Vietoris functor $V: \text{CompHaus} \rightarrow \text{CompHaus}$ admits a final coalgebra (and the same is true for $V: \text{PosComp} \rightarrow \text{PosComp}$).
- *A bit more general*: the compact Vietoris functor $V_c: \text{Top} \rightarrow \text{Top}$ admits a final coalgebra.
- *A bit surprising*?: Also the lower Vietoris functor $V: \text{Top} \rightarrow \text{Top}$ admits a final coalgebra.
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Questions

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- the upset functor $\text{Up}: \text{Ord} \to \text{Ord}$?
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- the upset functor $\text{Up}: \text{Ord} \to \text{Ord}$?
- liftings of $\text{Set}$-functors to $\text{Met}$ (or, more general, to $\mathcal{V}$-$\text{Cat}$)?

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\[ \mathcal{V}$-$\text{Cat} \xrightarrow{T} \mathcal{V}$-$\text{Cat} \]

\[ \downarrow \quad \downarrow \]

\[ \text{Set} \xrightarrow{T} \text{Set}. \]

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Questions

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- the upset functor \( \text{Up} : \text{Ord} \to \text{Ord} \)?
- liftings of \( \text{Set} \)-functors to \( \text{Met} \) (or, more general, to \( \mathcal{V}\text{-Cat} \))?  
- (in particular) the Hausdorff functor? \(^{abc}\)

\[ \text{H} : \text{Met} \to \text{Met} \]


Questions

What about “power functors” on other (topological) base categories? For instance,

- the upset functor $\text{Up}: \text{Ord} \to \text{Ord}$?
- liftings of $\text{Set}$-functors to $\text{Met}$ (or, more general, to $\mathcal{V}$-$\text{Cat}$)?
- (in particular) the Hausdorff functor? $^a$

$$H: \mathcal{V}$-$\text{Cat} \longrightarrow \mathcal{V}$-$\text{Cat}$$

Here $H(a(A, B)) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y)$, for a $\mathcal{V}$-category $(X, a)$.

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$^b$ Isar Stubbe. ““Hausdorff distance” via conical cocompletion”. In: Cahiers de Topologie et Géométrie Différentielle Catégoriques 51(1) (2010), pp. 51–76.
Some "powerful functors"
Theorem

Let $X$ be a partially ordered set. Then there is no embedding $\varphi : \text{Up}(X) \to X$.  


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**Corollary**

The upset functor $\text{Up} : \text{Ord} \to \text{Ord}$ does not admit a final coalgebra.
Theorem

Let $X$ be a partially ordered set. Then there is no embedding $\varphi : \text{Up}(X) \rightarrow X$.

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Corollary

The upset functor $\text{Up} : \text{Ord} \rightarrow \text{Ord}$ does not admit a final coalgebra.

Remark

The category $\text{CoAlg}(\text{Up})$ has equalisers.
Theorem

Consider the following commutative diagram of functors.

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\begin{array}{ccc}
X & \xrightarrow{\bar{F}} & X \\
\downarrow U & & \downarrow U \\
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1. If \( \bar{F} \) has a fix-point, then so has \( F \). Hence, if \( F \) does not have a fix-point, then neither does \( \bar{F} \).
Theorem

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1. If $\bar{F}$ has a fix-point, then so has $F$. Hence, if $F$ does not have a fix-point, then neither does $\bar{F}$.

2. If $U : X \to A$ is topological, then so is $U : \text{CoAlg}(\bar{F}) \to \text{CoAlg}(F)$.
Theorem

Consider the following commutative diagram of functors.

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1. If \(\bar{F}\) has a fix-point, then so has \(F\). Hence, if \(F\) does not have a fix-point, then neither does \(\bar{F}\).

2. If \(U : X \rightarrow A\) is topological, then so is \(U : \text{CoAlg}(\bar{F}) \rightarrow \text{CoAlg}(F)\).

In particular, the category \(\text{CoAlg}(\bar{F})\) has limits of shape I if and only if \(\text{CoAlg}(F)\) has limits of shape I.
**Definition**

Let \( f : (X, a) \to (Y, b) \) be a \( \mathcal{V} \)-functor.

1. For every \( A \subseteq X \), put \( \uparrow^a A = \{ y \in X \mid k \leq \bigvee_{x \in A} a(x, y) \} \).
**Definition**

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2. We call a subset $A \subseteq X$ of $(X, a)$ **increasing** whenever $A = \uparrow^a A$. 

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**The Hausdorff Monad on $\mathcal{V}$-Cat**

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2. We call a subset $A \subseteq X$ of $(X, a)$ **increasing** whenever $A = \uparrow^a A$.
3. We consider the $\mathcal{V}$-category $H X = \{ A \subseteq X \mid A \text{ is increasing} \}$, equipped with $H a(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y)$, for all $A, B \in H X$. 


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4. The map \( Hf : H(X, a) \to H(Y, b) \) sends an increasing subset \( A \subseteq X \) to \( \uparrow^b f(A) \).
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H_a(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y), \text{ for all } A, B \in HX.
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5. The functor \( H \) is part of a Kock–Zöberlein monad \( \mathbb{H} = (H, w, h) \) on \( \mathcal{V}\text{-Cat} \).

\[
\begin{align*}
  h_X : X &\to HX, \\
  x &\mapsto \uparrow^a x \\
  w_X : HHX &\to HX, \\
  A &\mapsto \bigcup A
\end{align*}
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THE HAUSDORFF MONAD ON $\mathcal{V}$-Cat

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6. $\mathbb{H} = (H, \omega, \hat{\iota})$ is a submonad of the covariant presheaf monad on $\mathcal{V}$-$\text{Cat}$; in fact, $\mathbb{H}$ is the monad of “conical limit weights”.
For metric spaces

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### Theorem

Let $\mathcal{V}$ be a non-trivial quantale and $(X, a)$ be a $\mathcal{V}$-category. There is no embedding of type $\mathcal{H}(X, a) \rightarrow (X, a)$. 

### Corollary

Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $\mathcal{H}: \mathcal{V}-\text{Cat} \rightarrow \mathcal{V}-\text{Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}-\text{Cat}$. 

### Remark

In particular, the (non-symmetric) Hausdorff functor on $\text{Met}$ does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces. Passing to the symmetric version of the Hausdorff functor does not remedy the situation.
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Corollary
Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $H: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}\text{-Cat}$. 

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ADDING TOPOLOGY
Extending the Ultrafilter monad

We assume that $\mathcal{V}$ is a completely distributive quantale, then

$$\xi: U\mathcal{V} \rightarrow \mathcal{V}, \quad v \mapsto \bigwedge \bigvee_{A \in v} A$$

is the structure of an $U$-algebra on $\mathcal{V}$ (the Lawson topology).
Extending the Ultrafilter monad

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is the structure of an \( \mathcal{U} \)-algebra on \( \mathcal{V} \) (the Lawson topology). Therefore we obtain a lax extension of the ultrafilter monad to \( \mathcal{V}\text{-Rel} \) that induces a monad on \( \mathcal{V}\text{-Cat} \).

Here:
\[
\mathcal{U}a(x, y) = \bigwedge \bigvee a(x, y), \quad (X, a) \mapsto (UX, Ua).
\]
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Its algebras are $\mathcal{V}$-categories equipped with a compatible compact Hausdorff topology $ab$; we call them $\mathcal{V}$-categorical compact Hausdorff spaces, and denote the corresponding Eilenberg–Moore category by $\mathcal{V}$-$\textbf{CatCH}$.

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Theorem

For an ordered set $(X, \leq)$ and a $U$-algebra $(X, \alpha)$, the following are equivalent.

(i) $\alpha : (UX, U\leq) \rightarrow (X, \leq)$ is monotone.
### Generalised Nachbin Spaces

#### Extending the Ultrafilter monad

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#### Theorem

For an ordered set $(X, \leq)$ and a $U$-algebra $(X, \alpha)$, the following are equivalent.

1. $\alpha : (UX, U\leq) \to (X, \leq)$ is monotone.
2. $G_{\leq} \subseteq X \times X$ is closed.

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{{namespace|Generalised Nachbin Spaces}}

{{section|Extending the Ultrafilter monad}}

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Extending the Ultrafilter monad

We assume that $\mathcal{V}$ is a completely distributive quantale, then

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Theorem

For a $\mathcal{V}$-category $(X, a)$ and a $U$-algebra $(X, \alpha)$, the following are equivalent.

(i) $\alpha : U(X, a) \rightarrow (X, a)$ is a $\mathcal{V}$-functor.
(ii) $a : (X, \alpha) \times (X, \alpha) \rightarrow (\mathcal{V}, \xi \leq)$ is continuous.
Theorem
For an ordered compact Hausdorff space $X$, the ordered set $X$ is directed complete.

Proof.
$\text{OrdCH}_{\text{Top}_{K}(X)}$ is sober, . . .

Corollary
Every compact metric space is Cauchy complete.

Example
$(U_{X}, U_{d})$ is Cauchy complete.

Consider $(U_{X}, U_{d}, m_{X})$ . . . and close.)
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Example

$((U, d_X), (U, d_X), m_X)$
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Proof.

$\text{OrdCH} \xrightarrow{K} \text{Top} \xrightarrow{|-|} \text{Ord}$

$K(X)$ is sober, ...

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Theorem
For a metric compact Hausdorff space $X$, the metric space $X$ is Cauchy complete.

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**Theorem**

For a metric compact Hausdorff space $X$, the metric space $X$ is **Cauchy** complete.

**Proof.**

$$\text{MetCH} \xrightarrow{K} \text{App} \quad \text{K}(X) \text{ is sober, ...}$$

$$|X| \text{ is directed complete} \quad \Box$$

Approach space = “metric” topological space.


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\vert-\vert & \downarrow S \\
\text{Met} & \quad |X| \text{ is directed complete}
\end{align*}
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$K(X)$ is sober, ...

**Theorem (Lawvere (1973))**

A metric space $X$ is Cauchy-complete if and only if every left adjoint distributors $\varphi: 1 \xrightarrow{\sim} X$ is representable (i.e. $\varphi = x_\ast$).
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$K(X)$ is Cauchy complete, \ldots

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For a metric compact Hausdorff space $X$, the metric space $X$ is Cauchy complete.

Proof.

\[ \text{MetCH} \xrightarrow{K} \text{App} \]

\[ \xrightarrow{K(X) \text{ is Cauchy complete, ...}} \]

\[ \text{Met} \xrightarrow{\mid - \mid} \]

\[ \xrightarrow{\mid X \mid \text{ is Cauchy complete}} \]

Corollary

Every compact metric space is Cauchy complete.
**Theorem**

For a metric compact Hausdorff space $X$, the metric space $X$ is **Cauchy** complete.

**Proof.**

$\text{MetCH} \xrightarrow{K} \text{App} \xrightarrow{\text{S}} \text{Met} \xrightarrow{|-|} K(X) \text{ is Cauchy complete, } ... \xrightarrow{|-|} |X| \text{ is Cauchy complete} \quad \Box$

**Corollary**

Every **compact metric** space is Cauchy complete.

**Example**

Every discrete metric space is Cauchy complete (any compact Hausdorff topology).
Theorem
For a metric compact Hausdorff space $X$, the metric space $X$ is Cauchy complete.

Proof.

$$\text{MetCH} \xrightarrow{K} \text{App}$$

$$\text{Met} \xrightarrow{\text{S}} |X|$$

$K(X)$ is Cauchy complete, ...

$|X|$ is Cauchy complete.

Corollary
Every compact metric space is Cauchy complete.

Example
$(UX, Ud)$ is Cauchy complete.
**Theorem**

For a metric compact Hausdorff space $X$, the metric space $X$ is Cauchy complete.

**Proof.**

\[
\text{MetCH} \xrightarrow{K} \text{App} \quad \text{K}(X) \text{ is Cauchy complete, } \ldots
\]

\[
\text{Met} \quad \downarrow \quad S \quad \quad \quad \quad \quad \quad \quad \quad \downarrow
\]

\[
|X| \text{ is Cauchy complete}
\]

**Corollary**

Every compact metric space is Cauchy complete.

**Example**

$(UX, Ud)$ is Cauchy complete. Consider $(UX, Ud, m_X) \ldots$ and close “)”
**Lemma**

Let $(X, a, \alpha)$ be a $\mathcal{V}$-categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \emptyset$, $A$ is increasing and compact in $(X, \alpha \leq)^{op}$ and $B$ is compact in $(X, \alpha \leq)$. Then there exists some $u \ll k$ so that, for all $x \in A$ and $y \in B$, $u \not\leq a(x, y)$. 

**Corollary**

For every compact subset $A \subseteq X$ of $(X, \alpha \leq)^{op}$, $\uparrow a A = \uparrow \leq A$. In particular, for every closed subset $A \subseteq X$ of $(X, \alpha \leq)$, $\uparrow a A = \uparrow \leq A$.

**Theorem (Nachbin)**

Let $A \subseteq X$ be closed and decreasing and $B \subseteq X$ be closed and increasing with $A \cap B = \emptyset$. Then there exist $V \subseteq X$ open and co-increasing and $W \subseteq X$ open and co-decreasing with $A \subseteq V$, $B \subseteq W$, $V \cap W = \emptyset$. 

Towards “Urysohn”
Lemma

Let \((X, a, \alpha)\) be a \(\mathcal{V}\)-categorical compact Hausdorff space and \(A, B \subseteq X\) so that \(A \cap B = \emptyset\), \(A\) is increasing and compact in \((X, \alpha \leq)^{\text{op}}\) and \(B\) is compact in \((X, \alpha \leq)\). Then there exists some \(u \ll k\) so that, for all \(x \in A\) and \(y \in B\), \(u \not\leq a(x, y)\).

Corollary

For every compact subset \(A \subseteq X\) of \((X, \alpha \leq)^{\text{op}}\), \(\uparrow^a A = \uparrow^\leq A\). In particular, for every closed subset \(A \subseteq X\) of \((X, \alpha)\), \(\uparrow^a A = \uparrow^\leq A\).
Lemma

Let \((X, a, \alpha)\) be a \(\mathcal{V}\)-categorical compact Hausdorff space and \(A, B \subseteq X\) so that \(A \cap B = \emptyset\), \(A\) is increasing and compact in \((X, \alpha \leq)\)\(^{op}\) and \(B\) is compact in \((X, \alpha \leq)\). Then there exists some \(u \ll k\) so that, for all \(x \in A\) and \(y \in B\), \(u \not\leq a(x, y)\).

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For every compact subset \(A \subseteq X\) of \((X, \alpha \leq)\)\(^{op}\), \(\uparrow^a A = \uparrow^\leq A\). In particular, for every closed subset \(A \subseteq X\) of \((X, \alpha)\), \(\uparrow^a A = \uparrow^\leq A\).

Theorem (Nachbin)

Let \(A \subseteq X\) be closed and decreasing and \(B \subseteq X\) be closed and increasing with \(A \cap B = \emptyset\). Then there exist \(V \subseteq X\) open and co-increasing and \(W \subseteq X\) open and co-decreasing with \(A \subseteq V\), \(B \subseteq W\), \(V \cap W = \emptyset\).
Definition

For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$HX = \{ A \subseteq X \mid A \text{ is closed and increasing} \}$$

with the restriction of the Hausdorff structure to $HX$ and the hit-and-miss topology (Vietoris topology). That is, the topology generated by the sets

$$V^\lozenge = \{ A \in HX \mid A \cap V \neq \emptyset \} \quad (V \text{ open, co-increasing})$$

and

$$W^\square = \{ A \in HX \mid A \subseteq W \} \quad (W \text{ open, co-decreasing}).$$
**Definition**
For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$HX = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

with the restriction of the Hausdorff structure to $HX$ and the hit-and-miss topology (Vietoris topology).

**Proposition**
For every $\mathcal{V}$-categorical compact Hausdorff space $X$, $HX$ is a $\mathcal{V}$-categorical compact Hausdorff space.

**Compare with:**
For a compact metric space, the Hausdorff metric induces the Vietoris topology.
**THE HAUSDORFF MONAD (AGAIN)**

**Definition**
For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

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For every $\mathcal{V}$-categorical compact Hausdorff space $X$, $HX$ is a $\mathcal{V}$-categorical compact Hausdorff space.

**Theorem**
The construction above defines a functor $H: \mathcal{V}\text{-CatCH} \to \mathcal{V}\text{-CatCH}$. 
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**Theorem**

The construction above defines a functor $H : \mathcal{V}\text{-CatCH} \to \mathcal{V}\text{-CatCH}$.

In fact, we obtain a Kock–Zöberlein monad.
Theorem (Lawvere (1973))
A metric space $X$ is Cauchy-complete if and only if every left adjoint distributor $\varphi : 1 \rightarrow X$ is representable (i.e. $\varphi = x_*$).
### Theorem (Lawvere (1973))

A metric space $X$ is Cauchy-complete if and only if every left adjoint distributor $\varphi : 1 \longrightarrow X$ is representable (i.e. $\varphi = x_*$).

### Definition

For a $\mathcal{V}$-category $X$, $A \subseteq X$ and $x \in X$, we define $x \in \overline{A}$ whenever "$x$ represents a left adjoint distributor $1 \longrightarrow A$".
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Remark
- Under suitable conditions, this closure operator is topological.
**Theorem (Lawvere (1973))**

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For a $\mathcal{V}$-category $X$, $A \subseteq X$ and $x \in X$, we define $x \in \overline{A}$ whenever “$x$ represents a left adjoint distributor $1 \rightarrow A$”.

**Remark**

- Under suitable conditions, this closure operator is topological.
- Moreover, if $X$ is separated, then this topology is Hausdorff.
## Compact $\mathcal{V}$-categories

### Theorem (Lawvere (1973))

A metric space $X$ is Cauchy-complete if and only if every left adjoint distributor $\varphi : 1 \longrightarrow X$ is representable (i.e. $\varphi = x_*$).

### Definition

For a $\mathcal{V}$-category $X$, $A \subseteq X$ and $x \in X$, we define $x \in \overline{A}$ whenever “$x$ represents a left adjoint distributor $1 \longrightarrow A$”.

### Remark

- Under suitable conditions, this closure operator is topological.
- Moreover, if $X$ is separated, then this topology is Hausdorff.
- With $\mathcal{V}\text{-Cat}_{\text{ch}}$ denoting the full subcategory of $\mathcal{V}\text{-Cat}_{\text{sep}}$ defined by those $\mathcal{V}$-categories with compact topology, we obtain a functor $\mathcal{V}\text{-Cat}_{\text{ch}} \to \mathcal{V}\text{-CatCH}$. 
Proposition

For the $\mathcal{V}$-categorical compact Hausdorff space induced by a compact separated $\mathcal{V}$-category $X$, the hit-and-miss topology on $HX$ coincides with the topology induced by the Hausdorff structure on $HX$.  

Theorem

The functor $H: \mathcal{V}-\text{Cat} \to \mathcal{V}-\text{Cat}$ restricts to the category $\mathcal{V}-\text{Cat}^\text{ch}$, moreover, the diagram $\mathcal{V}-\text{Cat}^\text{ch} \to \mathcal{V}-\text{Cat}^\text{ch} \to \mathcal{V}-\text{Cat}^\text{CH} \to \mathcal{V}-\text{Cat}^\text{CH}$ commutes.
Proposition

For the $\mathcal{V}$-categorical compact Hausdorff space induced by a compact separated $\mathcal{V}$-category $X$, the hit-and-miss topology on $HX$ coincides with the topology induced by the Hausdorff structure on $HX$.

Theorem

The functor $H : \mathcal{V}\text{-}\text{Cat} \to \mathcal{V}\text{-}\text{Cat}$ restricts to the category $\mathcal{V}\text{-}\text{Cat}_{\text{ch}}$, moreover, the diagram

\[
\begin{array}{ccc}
\mathcal{V}\text{-}\text{Cat}_{\text{ch}} & \xrightarrow{H} & \mathcal{V}\text{-}\text{Cat}_{\text{ch}} \\
\downarrow & & \downarrow \\
\mathcal{V}\text{-}\text{CatCH} & \xrightarrow{H} & \mathcal{V}\text{-}\text{CatCH}
\end{array}
\]

commutes.
Proposition

The diagrams of functors commutes.

\[ \text{OrdCH} \xrightarrow{H} \text{OrdCH} \]
\[ \text{V-CatCH} \xrightarrow{H} \text{V-CatCH} \]
Proposition

The diagrams of functors commute.

\[
\begin{array}{c}
\text{OrdCH} \xrightarrow{H} \text{OrdCH} \\
\nu\text{-CatCH} \xrightarrow{H} \nu\text{-CatCH}
\end{array}
\]

Proposition

The Hausdorff functor on \(\nu\text{-CatCH}\) preserves codirected initial cones with respect to the forgetful functor \(\nu\text{-CatCH} \to \text{CompHaus}\).
**Proposition**

The diagrams of functors commutes.

\[ \begin{array}{ccc} \text{OrdCH} & \xrightarrow{H} & \text{OrdCH} \\ \downarrow \downarrow & & \downarrow \downarrow \\
\mathcal{V}\text{-CatCH} & \xrightarrow{H} & \mathcal{V}\text{-CatCH} \end{array} \]

**Proposition**

The Hausdorff functor on \( \mathcal{V}\text{-CatCH} \) preserves codirected initial cones with respect to the forgetful functor \( \mathcal{V}\text{-CatCH} \to \text{CompHaus} \).

**Theorem**

The Hausdorff functor \( H : \mathcal{V}\text{-CatCH} \to \mathcal{V}\text{-CatCH} \) preserves codirected limits.
Theorem

For $H: \mathcal{V}\text{-CatCH} \to \mathcal{V}\text{-CatCH}$, the forgetful functor $\text{CoAlg}(H) \to \mathcal{V}\text{-CatCH}$ is comonadic. Moreover, $\mathcal{V}\text{-CatCH}$ has equalisers and is therefore complete.
**Theorem**

For $H : \mathcal{V}\text{-CatCH} \to \mathcal{V}\text{-CatCH}$, the forgetful functor $\text{CoAlg}(H) \to \mathcal{V}\text{-CatCH}$ is comonadic. Moreover, $\mathcal{V}\text{-CatCH}$ has equalisers and is therefore complete.

**Theorem**

The category of coalgebras of a Hausdorff polynomial functor on $\mathcal{V}\text{-CatCH}$ is (co)complete.

**Definition**

We call a functor **Hausdorff polynomial** whenever it belongs to the smallest class of endofunctors on $\mathcal{V}\text{-Cat}$ that contains the identity functor, all constant functors and is closed under composition with $H$, products and sums of functors.
An ordered compact Hausdorff space is a Priestley space whenever the cone \( \text{PosComp}(X, \mathbf{2}) \) is an initial monocone. \(^a\)\(^b\)


Recall

An ordered compact Hausdorff space is a Priestley space whenever the cone $\text{PosComp}(X, 2)$ is an initial monocone. $^ab$

---


Assumption

From now on we assume that the maps $\text{hom}(u, -): \mathcal{V} \to \mathcal{V}$ are continuous.
**Definition**

We call a $\mathcal{V}$-categorical compact Hausdorff space $X$ **Priestley** if the cone $\mathcal{V}$-$\text{CatCH}(X, \mathcal{V}^{\text{op}})$ is initial (and mono).

**Corollary**

The Hausdorff functor restricts to a functor $H : \mathcal{V}$-$\text{Priest} \rightarrow \mathcal{V}$-$\text{Priest}$, hence the Hausdorff monad $H$ restricts to $\mathcal{V}$-$\text{Priest}$.
Definition

We call a $\mathcal{V}$-categorical compact Hausdorff space $X$ Priestley if the cone $\mathcal{V}$-$\text{CatCH}(X, \mathcal{V}^{\text{op}})$ is initial (and mono).

$\mathcal{V}$-$\text{Priest}$ denotes the full subcategory defined by all Priestley spaces.
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**Proposition**

Let $X$ be a $\mathcal{V}$-categorical compact Hausdorff space. Consider a $\mathcal{V}$-subcategory $\mathcal{R} \subseteq \mathcal{V}^X$ that is closed under finite weighted limits and such that $(\psi: X \to \mathcal{V}^{\text{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\text{CatCH} \to \text{CompHaus}$. Then the cone $(\psi \ast: HX \to \mathcal{V}^{\text{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\text{CatCH} \to \text{CompHaus}$.

**Corollary**

The Hausdorff functor restricts to a functor $H: \mathcal{V}$-$\text{Priest} \to \mathcal{V}$-$\text{Priest}$, hence the Hausdorff monad $H$ restricts to $\mathcal{V}$-$\text{Priest}$. 
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Then the cone $\left( \psi^{\text{op}} : HX \to \mathcal{V}^{\text{op}} \right)_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\text{CatCH} \to \text{CompHaus}$. 
**Definition**

We call a $\mathcal{V}$-categorical compact Hausdorff space $X$ **Priestley** if the cone $\mathcal{V}\text{-CatCH}(X, \mathcal{V}^{\text{op}})$ is initial (and mono).

$\mathcal{V}\text{-Priest}$ denotes the full subcategory defined by all Priestley spaces.

**Proposition**

Let $X$ be a $\mathcal{V}$-categorical compact Hausdorff space. Consider a $\mathcal{V}$-subcategory $R \subseteq \mathcal{V}^X$ that is closed under finite weighted limits and such that $(\psi: X \to \mathcal{V}^{\text{op}})_{\psi \in R}$ is initial with respect to $\mathcal{V}\text{-CatCH} \to \text{CompHaus}$.

Then the cone $(\psi^\circ: HX \to \mathcal{V}^{\text{op}})_{\psi \in R}$ is initial with respect to $\mathcal{V}\text{-CatCH} \to \text{CompHaus}$.

**Corollary**

The Hausdorff functor restricts to a functor $H: \mathcal{V}\text{-Priest} \to \mathcal{V}\text{-Priest}$, hence the Hausdorff monad $\mathbb{H}$ restricts to $\mathcal{V}\text{-Priest}$. 
Duality theory
Theorem

$\text{CoAlg}(H) \simeq \text{DLO}^{\text{op}}$ (distributive lattices with operator).\(^{ab}\)


Theorem

\[ \text{Priest} \cong \text{DL}^{\text{op}} \] (induced by 2).\(^a\)


Theorem

\[ \text{CoAlg}(H) \cong \text{DLO}^{\text{op}} \] (distributive lattices with operator).\(^ab\)


### Some Classical Results

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Theorem

\[ C : \text{StablyComp}_V \rightarrow \text{LaxMon}([0, 1]-\text{FinSup})^{\text{op}} \text{ is fully faithful}. \]

\[ a \]

### Theorem

\( C : \text{StablyComp}_V \longrightarrow \text{LaxMon}([0, 1]-\text{FinSup})^\text{op} \) is fully faithful.

### Remark

- \( A \subseteq X \) closed \((1 \hookrightarrow X) \) \( \iff \) \( \Phi : CX \longrightarrow [0, 1] \).
### Theorem

\( C : \text{StablyComp}_V \rightarrow \text{LaxMon}([0, 1]-\text{FinSup})^{\text{op}} \) is fully faithful.

### Remark

- \( A \subseteq X \) closed (1 \( \rightarrow \) \( X \)) \( \iff \) \( \Phi : CX \rightarrow [0, 1] \).
- \( A \) is irreducible \( \iff \) \( \Phi \) is in \( \text{Mon}([0, 1]-\text{FinSup}) \).
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- Every \( X \) in \( \text{StablyComp} \) is sober.
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| Theorem                               |                                                                                              |                                                                                              |
|\( C \) \( : ([0,1]-\text{Priest})_V \rightarrow ([0,1]-\text{FinSup})^{\text{op}} \) is fully faithful. |                                                                                              |                                                                                              |
### From 2 to \([0, 1]\)

**Theorem**

\[ C : \text{StablyComp}_V \longrightarrow \text{LaxMon}([0, 1]-\text{FinSup})^{\text{op}} \text{ is fully faithful.} \]

**Remark**

- \( A \subseteq X \) closed \((1 \rightarrow X) \iff \Phi : CX \rightarrow [0, 1]. \)
- \( A \) is irreducible \(\iff\) \( \Phi \) is in \(\text{Mon}([0, 1]-\text{FinSup})\).
- Every \( X \) in \text{StablyComp} is sober.

**Theorem**

\[ C : ([0, 1]-\text{Priest})_V \longrightarrow ([0, 1]-\text{FinSup})^{\text{op}} \text{ is fully faithful.} \]

**Remark**

- \( \varphi : X \rightarrow [0, 1] \) \((1 \varphi X) \iff \Phi : CX \rightarrow [0, 1]. \)
### From 2 to $[0, 1]$

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<td>• $\varphi : X \rightarrow [0, 1]$ ($1 \varphi \rightarrow X$) $\iff \Phi : CX \rightarrow [0, 1]$.</td>
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<td>• $\varphi : X \rightarrow [0, 1]$ is irreducible(?)) $\iff$ $\Phi$ is ????.</td>
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Proposition

An distributor \( \varphi : X \rightarrow [0, 1] \) is left adjoint \iff the \([0, 1]\)-functor \( \varphi, - \) preserves tensors and finite suprema.

For Łukasiewicz \( \otimes = \odot \) is a Girard quantale: for every \( u \in [0, 1] \), \( u = u \perp \perp \), \( \text{hom}(u, \perp) = 1 - u =: u \perp \).

Furthermore, the diagram

\[
\begin{array}{ccc}
CX & \xrightarrow{\varphi} & \Xi [0, 1] \\
\downarrow & & \downarrow \\
\text{Fun}(X, [0, 1]) & \xrightarrow{\varphi, -} & [0, 1]
\end{array}
\]

commutes in \([0, 1]\)-Cat and \( CX \hookrightarrow \text{Fun}(X, [0, 1]) \) is \( \bigvee \)-dense.

Conclusion: \( \varphi : [0, 1] \dashv \bowtie X \) is left adjoint \iff \( \Phi \) preserves finite weighted limits.
**Proposition**

An distributor $\varphi : X \to [0, 1]$ is left adjoint $\iff$ the $[0, 1]$-functor $[\varphi, -] : \text{Fun}(X, [0, 1]) \to [0, 1]$ preserves tensors and finite suprema.\(^a\)\(^b\)


Proposition

An distributor \( \varphi : X \to [0, 1] \) is left adjoint \( \iff \) the \([0, 1]\)-functor \( \varphi, - : \text{Fun}(X, [0, 1]) \to [0, 1] \) preserves tensors and finite suprema.

For Łukasiewicz \( \otimes = \odot \)

\([0, 1]\) is a Girard quantale: for every \( u \in [0, 1] \), \( u = u \perp \perp \), \( \text{hom}(u, \bot) = 1 - u =: u \perp \).

Furthermore, the diagram

\[
\begin{align*}
& C \xleftarrow{\varphi} \text{Fun}(X, [0, 1])^{\text{op}} \xrightarrow{(-)\perp} \text{Fun}(X, [0, 1])^{\text{op}} \\
& \Phi \xrightarrow{-\cdot \varphi} \xrightarrow{\varphi, -} \text{Fun}(X, [0, 1])^{\text{op}} \\
& [0, 1] \xrightarrow{(-)\perp} [0, 1]^{\text{op}}
\end{align*}
\]

commutes in \([0, 1]-\text{Cat}\).
**Proposition**

An distributor $\varphi : X \to [0, 1]$ is left adjoint $\iff$ the $[0, 1]$-functor $[\varphi, -] : \text{Fun}(X, [0, 1]) \to [0, 1]$ preserves tensors and finite suprema.

**For Łukasiewicz $\otimes = \odot$**

$[0, 1]$ is a Girard quantale: for every $u \in [0, 1]$, $u = u \bot \bot$, $\text{hom}(u, \bot) = 1 - u =: u \bot$.

Furthermore, the diagram

$$
\begin{array}{ccc}
CX & \xrightarrow{(-) \bot} & \text{Fun}(X, [0, 1])^{\text{op}} \\
\Phi \downarrow & & \downarrow [\varphi, -]^{\text{op}} \\
[0, 1] & \xrightarrow{(-) \bot} & [0, 1]^{\text{op}}
\end{array}
$$

commutes in $[0, 1]$-$\text{Cat}$ and $CX \hookrightarrow \text{Fun}(X, [0, 1]^{\text{op}})$ is $\bigvee$-dense.
**Proposition**

An distributor $\varphi : X \to [0, 1]$ is left adjoint $\iff$ the $[0, 1]$-functor $[\varphi, -] : \text{Fun}(X, [0, 1]) \to [0, 1]$ preserves tensors and finite suprema.

**For Łukasiewicz $\otimes = \odot$**

$[0, 1]$ is a Girard quantale: for every $u \in [0, 1]$, $u = u \perp \perp$, $\text{hom}(u, \perp) = 1 - u =: u \perp$. Furthermore, the diagram

$$
\begin{array}{ccc}
CX & \xrightarrow{\Phi} & \text{Fun}(X, [0, 1])^\text{op} \\
\downarrow (- \cdot \varphi) & & \downarrow [\varphi, -]^\text{op} \\
[0, 1] & \xrightarrow{(-) \perp} & [0, 1]^\text{op}
\end{array}
$$

commutes in $[0, 1]$-$\text{Cat}$ and $CX \hookrightarrow \text{Fun}(X, [0, 1]^\text{op})$ is $\vee$-dense.

**Conclusion:** $\varphi : 1 \dashv X$ is left adjoint $\iff$ $\Phi$ preserves finite weighted limits.