Left Cancellative Categories and Ordered Groupoids

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Outline

Representations of Etendues

Sheaves on
Etale Groupoids

Sheaves on
Grothendieck sites
with monic maps

Sheaves on
Ordered Groupoids
with an Ehresmann
Topology
[Lawson-Steinberg,
2004]

Representations of Etendues

Etale Groupoids

Generalized Maps

$$\mathcal{G} \leftarrow \mathcal{G}' \rightarrow \mathcal{H}$$

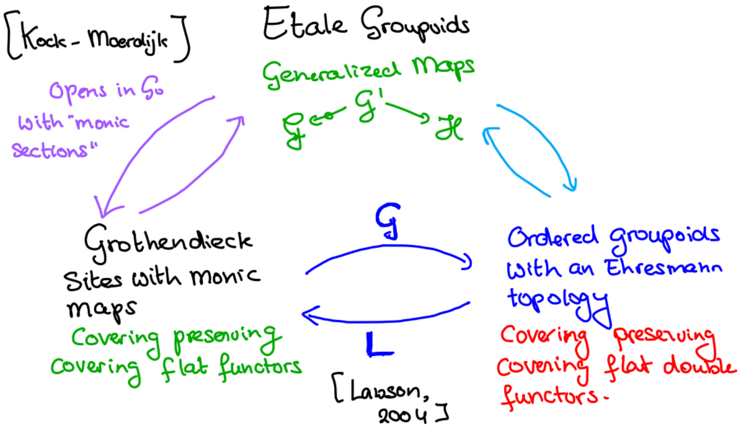
Grothendieck Sites with monic maps

Covering preserving covering flat functors

Ordered groupoids with an Ehresmann topology

Covering preserving covering flat double functors.

Representations of Etendues



Ordered Groupoids - the Internal Approach

- Ordered groupoids are internal groupoids in the category of posets such that the domain arrow is a fibration: they form double categories that are horizontally groupoidal and vertically posetal.
- Double cells are of the form

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\leq"] & B' \arrow[d, "\bullet" marking] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

  where $f' \leq f$.
- Given f and $A' \leq A$, there is exactly one such cell and we write $f' = f|_{A'}$.
- Morphisms are double functors.

Lawson: from **oGpd** to **lcCat**

The functor $\mathbf{L}\colon \mathbf{oGpd} \to \mathbf{lcCat}$ maps an ordered groupoid $\mathcal{G}$ to the category $\mathbf{L}(\mathcal{G})$ defined by:

- **Objects**: those of $\mathcal{G}$;
- **Arrows**: $A \to B$ in $\mathbf{L}(\mathcal{G})$ are formal composites

$$\begin{tikzcd} A \arrow[r, "h"] & B' \arrow[r, "\bullet" marking] & B \end{tikzcd}$$

  of a horizontal and vertical arrow in $\mathcal{G}$.
- **Composition** uses the fibration property of $\mathcal{G}$,

$$\begin{tikzcd} A \arrow[r, "h"] & B' \arrow[r, dashed, "k|_{B'}"] \arrow[d, "\bullet" marking] & C'' \arrow[d, "\bullet" marking] \\ & B \arrow[r, "k"] & C' \arrow[d, "\bullet" marking] \\ & & C \end{tikzcd}$$

Lawson: from **lcCat** to **oGpd**

The functor $\mathbf{G}\colon \mathbf{lcCat} \to \mathbf{oGpd}$ mapping a left cancellative category $\mathcal{C}$ to the ordered groupoid $\mathbf{G}(\mathcal{C})$ is defined by:

- **Objects**: subobjects $[m\colon A \to B]$ in $\mathcal{C}$; i.e., $[m\colon A \to B] = [m'\colon A' \to B]$ if there is an isomorphism $k\colon A \xrightarrow{\sim} A'$ such that $m'k = m$;
- **Horizontal arrows**: isomorphism classes of spans,

$$[m,n]\colon [m] \to [n], \qquad B \xleftarrow{m} A \xrightarrow{n} C$$

and $[m,n] = [m',n']$ if and only if there is an isomorphism h giving a commutative diagram,

$$\begin{tikzcd} & A \arrow[dl, "m"'] \arrow[dr, "n"] \arrow[dd, "h", "\wr"'] & \\ B & & C \\ & A' \arrow[ul, "m'"] \arrow[ur, "n'"'] & \end{tikzcd}$$

Vertical Arrows and Double Cells

- **Vertical Arrows:** there is a unique

$$[m' : A' \to B] \longrightarrow\!\!\bullet\!\!\longrightarrow [m : A \to B]$$

  if there is an arrow $k : A' \to A$ in $\mathcal{C}$ such that $mk = m'$.
- There is a (unique) double cell

$$\begin{tikzcd} {[m']} \arrow[r, "{[m',n']}"] \arrow[d, "\le"'] \arrow[dr, phantom, "\le"] & {[n']} \arrow[d, "\le"] \\ {[m]} \arrow[r, "{[m,n]}"'] & {[n]} \end{tikzcd}$$

  if there is an arrow k giving a commutative diagram,

$$\begin{tikzcd} & A' \arrow[dl, "m'"'] \arrow[dr, "n'"] \arrow[d, "k"] & \\ B & A \arrow[l, "m"] \arrow[r, "n"'] & C \end{tikzcd}$$

The composite $\mathbf{LG}\colon \mathbf{lcCat} \to \mathbf{lcCat}$

- The objects of $\mathbf{LG}(\mathcal{C})$ are subobjects in $\mathcal{C}$: $[m\colon A \to B]$.
- The arrows are constructed as

$$[m, n']\colon [m] \to [n'] \leq [n]$$

and this corresponds to a diagram

$$\begin{tikzcd}
 & A' \arrow[r, "h"] \arrow[dl, "m"'] \arrow[dr, "n'"'] & A \arrow[d, "n"] \\
B & & C
\end{tikzcd}$$

Arrows in $\mathbf{LG}(\mathcal{C})$

An arrow $[h]\colon [m\colon A' \to B] \to [n\colon A \to C]$ is given by an arrow $h\colon A' \to A$. Furthermore,

$$\left([m] \xrightarrow{[h]} [n]\right) \equiv \left([m'] \xrightarrow{[h']} [n']\right)$$

if and only if

$$\begin{tikzcd}
 & {} \arrow[dl, "m"'] \arrow[r, "h"] \arrow[dd, "k", "\wr"'] & {} \arrow[dd, "\ell"', "\wr"] \arrow[dr, "n"] & \\
{} & & & {} \\
 & {} \arrow[ul, "m'"] \arrow[r, "h'"'] & {} \arrow[ur, "n'"'] &
\end{tikzcd}$$

Composition in $\mathbf{LG}(\mathcal{C})$

Composition of $[m] \xrightarrow{[h_1]} [n]$ and $[n'] \xrightarrow{[h_2]} [p]$ is defined when there is an arrow k as in the diagram

$$\begin{tikzcd}
& {} \arrow[dl, "m"'] \arrow[r, "h_1"] & {} \arrow[rr, "k"] \arrow[dr, "n"'] & & {} \arrow[dl, "n'"] \arrow[r, "h_2"] & {} \arrow[dr, "p"] & \\
{} & & & {} & & & {}
\end{tikzcd}$$

and the composition is

$$[h_2 k h_1] \colon [m] \to [p].$$

The Natural Transformation $\eta\colon \mathrm{Id}_{\mathbf{lcCat}} \Rightarrow \mathbf{LG}$

The natural transformation $\eta\colon \mathrm{Id} \Rightarrow \mathbf{LG}$ has components

$$\eta_{\mathcal{C}}\colon \mathcal{C} \to \mathbf{LG}(\mathcal{C})$$

defined by

- on objects,
$$A \mapsto [1_A]$$
- on arrows,
$$(h\colon A \to B) \mapsto [h]\colon [1_A] \to [1_B])$$

Each $\eta_{\mathcal{C}}$ is a (weak) equivalence of categories ($[m\colon A \to B] \cong [1_A]$ and $\eta_{\mathcal{C}}$ is full and faithful) and subject to the axiom of choice there is a pseudo inverse.

The composite $\mathbf{GL}\colon \mathbf{oGpd} \to \mathbf{oGpd}$

For an ordered groupoid $\mathcal{G}$, the double category $\mathbf{GL}(\mathcal{G})$ is given by:

- Objects: (B', B) with $B' \longrightarrow\!\!\bullet\!\!\rightarrow B$ in $\mathcal{G}$
- Horizontal Arrows: $(B', B) \xrightarrow{h} (C', C)$ where $h\colon B' \to C'$ in $\mathcal{G}$.
- Vertical Arrows: $(B', B) \longrightarrow\!\!\bullet\!\!\rightarrow (D', D)$ if and only if $B = D$ and $B' \leq D'$.
- Double Cells:

$$\begin{tikzcd}
(A', A) \arrow[r, "h"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\leq"] & (B', B) \arrow[d, "\bullet" marking] \\
(A'', A) \arrow[r, "k"] & (B'', B)
\end{tikzcd}$$

if and only if $k|_{A'} = h$ in $\mathcal{G}$.

The Natural Transformation $\kappa\colon \mathbf{GL} \to \mathrm{Id}_{\mathbf{oGpd}}$

For each ordered groupoid $\mathcal{G}$, there is a double functor $\kappa_{\mathcal{G}}\colon \mathbf{GL}(\mathcal{G}) \to \mathcal{G}$,

- On objects: $(B', B) \mapsto B'$
- On horizontal arrows: $(B', B) \xrightarrow{h} (C', C) \quad \mapsto \quad B' \xrightarrow{h} C'$.
- On vertical arrows:

$$\begin{tikzcd} (A', A) \arrow[d, "\bullet" marking] \\ (A'', A) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A' \arrow[d, "\bullet" marking] \\ A'' \end{tikzcd}$$

- On double cells:

$$\begin{tikzcd} (A', A) \arrow[r, "h"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\leq"] & (B', B) \arrow[d, "\bullet" marking] \\ (A'', A) \arrow[r, "k"'] & (B'', B) \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} A' \arrow[r, "h"] \arrow[d, "\bullet" marking] \arrow[dr, phantom, "\leq"] & B' \arrow[d, "\bullet" marking] \\ A'' \arrow[r, "k"'] & B'' \end{tikzcd}$$

Properties of κ

- The components $\kappa_{\mathcal{G}}$ are weak equivalences of internal categories.
- If the ordered groupoid $\mathcal{G}$ has maximal objects, $\kappa_{\mathcal{G}}$ has a pseudo inverse.
- Do **L** and **G** define a 2-adjunction/equivalence?
- What are the 2-cells between ordered functors?

Properties of κ

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lcCat and **oGpd** as 2-categories

- The 2-cells in **lcCat** are natural transformations, $\alpha\colon F \Rightarrow F'\colon \mathcal{C} \rightrightarrows \mathcal{D}$.
- For ordered groupoids $\mathcal{G}$ and $\mathcal{H}$, the double category **DblFun**$(\mathcal{G},\mathcal{H})$ of double functors $\mathcal{G} \to \mathcal{H}$, horizontal transformations, vertical transformations and modifications, is again an ordered groupoid.
- Apply the functor **L** to obtain a notion of transformation between double functors that is a formal composition of a horizontal and a vertical transformation.

2-Adjunction

Theorem

- *There is a 2-adjunction,*

$$\begin{tikzcd} \textbf{lcCat} \arrow[r, shift left, "\mathbf{G}"] \arrow[r, phantom, "\perp"] & \textbf{oGpd} \arrow[l, shift left, "\mathbf{L}"] \end{tikzcd}$$

  The unit and counit of this adjunction have components that are essential equivalences.

- *This 2-adjunction restricts to a 2-equivalence*

$$\textbf{lcCat} \simeq_2 \textbf{oGpd}_{max}$$

  where $\textbf{oGpd}_{max}$ is the full subcategory on ordered groupoids with maximal objects.

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Presheaves on Ordered Groupoids

- Let
$$\mathbb{Q}\mathbf{Set}$$
be the quartet double category on the category of sets (double cells are commutative squares).
- A presheaf F on an ordered groupoid $\mathcal{G}$ is a functor
$$F\colon \mathcal{G}^{\mathrm{op,\,op}} \to \mathbb{Q}\mathbf{Set}$$
that is contravariant in both the horizontal and vertical directions.
- Write $\mathbf{PreSh}\,(\mathcal{G})$ for the category of presheaves on $\mathcal{G}$ and (horizontal/vertical) transformations.
- **Note**: Horizontal and vertical transformations are the same in this case and could be viewed as transformations that are natural in the horizontal and the vertical direction.

Results

- [Lawson-Steinberg, 2004] There is an isomorphism of categories,

$$\mathbf{PreSh}(\mathcal{G}) \cong \mathbf{PreSh}(\mathbf{L}(\mathcal{G})).$$

- Note further:

$$\mathbf{PreSh}(\mathcal{C}) \simeq \mathbf{PreSh}(\mathbf{LG}(\mathcal{C})),$$

  so

$$\mathbf{PreSh}(\mathbf{G}(\mathcal{C})) \simeq \mathbf{PreSh}(\mathcal{C}).$$

- Also, an internal weak equivalence of ordered groupoids $\mathcal{G} \to \mathcal{G}'$ induces an equivalence or presheaf categories,

$$\mathbf{PreSh}(\mathcal{G}) \simeq \mathbf{PreSh}(\mathcal{G}').$$

Vertical Sieves for an Ordered Groupoid

- Coverings for an Ehresmann topology on an ordered groupoid are sieves of vertical arrows; i.e., **vertical sieves**.
- We introduce the following notation: for a vertical sieve $\mathcal{B}$ on B and a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B' \arrow[d, "\bullet" marking] \\ & B, \end{tikzcd}$$

we define

$$f^*\mathcal{B} = \left\{ A' \overset{\bullet}{\longrightarrow} A \,\middle|\, f|_{A'} : A' \to B'' \text{ with } (B'' \overset{\bullet}{\longrightarrow} B) \in \mathcal{B} \right\}$$

An Ehresmann Topology on an Ordered Groupoid

An **Ehresmann topology** on an ordered groupoid $\mathcal{G}$ is given by an assignment of a collection $T(A)$ of vertical sieves to each object A, such that:

- ET1 The trivial sieve $(\downarrow A) \in T(A)$.
- ET2 If $\mathcal{B} \in T(B)$ and $f\colon A \to B'$ with $B' \longrightarrow\!\!\!\!\!\!\!\bullet\;\; B$, then $f^*\mathcal{B} \in T(A)$.
- ET3 Let $\mathcal{A} \in T(A)$ and let $\mathcal{B}$ be any vertical sieve on A. If for each

$$\begin{tikzcd} C \arrow[r, "f"] & A' \arrow[d, "\bullet"] \\ & A \end{tikzcd}$$

with $(A' \longrightarrow\!\!\!\!\!\!\!\bullet\;\; A) \in \mathcal{A}$, $f^*\mathcal{B} \in T(C)$, then $\mathcal{B} \in T(A)$.

Results [Lawson-Steinberg,2004]

- Sieves on a left cancellative category $\mathcal{C}$ are in one-to-one correspondence to vertical sieves on $\mathbf{G}(\mathcal{G})$.
- Vertical sieves on $\mathcal{G}$ are in one-to-one correspondence to sieves on $\mathbf{C}(\mathcal{G})$.
- Grothendieck topologies on a left cancellative category $\mathcal{C}$ are in one-to-one correspondence to Ehresmann topologies on $\mathbf{G}(\mathcal{G})$.
- Ehresmann topologies on $\mathcal{G}$ are in one-to-one correspondence to sieves on $\mathbf{C}(\mathcal{G})$.
- There is an isomorphism of categories,

$$\mathbf{Sh}(\mathcal{G}, T) \cong \mathbf{Sh}(\mathbf{L}(\mathcal{G}), J_T).$$

Functors Between Categories of Sites

- $\mathbf{G}(\mathcal{C}, J) = (\mathbf{G}(\mathcal{C}), T_J)$ where

$$T_J([m\colon A \to B]) = \{[mS]; S \in J(A)\}$$

and

$$[mS] = \{\, [mn] \longrightarrow\!\!\!\!\!\!\bullet\;\;\, [m] \mid n \in S\}.$$

- $\mathbf{L}(\mathcal{G}, T) = (\mathbf{L}(\mathcal{G}), J_T)$ where

$$\{\, B_i \xrightarrow{m_i} A'_i \longrightarrow\!\!\!\!\!\!\bullet\;\;\, A \mid i \in I\} \in J_T(A)$$

if and only if

$$\{\, A'_i \longrightarrow\!\!\!\!\!\!\bullet\;\;\, A \mid i \in I\} \in T(A).$$

Morphisms of Ehresmann Sites

A **morphism of Ehresmann sites** $(\mathcal{G}, T) \to (\mathcal{G}', T')$ is a double functor $\mathcal{G} \to \mathcal{G}'$ which satisfies the following two conditions:

- **Cover preserving:** If $\mathcal{A} \in T(A)$ then $F\mathcal{A} \in T'(FA)$.
- **Covering-flat:** For each collection of objects $(X_i)_{i\in I}$ in $\mathcal{G}$ and each cone $\begin{tikzcd} C \arrow[r, "f_i"] & Y_i \arrow[d, "\bullet" marking] \\ & FX_i \end{tikzcd}$ for $i \in I$, there is a covering sieve

$$\left\{ C_j' \xrightarrow{\bullet} C \,|\, j \in J \right\} \in T(C)$$

and a cone $\begin{tikzcd} E_j \arrow[r, "h_{ij}"] & X_{ij}' \arrow[d, "\bullet" marking] \\ & X_i \end{tikzcd}$ for each $j \in J$, such that $C_j' \xrightarrow{\bullet} FE_j$ and

$f_i|_{C_j'} = Fh_{ij}|_{C_j'}$.

Results

Proposition

Both functors **L** *and* **G** *preserve and reflect covering preservation and covering flatness.*

The Comparison Lemma for Grothendieck Topologies

A functor $F\colon \mathcal{C} \to \mathcal{C}'$ induces an equivalence of topoi, $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{C}', J')$ if it satisfies the following five conditions:

1. **Cover preserving** F sends J-covers to J'-covers.
2. **Locally full** If $\varphi\colon FA \to FB$ in $\mathcal{C}'$, then there is a cover $\{\alpha_i\colon A_i \to A | i \in I\} \in J(A)$ with arrows $f_i\colon A_i \to B$ such that $\varphi \circ F(\alpha_i) = Ff_i$ for all $i \in I$.
3. **Locally faithful** If $F(f) = F(g)$ for $f, g\colon A \rightrightarrows B$ in $\mathcal{C}$, then there is a cover $\{\alpha_i\colon A_i \to A | i \in I\} \in J(A)$ such that $f\alpha_i = g\alpha_i$ for all $i \in I$.
4. **Locally surjective on objects** For each object C' in $\mathcal{C}'$ there is a covering $\{\gamma_i\colon F(C_i) \to C'\} \in J'(C')$.
5. **Co-continuous** If $\{\gamma_i\colon C_i' \to F(C)\} \in J'(F(C))$ then the set $\{f\colon D \to C | F(f)$ factors through γ_i for some $i \in I\}$ is in $J(C)$.

The Comparison Lemma for Ehresmann Topologies

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- **Cover preserving** F sends T-covers to T'-covers.
- **Locally full** For any diagram $\begin{array}{ccc} FA & \xrightarrow{f} & B' \\ & & \downarrow \\ & & FB \end{array}$ in $\mathcal{G}'$ there is a cover

  $\left\{ A_i \longrightarrow A \mid i \in I \right\} \in T(A)$ and arrows $\begin{array}{ccc} A_i & \xrightarrow{\varphi_i} & B_i' \\ & & \downarrow \\ & & B \end{array}$ in $\mathcal{G}$ such that

  $f|_{FA_i} = F\varphi_i$.

The Comparison Lemma for Ehresmann Topologies, continued

- **Locally faithful** For any two horizontal arrows as in
$$\begin{tikzcd} A \arrow[r, "f_i"] & B_i \arrow[d, "\bullet" marking] \\ & B \end{tikzcd}$$
with $Ff_1 = Ff_2$ there is a vertical cover $\left\{A_j \longrightarrow\!\!\bullet\!\!\rightarrow A | j \in J\right\} \in T(A)$ such that $f_1|_{A_j} = f_2|_{A_j}$ for all $j \in J$.
- **Locally surjective on objects** For each object A in $\mathcal{G}'$ there is a cover $\left\{A_i \longrightarrow\!\!\bullet\!\!\rightarrow A | i \in I\right\} \in T(A)$ such that for each A_i there is a horizontal arrow of the form $FB_i \longrightarrow A_i$.
- **Co-continuous** For any cover $\left\{B_i \longrightarrow\!\!\bullet\!\!\rightarrow FA | i \in I\right\} \in T(A)$ in $\mathcal{G}'$ the collection of arrows $\left\{ A' \longrightarrow\!\!\bullet\!\!\rightarrow A \,|\, FA' \longrightarrow\!\!\bullet\!\!\rightarrow A_i \text{ for some } i \in I\right\}$ is in $T(A)$.

Results

Proposition

- *$F\colon (\mathcal{C}, J) \to (\mathcal{C}', J')$ satisfies the conditions of the comparison lemma for Grothendieck topologies if and only if $\mathbf{G}(F)\colon (\mathbf{G}(\mathcal{C}), T_J) \to (\mathbf{G}(\mathcal{C}'), T_{J'})$ satisfies the conditions of the comparison lemma for Ehresmann topologies.*
- *$\varphi\colon (\mathcal{G}, T) \to (\mathcal{G}', T')$ satisfies the conditions of the comparison lemma for Ehresmann topologies if and only if $\mathbf{L}(\varphi)\colon (\mathbf{L}(\mathcal{G}), J_T) \to (\mathbf{L}(\mathcal{G}'), J_{T'})$ satisfies the conditions of the comparison lemma for Grothendieck topologies.*
- *The components of η and κ satisfy the conditions of the respective comparison lemmas.*

Concluding Remarks

- The 2-adjunctions and equivalences between left cancellative categories and ordered groupoids are 2-adjunctions and equivalences over the 2-category of topoi.
- It would be nice to have similar results between the 2-category of étale groupoids and the 2-category of Ehresmann sites.
- So far we have nice assignments on objects for the latter.

Weak Equivalences of Double Categories, Part 1

A double functor $F\colon \mathbb{C} \to \mathbb{D}$ is a **weak equivalence** if it satisfies the following two conditions:

- It is **essentially surjective on objects**: the composition $d_1\pi_2$ in

$$\begin{tikzcd} \mathbb{C}_0 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathrm{Iso}(\mathbb{D}_1) \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathbb{D}_0 \\ \mathbb{C}_0 \arrow[r, "F_0"'] & \mathbb{D}_0 & \end{tikzcd}$$

  is surjective on objects, arrows, and composable pairs of arrows.

Weak Equivalences of Double Categories, Part 2

- It is **full and faithful**: the square

$$\begin{tikzcd} \mathbb{C}_1 \arrow[r, "F_1"] \arrow[d, "(d_0{,}d_1)"'] & \mathbb{D}_1 \arrow[d, "(d_0{,}d_1)"] \\ \mathbb{C}_0 \times \mathbb{C}_0 \arrow[r, "F_0\times F_0"'] & \mathbb{D}_0 \times \mathbb{D}_0 \end{tikzcd}$$

  is a pullback of categories.

$\mathcal{G}$ and $\mathbf{GL}(\mathcal{G})$ are weakly equivalent

- For $F = \kappa_{\mathcal{G}}$,
$$d_1\pi_2 \colon \mathbf{GC}(\mathcal{G})_0 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$$
is surjective on objects, since $\kappa_{\mathcal{G}}(B \leq B) = B$;
- $d_1\pi_2 \colon \mathbf{GL}(\mathcal{G})_0 \times_{\mathcal{G}_0} \mathcal{G}_1 \to \mathcal{G}_0$ is surjective on arrows, since $\kappa_{\mathcal{G}}\left((B' \leq B') \xrightarrow{\leq} (D' \leq D')\right) = \left(B' \xrightarrow{\leq} D'\right)$
- It is also easy to see that it is surjective on composable pairs of inequalities, $A' \leq B' \leq C'$.
- So $\kappa_{\mathcal{G}}$ is essentially surjective on objects.
- $\kappa_{\mathcal{G}}$ is both order preserving and order reflecting; hence, fully faithful.

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Equivalence of **oGpd** and **lcCat**?

Since the arrows $\kappa_{\mathcal{G}} : \mathbf{GL}(\mathcal{G}) \to \mathcal{G}$ and $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbf{LG}(\mathcal{C})$ are only weak equivalences, without inverse arrows, we need to compare these categories as **2-categories**.