Duality, definability and conceptual completeness for $\kappa$-pretoposes

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The completeness theorem

Definition

A \( \kappa \)-topos is a topos of sheaves on a site with \( \kappa \)-small limits in which the covers of the topology satisfy in addition the transfinite transitivity property (a transfinite version of the transitivity property), i.e., transfinite composites of covering families form a covering family.
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A $\kappa$-topos is $\kappa$-separable if the underlying category of the site has at most $\kappa$ many objects and morphisms, and where the Grothendieck topology is generated by at most $\kappa$ many covering families.
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A \( \kappa \)-point of a \( \kappa \)-topos is a point whose inverse image preserves all \( \kappa \)-small limits.
The completeness theorem

**Theorem**

(E.) Let \( \kappa \) be a regular cardinal such that \( \kappa^{<\kappa} = \kappa \). Then a \( \kappa \)-separable \( \kappa \)-topos has enough \( \kappa \)-points.

This is an infinitary version of Deligne completeness theorem. When \( \kappa \) is strongly compact (e.g., \( \kappa = \omega \)), we recover the usual version: a \( \kappa \)-coherent topos has enough \( \kappa \)-points.
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\[
\phi_f \vdash y_f \quad \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \exists x_g \phi_g \quad \beta < \kappa, f \in \gamma^\beta
\]

\[
\phi_f \not\vdash y_f \quad \bigwedge_{\alpha < \beta} \phi_f|_\alpha \quad \beta < \kappa, \text{ limit } \beta, f \in \gamma^\beta
\]

\[
\phi_\emptyset \vdash y_\emptyset \quad \bigvee_{f \in B} \exists \beta < \delta_f x_f|_{\beta+1} \bigwedge_{\beta < \delta_f} \phi_f|_{\beta+1}
\]
$\kappa$-geometric logic

Extension of geometric logic in which we have:

- Arities of cardinality less than $\kappa$.
- Conjunction of less than $\kappa$ many formulas.
- Existential quantification of less than $\kappa$ many variables.

More logical axioms or rules are needed.

Example: the theory of well-orderings:

\[ \top \vdash x < y \lor y < x \lor x = y \]

\[ \exists x_0 x_1 x_2 \ldots \bigwedge_{n \in \omega} x_{n+1} < x_n \vdash \bot \]

López-Escobar: the theory of well-orderings is not axiomatizable in $L^{\kappa,\omega}$ for any $\kappa$. 

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Every $\kappa$-geometric theory has a $\kappa$-classifying topos:

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\begin{array}{ccc}
\mathcal{C}_T & \xrightarrow{y} & \text{Sh}(\mathcal{C}_T, \tau) \\
\downarrow M & & \downarrow f^* \\
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Let $\kappa$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ (resp. $\kappa$ is weakly compact).
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Let $\kappa$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ (resp. $\kappa$ is weakly compact). Let $T$ be a theory in a $\kappa$-fragment of $\mathcal{L}_{\kappa^+,\kappa}$ (resp. in $\mathcal{L}_{\kappa,\kappa}$) with at most $\kappa$ many axioms.
Every $\kappa$-geometric theory has a $\kappa$-classifying topos:

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Let $\kappa$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ (resp. $\kappa$ is weakly compact). Let $\mathcal{T}$ be a theory in a $\kappa$-fragment of $\mathcal{L}_{\kappa^+ \kappa}$ (resp. in $\mathcal{L}_{\kappa \kappa}$) with at most $\kappa$ many axioms. Let $\lambda > \kappa$ be regular and satisfy $\lambda^{<\lambda} = \lambda$. Let $\text{Mod}_\lambda(\mathcal{T})$ be the full subcategory of $\lambda$-presentable models.
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The \( \lambda \)-classifying topos of a \( \kappa \)-theory

Let \( \mathbb{T}' \) be the theory in \( \mathcal{L}_{\lambda^+, \lambda} \) with the same axioms as those of \( \mathbb{T} \).
Let $\mathbb{T}'$ be the theory in $\mathcal{L}_{\lambda^+\lambda}$ with the same axioms as those of $\mathbb{T}$.

**Theorem**

(E.) The $\lambda$-classifying topos of $\mathbb{T}'$ is equivalent to the presheaf topos $\text{Set}^{\text{Mod}_{\lambda}(\mathbb{T})}$. Moreover, the canonical embedding of the syntactic category $\mathcal{C}_{\mathbb{T}'} \hookrightarrow \text{Set}^{\text{Mod}_{\lambda}(\mathbb{T})}$ is given by the evaluation functor, which on objects acts by sending $(x, \phi)$ to the functor $\{M \mapsto [[\phi]]^M\}$. 
The \( \lambda \)-classifying topos of a \( \kappa \)-theory

The first consequence is a positive result regarding definability theorems for infinitary logic. If \( C_T \) is the syntactic category of \( T \) considered in \( \mathcal{L}_{\lambda^+,\lambda} \), we have that

\[
ev : C_T \longrightarrow \text{Set}^{\text{Mod}_\lambda(T)}
\]

can be identified with Yoneda embedding

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Theorem

(Infinitary Beth) Let $\phi(R)$ be a formula in $\mathcal{L}_{\kappa^+,\kappa}$ over the language $\mathcal{L} \cup R$ containing the predicate $R$. If every $\mathcal{L}$-structure has a unique expansion to a model of $\phi(R)$ and the interpretation of $R$ in each such model is preserved by $\mathcal{L}$-homomorphisms, then there is an $\mathcal{L}$-formula $\psi$ of $\mathcal{L}_{\lambda^+,\lambda}$ such that $R \vDash_x \psi$. 
Another consequence is the conceptual completeness theorem for $\mathcal{L}_{\kappa^+, \kappa}$:

**Theorem**

*(Infinitary conceptual completeness)* If a $\lambda^+$-coherent functor $I : \mathcal{P} \to \mathcal{S}$, where $\mathcal{P}$ is a $\lambda^+$-pretopos, induces an equivalence between their categories of models $I^* : \text{Mod}(\mathcal{S}) \to \text{Mod}(\mathcal{P})$, then $I$ is itself an equivalence.
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**Theorem**

*(Infinitary Joyal)* If $\mathcal{T}$ is intuitionistic first-order, the functor:

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This version of Joyal’s theorem provides a proof of completeness with respect to Kripke models for theories in $\mathcal{L}_{\kappa^+,\kappa,\kappa}$. 
Consider a \( \lambda \)-accessible category \( K \) and the subcategory \( \text{Pres}_\lambda(K) \) of its \( \lambda \)-presentable objects. Then the category of \( \lambda \)-points of the presheaf topos \( \text{Set}^{\text{Pres}_\lambda(K)} \) is equivalent to \( K \) itself.
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**Definition**

A functor $F : C \to D$ between $\lambda$-accessible categories is $\lambda$-coherent if the induced functor $F^* : FC_\lambda(D, \text{Set}) \to FC_\lambda(C, \text{Set})$ preserves $\lambda$-coherent objects.
Duality and descent

Theorem

(Infinitary Stone duality) Let $\lambda > \kappa$ be weakly compact. There is a (bi-)equivalence (given by homming into $\text{Set}$) between the following categories:

1. $A$: $\lambda$-pretopos completion of (syntactic categories of) theories in $\mathcal{L}_{\lambda,\lambda}$ with less than $\lambda$ axioms; $\lambda$-pretopos morphisms; natural transformations.

2. $B$: $\mu$-accessible categories for $\mu < \lambda$; $\lambda$-accessible, $\lambda$-coherent functors preserving $\lambda$-presentable objects; natural transformations.

Corollary

The category of $A$-morphisms between two objects $T$ and $S$, in $A$, is equivalent to the category of $B$-morphisms between $\text{Mod}(S)$ and $\text{Mod}(T)$.
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It turns out that the previous duality theorem is flexible enough to cast Zawadowski’s argument for the descent theorem, which simplifies his proof. We get:

**Theorem (Infinitary Zawadowski)** If $\kappa$ is strongly compact, conservative $\kappa$-pretopos morphisms between $\kappa$-pretoposes are of effective descent.
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Categoricity and the $\lambda$-classifying topos

The completeness theorem allows to generalize a result of Barr and Makkai on the classifying topos of categorical theories:

Theorem

Let $\kappa$ be a regular cardinal such that $\kappa < \kappa = \kappa$. Let $T$ be a theory in a $\kappa$-fragment of $L_{\kappa + \kappa}$. Then for any $\lambda \geq \kappa$ such that $\lambda < \lambda = \lambda$, $T$ is $\lambda$-categorical if and only if the $\lambda$-classifying topos of the theory:

$T_{\lambda} := T \cup \{"there are \lambda distinct elements"\}$

is two-valued and Boolean (alternatively, atomic and connected).

Corollary

A $\kappa$-separable $\kappa$-topos has a unique point of cardinality at most $\kappa$ (up to isomorphism) if and only if it is two-valued and Boolean (alternatively, atomic and connected).
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Shelah’s eventual categoricity conjecture

Shelah’s conjecture is an infinitary version of the behaviour of models of uncountable categorical theories:

**Theorem**

*(Morley)* If a countable theory $\mathbb{T}$ is categorical in an uncountable cardinal $\lambda$, then it is categorical in every uncountable cardinal $\lambda$.

Shelah extended this theorem to the case of uncountable theories and conjectured that, more generally, an eventual version holds for models of theories in $\mathcal{L}_{\omega_1,\omega}$ and even more general classes of models known as abstract elementary classes:

**Conjecture**

*(Shelah)* If a theory in $\mathcal{L}_{\omega_1,\omega}$ is categorical in a sufficiently high cardinal $\lambda \geq \kappa$, then it is categorical in all $\lambda \geq \kappa$. 
Shelah’s eventual categoricity conjecture

Let $\textbf{Set}[\mathbb{T}]_\lambda$ be the $\lambda$-classifying topos of $\mathbb{T}$. Suppose $\mathbb{T}$ is $\lambda$-categorical and let $M_0$ be its unique model of cardinality $\lambda$. 
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$\text{Set}[\mathbb{T}_\lambda]_{\lambda^+} \cong \text{Set}^{M_0^{\text{op}}}$

$\text{Set}[\mathbb{T}^1]_{\lambda^+} \cong \text{Set}^{\mathbb{M}_0^{\text{op}}}$

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$\text{Set}[\mathbb{T}^2]_{\lambda^+} \cong \text{Sh}(\mathbb{M}_1^{\text{op}}, \tau_D^1)$

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Thank you!