What is a monoid?

*How I learnt to stop worrying and love skewness*

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The notion of **monoid** can be defined in each of these settings.
Outline

1. Monoidal categories and multicategories

2. The world of skew
A **monoid** consists of

- an object $a$, the **carrier**
- a map $e : 1 \rightarrow a$, the **unit**
- a map $m : a \otimes a \rightarrow a$, the **multiplication**

Three diagrams must commute:

- Associativity
- Left unitality
- Right unitality.
Examples

- Monoid = monoid in $\text{Set}$.
- Ring = monoid in $\text{Ab}$.
- Algebra = monoid in $\text{Vect}_\mathbb{R}$.
- Quantale = monoid in $\text{CompSupLatt}$.
- Regular* cardinal = monoid in $\text{Card}$.
- Monad on $\mathcal{C}$ = monoid in $[\mathcal{C}, \mathcal{C}]$. 
Examples

- Monoid = monoid in Set.
- Ring = monoid in Ab.
- Algebra = monoid in Vectₓ.
- Quantale = monoid in CompSupLatt.
- Regular* cardinal = monoid in Card.
- Monad on C = monoid in [C, C].
- Monad on an object c of a bicategory.
In a multicategory, a morphism ("multi-map") goes from a list of objects to an object.

$$f: \vec{a} \rightarrow b$$

**Example**

Vector spaces and multilinear maps.

We have an identity maps \(\text{id}_a: a \rightarrow a\)

and can compose \(f: \vec{a} \rightarrow b_i\) with \(g: \vec{b} \rightarrow c\).

Four equations must be satisfied.
A virtual bicategory has

- objects
- morphisms—not composable
- 2-cells

\[ \begin{array}{cccccc}
& & a_1 & & & \\
& f_0 & \rightarrow & & a_n-1 & \rightarrow \\
& a_0 & \rightarrow & \cdots & \downarrow \alpha & \rightarrow \\
& & \rightarrow & a_n & \rightarrow & f_{n-1}
\end{array} \]
A virtual bicategory has

- objects
- morphisms—not composable
- 2-cells

Also: virtual double categories.
A monoid consists of an object $a$ and multi-maps $e: \rightarrow a$ and $m: a, a \rightarrow a$ satisfying associativity, left and right unitality.

A monad on $a$ consists of a 1-cell $g: a \rightarrow a$ and 2-cells $g \downarrow e$ and $m \downarrow g$ satisfying associativity, left and right unitality.
Often said

Small category $=$ monad in the bicategory $\text{Span}$.
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Small category = monad in the bicategory \textbf{Span}.

A category \( \mathcal{C} \) is \textcolor{red}{light} (or “moderate and locally small”)

when \( |\mathcal{C}| \) is a class, and each \( \mathcal{C}(a,b) \) is a set.

Light category = monad in ?
**Example: light categories**

**Often said**

Small category $= \text{monad in the bicategory Span.}$

A category $\mathcal{C}$ is **light** (or “moderate and locally small”)

when $|\mathcal{C}|$ is a class, and each $\mathcal{C}(a, b)$ is a set.

Light category $= \text{monad in ?}$

**Answer**

The virtual bicategory of classes and set-valued relations.

A **set-valued relation** $A \rightarrow B$ is a family of sets $(\mathcal{C}(a, b))_{a \in A, b \in B}$.

Composites don’t exist; they would be class-valued.
Let **Bimod** be the virtual bicategory of light categories and bimodules.

A (light) bimodule $\mathcal{C} \rightarrow \mathcal{D}$ is a functor $\mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \text{Set}$.

Composites of bimodules don’t exist: they would be functors to **Class**.
Example: bimodules

Let \textbf{Bimod} be the virtual bicategory of light categories and bimodules. A (light) bimodule $C \rightarrow D$ is a functor $C^{\text{op}} \times D \rightarrow \text{Set}$.

Composites of bimodules don’t exist: they would be functors to \textbf{Class}.

A monad in \textbf{Bimod} on $C$ is a (Heunen-Jacobs) \textbf{arrow} on $C$

i.e. an identity-on-objects functor $C \rightarrow D$. 
Let \textbf{Bimod} be the virtual bicategory of light categories and bimodules. A (light) bimodule \( C \to D \) is a functor \( C^{\text{op}} \times D \to \text{Set} \).

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We can adapt this example to include strength. (Freyd category)
In some multicategories, tensors don’t exist.

In others they exist but are complicated,

Compare:

- A quantale is a monoid in the monoidal category $\text{CompSupLatt}$.
- A quantale is a monoid in the multicategory $\text{CompSupLatt}$.

The latter is easy to unpack.
A left skew monoidal category consists of

- a category $\mathcal{C}$
- an object $1$
- a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- an associator $(a \otimes b) \otimes c \rightarrow a \otimes (b \otimes c)$
- a left unitor $1 \otimes c \rightarrow c$
- a right unitor $a \rightarrow a \otimes 1$

satisfying five coherence laws.
In a skew monoidal category, we can define monoids just as in a monoidal category.

Example: relative monads
Under certain size conditions:
relative monads are monoids in a skew monoidal category.
(Altenkirch, Chapman, Uustalu)
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- monoid in a right skew monoidal category
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Bourke and Lack introduced skew multicategories.
In a left skew multicategory $\mathcal{C}$, a morphism goes from $s[\overrightarrow{a}]$ where the house $\overrightarrow{a}$ is a list of objects.

and the left stoup $s$ is either nothing or an object.
Bourke, Lack; Veltri, Uustalu, Zeilberger

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A morphism $f$ from $c[\overrightarrow{a}]$ can be left-housed giving $f[\overrightarrow{a}]$ from $[c, \overrightarrow{a}]$. 
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A morphism $f$ from $c[\overrightarrow{d}]$ can be left-housed giving $f[\overrightarrow{d}]$ from $[c, \overrightarrow{d}]$.

When left-housing is invertible, $C$ is “just” a multicategory.
In a bi-skew multicategory, a morphism goes from $s[\overrightarrow{a}]t$.

The house $\overrightarrow{a}$ is a list of objects.

The left stoup $s$ is either nothing or an object.

The right stoup $t$ is either nothing or an object.
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We have left and right housing.

They commute for a morphism from $c[\vec{d}]d$. 
Bi-skew multicategories

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If right housing is an isomorphism, then it’s “just” left skew.
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3 kinds of composition, 3 kinds of identity.
A monoid consists of

an object $a$ and multi-maps $e : [] \to a$ and $m : a[]a \to a$

Associativity $a[a]a \to a$

Left unitality $a[] \to a$.

Right unitality $[]a \to a$. 
A (light) category on a span of classes consists of the following:

- For each \( x \in X, y \in Y \), a set \( C(x, y) \) of morphisms \( x \to y \).
- For each \( i \in I \), an identity \( \text{id}_i : \alpha(i) \to \beta(i) \).
- For each \( f : x \to \beta(i) \) and \( g : \alpha(i) \to y \), a composite \( f; g : x \to y \).

Equations:

- Left identity for \( g : \alpha(i) \to y \).
- Right identity for \( f : x \to \beta(i) \).
- Associativity for \( f : x \to \beta(i) \) and \( g : \alpha(i) \to \beta(j) \) and \( h : \alpha(j) \to y \).

Category on the span = monoid in ?
An object is a family of sets \((A(x, y))_{x \in X, y \in Y}\)

A map \(A[B]C \rightarrow D\) is a family of functions

\[ A(x, \beta(i)) \times B(\alpha(i), \beta(j)) \times C(\beta(j), y) \rightarrow D(x, y) \]

A map \(A[B] \rightarrow D\) is a family of functions

\[ A(x, \beta(i)) \times B(\alpha(i), \beta(j)) \rightarrow D(x, \beta(j)) \]
Let \( \mathcal{O} \) be a bimodule \( \mathcal{A} \rightarrow \mathcal{B} \).

For example \textbf{FinSet} \rightarrow \textbf{Set} giving function sets.

A relative monad on \( \mathcal{O} \) provides

- for each \( a \in \mathcal{A} \), an object \( Ta \in \mathcal{B} \) and unit \( \eta_a : a \rightarrow Ta \)
- for each \( \mathcal{O} \)-map \( f : a \rightarrow Tb \), a \( \mathcal{D} \)-map \( f^* : Ta \rightarrow Tb \)

subject to the three “Kleisli triple” laws.

Relative monad \( = \) monoid in ?
Answer: a left skew multicategory

An object is a function $|A| \rightarrow |B|$.

A map $S[T_0, T_1] \rightarrow U$ is a family of maps

$$O(a_0, T_0 a_1) \times O(a_1, T_1 a_2) \rightarrow D(Sa_0, Ua_2)$$

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$$O(a_0, T_0 a_1) \times O(a_1, T_1 a_2) \rightarrow O(a_0, Ua_2)$$
Two kinds of terms: values (e.g. variables) and computations.
Two kinds of type: value type $A$ and computation type $B$.
$FA$ is the type of computations that aim to return a value of type $A$.

$$
\Gamma \vdash^v V : A \quad \Gamma \vdash^c M : FA \quad \Gamma, x : A \vdash^c N : B
$$

Three laws

\begin{align*}
(M \text{ to } x. N) \text{ to } y. P & = M \text{ to } x. (N \text{ to } y. P) \\
(\text{return } V) \text{ to } x. M & = M[V/x] \\
M & = M \text{ to } x. \text{return } x
\end{align*}
In any bi-skew multicategory $C$, we have a notion of a monoid.
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By choosing an appropriate bi-skew multicategory, the following notions are monoid notions:

- category on a given span of classes
- model of the $F$ fragment of call-by-push-value
- relative monad on a bimodule
- guardedness predicate.
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THANKS FOR LISTENING!