Fixpoint toposes

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Motivation

(Freyd) The category whose objects are sets X endowed with an isomorphism $X \cong X \times X$ is a topos.

(Kennison) The category whose objects are sets X endowed with an isomorphism $X \cong X + X$ is a topos.

(Leinster) Let $M : \mathcal{C} \nrightarrow \mathcal{C}$ be a profunctor. The category whose objects are presheaves $X \in [\mathcal{C}^{op}, \mathrm{Set}]$ endowed with an isomorphism $X \cong \{M, X\}$ is a topos.

The question

Let $F : \mathcal{E} \longrightarrow \mathcal{E}$ be an endofunctor.

- The category **F-alg** has as objects, pairs $(X \in \mathcal{E},\ x : FX \to X)$
- The category **F-coalg** has as objects, pairs $(X \in \mathcal{E},\ x : X \to FX)$
- The category **Fix(F)** is the full subcaty of F-coalg on those (X, x) with x invertible

Suppose $\mathcal{E}$ is a topos.
When is Fix(F) also a topos?

The answer

Theorem (Paré, Rosebrugh, Wood 1989)
If $F : \mathcal{E} \to \mathcal{E}$ is left exact and idempotent, then $\mathcal{E}$ a topos $\Rightarrow$ $\mathrm{Fix}(F)$ a topos.

Our goal today:

Theorem
If $F : \mathcal{E} \to \mathcal{E}$ preserves pullbacks and generates a cofree comonad, then $\mathcal{E}$ a topos $\Rightarrow$ $\mathrm{Fix}(F)$ a topos.

When is F-coalg a topos?

We say $F : \mathcal{E} \to \mathcal{E}$ generates a cofree comonad when $\exists R$ in:

$$\begin{tikzcd} F\text{-coalg} \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "R"] \end{tikzcd}$$

Examples: (1) F idempotent, $\mathcal{E}$ has products

(2) F accessible, $\mathcal{E}$ locally presentable

(3) F polynomial, $\mathcal{E}$ a slice topos with NNO

The cofree comonad in question is $Q_F := UR$.

Cofreeness says that $F\text{-coalg} \cong Q_F\text{-Coalg}$

Eilenberg-Moore coalgebras

When is F-coalg a topos?

Theorem (Johnstone, Power, Tsujishita, Watanabe, Worrell 2001)

If $F : \mathcal{E} \to \mathcal{E}$ preserves pullbacks and generates a cofree comonad, then

$\mathcal{E}$ a topos $\Rightarrow$ F-coalg a topos.

Proof: If F preserves <u>all</u> finite limits, then $U : F\text{-coalg} \to \mathcal{E}$ creates them; so $Q_F = UR$ is a lex comonad on a topos. Thus $F\text{-coalg} \cong Q_F\text{-Coalg}$ is a topos.

If F only preserves pullbacks, we first slice judiciously. $\square$

When is Fix(F) a topos?

A **well-pointed endofunctor** (T,τ) on $\mathcal{E}$ is $T : \mathcal{E} \longrightarrow \mathcal{E}$ and $\eta : 1_{\mathcal{E}} \Rightarrow T$ such that $T\eta = \eta T : T \Rightarrow TT$. An **algebra** for (T,η) is $(X \in \mathcal{E},\ x : X \longrightarrow TX)$ with $x \circ \eta_X = 1_X$.

Lemma (Wolff 1974)
(T,η)-alg is $\cong$ to the full subcat$_y$ of $\mathcal{E}$ on those X with $\eta_X : X \to TX$ invertible.

<u>Example</u> If $F : \mathcal{E} \to \mathcal{E}$, there's a well-ptd $(\bar{F},\varphi)$ on F-coalg with:

$$\bar{F}(X,x) = (FX, Fx) \qquad \text{and} \qquad \varphi_{(X,x)} = x : (X,x) \longrightarrow (FX, Fx)$$

Here, $(\bar{F},\varphi)$-alg $\cong$ Fix(F); and if F preserves pullbacks, so does $\bar{F}$.

When is Fix(F) a topos?

So the main theorem ($\mathcal{E}$ a topos $\Rightarrow$ Fix(F) a topos)
will follow from [JPTWW] ($\mathcal{E}$ a topos $\Rightarrow$ F-coalg a topos)
on taking $(T,\eta) = (\bar{F},\varphi)$ in:

Proposition

If $(T,\eta): \mathcal{E} \to \mathcal{E}$ is a pullback-preserving well-pointed endofunctor, then
$\mathcal{E}$ a topos $\Rightarrow$ (T,η)-alg a topos.

When is $\mathrm{Fix}(F)$ a topos?

$$\boxed{\begin{array}{l}\text{If } (T,\eta):\mathcal{E}\to\mathcal{E} \text{ is a pullback-preserving} \\ \text{well-pointed endofunctor, then} \\ \mathcal{E} \text{ a topos} \Rightarrow (T,\eta)\text{-alg a topos.}\end{array}}$$

<u>Proof</u> Call a mono $m: X \rightarrowtail Y$ T-proximal if:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & TX \arrow[d, tail, "Tm"] \\ Y \arrow[r, "\eta_Y"'] & TY \end{tikzcd}$$

If $Y \in \mathcal{E}$, then both $\mathrm{im}(\eta_Y) \rightarrowtail TY$ and $Y \rightarrowtail Y \times_{TY} Y$ are T-proximal. Thus:

$$\boxed{X \in (T,\eta)\text{-alg} \iff X \text{ orthogonal to all T-proximals}}$$

When is $\mathrm{Fix}(F)$ a topos?

If $(T,\eta):\mathcal{E}\to\mathcal{E}$ is a pullback-preserving well-pointed endofunctor, then

$\mathcal{E}$ a topos $\Rightarrow$ (T,η)-alg a topos.

Proof $\quad X \in (T,\eta)$-alg $\Leftrightarrow$ X orthogonal to all T-proximals

There's a generic T-proximal $1 \rightarrowtail P$ given by

$$\begin{tikzcd}
1 \arrow[r, dashed, tail] \arrow[rr, bend left, "\eta_1"] \arrow[dr, tail, "\tau"'] & P \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & T1 \arrow[d, "T\tau"] \\
 & \Omega \arrow[r, "\eta_\Omega"'] & T\Omega .
\end{tikzcd}$$

By an argument of Joyal, the objects orthogonal to a classified class of monos in a topos form a subtopos. $\square$

Site presentations

Special case of main theorem:

$\boxed{\text{If } \mathcal{E} \text{ is a topos with NNO, and } F : \mathcal{E}/_I \longrightarrow \mathcal{E}/_I \text{ is polynomial, then } \mathrm{Fix}(F) \text{ is a topos.}}$

In this case, the cofree comonad Q_F is also polynomial;

$\Rightarrow$ $F\text{-coalg} \simeq Q_F\text{-Coalg} \simeq \mathcal{E}^{\mathbb{I}^{op}}$ for some $\mathbb{I} \in \mathrm{Cat}(\mathcal{E})$.

Now $\mathrm{Fix}(F)$ a subtopos of F-coalg $\Rightarrow$ $\mathrm{Fix}(F) \simeq \mathrm{Sh}(\mathbb{I}, J)$.

Direct calculation shows all maps in $\mathbb{I}$ are monos; so

$\mathrm{Fix}(F)$ is étendue over $\mathcal{E}$.

Examples

(1) $\mathcal{E} = \mathbf{Set}$, $\quad F(X) = X \times X$

Here, $\mathbb{I}$ = free monoid on two generators l, r;

coverage generated by $\quad * \xrightarrow{l} * \xleftarrow{r} * \quad$ (Freyd's presentation).

(2) $\mathcal{E} = \mathbf{Set}$, $\quad F(X) = X + X$

Now objects of $\mathbb{I}$ are infinite binary strings $W \in \{l, r\}^{\omega}$.

Maps are freely generated by

$$W \longrightarrow aW \qquad\qquad (a \in \{l, r\},\ W \in \{l, r\}^{\omega}).$$

Coverage generated by <u>all</u> singletons.

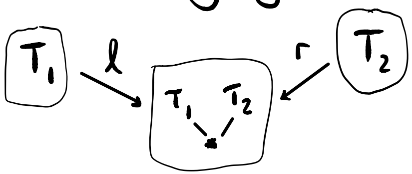
In fact, $\mathrm{Sh}(\mathbb{X}) \simeq \mathrm{PSh}(\pi_1(\mathbb{X}))$. Objects of $\pi_1(\mathbb{X})$ same as $\mathbb{I}$;

maps $V \longrightarrow W$ are integers i st eventually $V_n = W_{n+i}$.

Examples

(3) $\mathcal{E} = \mathbf{Set}$, $F(X) = X \times X + 1$

Now $\mathbb{I}$ has as objects, finite or infinite binary trees;
maps are freely generated by



Coverage generated by the doubletons above, plus empty cover of $\boxed{*}$

(4) $\mathcal{E} = \mathbf{Set}^{\mathcal{C}^{op}}$, $F(X) = \{M, X\}$ for some $M : \mathcal{C} \nrightarrow \mathcal{C}$ (Leinster)

(5) $\mathcal{E} = \mathbf{Set}$, $F(X) = \prod_{\mathcal{F}} X$ for some filter $\mathcal{F} \subseteq \mathcal{P}U$.

(6) $\mathcal{E} = \mathrm{Sh}(A)$, $F = f^*$ for some cts function $f : A \longrightarrow A$.

Applications

When $\mathcal{E} = \mathrm{Set}$, $F(X) = X \times X$, $\mathrm{Fix}(F)$ is the Jonsson-Tarski topos JT.

$\mathrm{JT} = \mathrm{Sh}(M)$ is étendue over the representable sheaf $B = ay(*)$.

From B we can construct various interesting "self-similar objects"

Proposition (Freyd)
The locale of subobjects of $B \in \mathrm{JT}$ is Cantor space $\{0,1\}^{\omega}$

Proposition (Higman)
The automorphism group of $B \in \mathrm{JT}$ is Thompson's group V.

Applications

If R is a commutative ring, the Leavitt algebra $L_{2,1}(R)$ is

$$R\langle \ell, r, \ell^*, r^* \rangle \Big/ \left\{ \begin{pmatrix} \ell & r \end{pmatrix} \begin{pmatrix} \ell^* \\ r^* \end{pmatrix} = 1, \quad \begin{pmatrix} \ell^* \\ r^* \end{pmatrix} \begin{pmatrix} \ell & r \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

It has the property that:

$$L_{2,1}(R)\text{-modules} \quad \leftrightsquigarrow \quad \begin{array}{c} R\text{-modules } M \text{ t/w} \\ M \xrightarrow{\cong} M \oplus M \end{array}$$

Proposition

The endomorphism ring of $R(B)$ in $\mathcal{T}$ is the Leavitt algebra $L_{2,1}(R)$

Applications

Let H be a separable Hilbert space, let $\ell, r \in B(H)$ be isometries satisfying $\ell\ell^* + rr^* = 1$.

The Cuntz C^*-algebra $\mathcal{O}_2$ is $\langle \ell, r \rangle \subseteq B(H)$.

It has the property that:

$\mathcal{O}_2$-representations $\leftrightsquigarrow$ Hilbert spaces H w/ $H \xrightarrow{\sim} H \oplus H$

Proposition

The C^*-algebra of adjointable operators on $\ell^2(B)$ in JT is the Cuntz C^*-algebra $\mathcal{O}_2$

Applications

Now let $G = E \overset{s}{\underset{t}{\rightrightarrows}} V$ be a directed graph.

Taking $\mathcal{E} = \mathrm{Set}^V$, $F = \Pi_t s^*$, we get a topos $\mathrm{Fix}(F) := \mathbb{J}\mathbb{T}_G$.

$\mathbb{J}\mathbb{T}_G = \mathrm{Sh}(\mathbb{I})$ is étendue over the sum of representables $B = \sum_{i \in \mathbb{I}} ay(i)$.

Proposition

The locale of subobjects of $B \in \mathbb{J}\mathbb{T}_G$ is the space of infinite paths in G.

The endomorphism ring of $R(B)$ in $\mathbb{J}\mathbb{T}_G$ is the Leavitt path algebra of G.

The adjointable operators on $\ell^2(B)$ in $\mathbb{J}\mathbb{T}_G$ give the Cuntz-Krieger C^*-algebra of G.

Applications

There are further examples which reconstruct algebras and C^*-algebras associated to:

- Self-similar group actions (Nekrashevych)
- Self-similar groupoid actions (Laca-Raeburn-Ramagge-Whittaker)
- Higher-rank graph actions (Kumjian-Pask)
- Discrete Conduché fibrations (Brown-Yetter)