Profinite Monads and Reiterman’s Theorem

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The Birkhoff Variety Theorem (1935)

- **The Birkhoff Theorem**
  - \( \mathcal{A} \) a full subcategory of \( \Sigma\text{-Alg} \):
  - \( \mathcal{A} \) presentable by equations
    - \( \iff \) variety \( (= \text{HSP class}) \)

\[
\begin{align*}
\text{regular quotients} & \quad \text{subobjects} \quad \text{products}
\end{align*}
\]
The Birkhoff Variety Theorem (1935)

- **The Birkhoff Theorem**
  \( \mathcal{A} \) a full subcategory of \( \Sigma\text{-Alg} \):
  \( \mathcal{A} \) presentable by equations
  \( \iff \) variety \((= \text{HSP class})\)

  - regular quotients
  - subobjects
  - products

- **Lawvere**: equations are pairs of \( n.t. \) \( \alpha : U^n \to U \) for

  \[ U : \Sigma\text{-Alg} \to \text{Set} \]

  An algebra \( A \) satisfies \( \alpha = \alpha' \) iff \( \alpha_A = \alpha'_A \)
The Reiterman Theorem (1982)

- **The Reiterman Theorem**
  - $\mathcal{A}$ a full subcategory of $(\Sigma\text{-Alg})_f$:
    - $\mathcal{A}$ presentable by pseudoequations
    - $\Leftrightarrow$ pseudovariety ($= \text{HSP}_f$ class)
The Reiterman Theorem (1982)

- **The Reiterman Theorem**
  \( \mathcal{A} \) a full subcategory of \((\Sigma\text{-Alg})_f\):
  - \( \mathcal{A} \) presentable by pseudoequations
  - \( \Leftrightarrow \) pseudovariety (= HSP\(_f\) class)

- **\( U_f : (\Sigma\text{-Alg})_f \rightarrow \text{Set}_f \)**
  - Pseudoequations are pairs of \( n.t. \) \( \alpha : U_{f}^{n} \rightarrow U_{f} \)
  - A finite algebra \( A \) satisfies \( \alpha = \alpha' \) iff \( \alpha_A = \alpha'_A \)
The Reiterman Theorem (1982)

- **Example** $Un$, unary algebras

  \[ \sigma : A \rightarrow A \]

  A finite $\Rightarrow \exists n : \sigma^n = (\sigma^n)^2$

  Notation : $\sigma^* = \sigma^n$

  Pseudoequation : $\sigma^*(x) = x$

  presents : finite algebras with $\sigma$ invertible
Banaschewski and Herrlich (1976)

- $\mathcal{D}$ a complete category
- $(\mathcal{E}, \mathcal{M})$ a proper factorization system (e.g. regular epi - mono)
  notation $\twoheadrightarrow$ and $\hookrightarrow$
- $\mathcal{D}$ has enough projectives $X: \forall D \exists X \twoheadrightarrow D$

**Definitions** An equation $e: X \twoheadrightarrow A$, $X$ projective.

It is satisfied by $D \in \mathcal{D}$ if $X \xrightarrow{\forall f} D \xrightarrow{\exists} A$

($D$ is $e$-injective)

**Theorem** A full subcategory $\mathcal{A}$ of $\mathcal{D}$:

$\mathcal{A}$ presentable by equations $\iff$ a variety ($= HSP$ class)
Assume: \( \mathcal{D} \) and \((\mathcal{E}, \mathcal{M})\) as above

\[ \mathcal{D}_f \subseteq \mathcal{D} \] full subcategory closed under \( S \) and \( P_f \)

'finite' objects
Assume: $\mathcal{D}$ and $(\mathcal{E}, \mathcal{M})$ as above
$\mathcal{D}_f \subseteq \mathcal{D}$ full subcategory closed under $S$ and $P_f$
'finite' objects

Definition A **pseudovariety** is a full subcategory of $\mathcal{D}_f$
closed under $\text{HSP}_f$. 
**Definition** A quasi-equation over $X$ (projective) is a semilattice $\Omega$ of finite quotients $e: X \rightarrow A$ ($A \in D_f$)

\[ \forall e, e' \in \Omega \]

\[ \bar{e} = e \wedge e' \in \Omega \]

\[ A' \rightleftharpoons A \leftleftharpoons A'' \]

\[ A' \times A'' \]
**Pseudovariety Presentation**

- **Definition** A *quasi-equation* over $X$ (projective) is a semilattice $\Omega$ of finite quotients $e : X \rightarrow A$ ($A \in D_f$)

- **Diagram**
  
  \[
  \begin{array}{c}
  X \\
  \downarrow e \quad \downarrow e' \\
  \downarrow \bar{e} \\
  \bar{A} \\
  \downarrow u' \quad \downarrow u'' \\
  A' \quad A'' \\
  \end{array}
  \]

  $\forall e, e' \in \Omega$

  $\bar{e} = e \land e' \in \Omega$

  $A' \times A''$

- **An object** $D$ satisfies $\Omega$ if it is injective:

  \[
  \begin{array}{c}
  X \\
  \downarrow \forall f \\
  \Downarrow \exists e \in \Omega \\
  \Downarrow \exists A \\
  \end{array}
  \]

  \[
  D
  \]

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Profinite Monads and Reiterman’s Theorem
Proposition $\mathcal{A}$ a full subcategory of $\mathcal{D}_f$:

$\mathcal{A}$ presentable by quasi-equations $\iff \mathcal{A}$ a pseudovariety
**Proposition** \( \mathcal{A} \) a full subcategory of \( D_f \): 
\( \mathcal{A} \) presentable by quasi-equations \( \iff \mathcal{A} \) a pseudovariety

**Proof** \( \iff \) For every \( X \) projective

\[ \Omega_X : X \to A(A \in \mathcal{A}) \]

\( \Omega_X \) semilattice \( \iff \mathcal{A} \) is \( SP_f \)-class
\( D \in \mathcal{A} \implies D \) satisfies \( \Omega_X \) \( \ldots \) trivial

\( D \) satisfies each \( \Omega_X \) \( \implies D \in \mathcal{A} \): choose \( X \)

\[ \xymatrix{ X \ar[r]^f & D \ar@/^1pc/[l] \ar@/_1pc/[l] } \]

\( X \) projective, \( e \in \mathcal{E} \), \( A \in \mathcal{A} \implies D \in \mathcal{A} \)
Our Goal

- Given: $\mathcal{D}$, $(\mathcal{E}, \mathcal{M})$ and $\mathcal{D}_f$ as above
  - $\mathcal{T}$ a monad on $\mathcal{D}$ preserving $\mathcal{E}$
- Describe pseudovarieties in $\mathcal{D}^\mathcal{T}$ by equations
  in some extension of $\mathcal{D}^\mathcal{T}$
Our Goal

- Given: \( \mathcal{D}, (\mathcal{E}, \mathcal{M}) \) and \( \mathcal{D}_f \) as above
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    in some extension of \( \mathcal{D}^\mathbb{T} \)

- \( \mathcal{D}^\mathbb{T} \) has the factorization system inherited from \( \mathcal{D} \)
  - it has enough projectives: \( (TX, \mu X) \) with \( X \) projective
  - \( \mathcal{D}^\mathbb{T} \) def all algebras \( (A, \alpha) \) with \( A \in \mathcal{D}_f \)
Our Goal

- Given: \( D, (E, M) \) and \( D_f \) as above
  - \( T \) a monad on \( D \) preserving \( E \)
- Describe pseudovarieties in \( D^T \) by **equations**
  - in some extension of \( D^T \)

- \( D^T \) has the factorization system inherited from \( D \)
  - it has enough projectives: \((TX, \mu X)\) with \( X \) projective
  - \( D^T \) def = all algebras \((A, \alpha)\) with \( A \in D_f \)
- Thus pseudovarieties are presentable by quasi-equations in \( D^T \)
The Category $\hat{\mathcal{D}}_f$

- **Profinite completion** $\text{Pro } \mathcal{D}_f = \hat{\mathcal{D}}_f$ (dual to $\text{Ind}$)
  - $\mathcal{D}_f$ finitely complete $\Rightarrow \hat{\mathcal{D}}_f$ complete
  - $\hat{\mathcal{E}} = \text{cofiltered limits of quotients in } \mathcal{D}_f$
  - $\hat{\mathcal{M}} = \text{cofiltered limits of subobjects in } \mathcal{D}_f$
The Category $\hat{D}_f$

- **Profinite completion** $\text{Pro } D_f = \hat{D}_f$ (dual to Ind)
  - $D_f$ finitely complete $\Rightarrow \hat{D}_f$ complete
  - $\hat{E} = \text{cofiltered limits of quotients in } D_f$
  - $\hat{M} = \text{cofiltered limits of subobjects in } D_f$

- **Wanted**: $\hat{D}_f$ has enough $\hat{E}$-projectives
  - $T$ yields (canonically) a monad $\hat{T}$ on $\hat{D}_f$ preserving $\hat{E}$

$\Rightarrow \hat{D}_f, (\hat{E}, \hat{M})$ and $\hat{T}$ satisfy all of our assumptions

Goal: quasi-equations in $D^T \Leftrightarrow$ equations in $(\hat{D}_f)^{\hat{T}}$

**Important**: $T$ and $\hat{T}$ have the same finite algebras

$$D^T_f \simeq \hat{D}_f^{\hat{T}}$$
**Profinite Factorization Systems**

**Definition** $(\mathcal{E}, \mathcal{M})$ is a **profinite** factorization system if $\mathcal{E}$ is closed under cofiltered limits of quotients in $D_f^\to$

**Examples** with $\mathcal{E} =$ surjective morphisms

- **Set** : $\hat{\text{Set}}_f = \text{Stone}$
**Definition** \((\mathcal{E}, \mathcal{M})\) is a **profinite** factorization system if \(\mathcal{E}\) is closed under cofiltered limits of quotients in \(D_f\)

**Examples** with \(\mathcal{E} = \text{surjective morphisms}\)

- **Set**: \(\hat{\text{Set}}_f = \text{Stone}\)
- **Pos**: with \(\mathcal{E} = \text{surjective monotone maps}\)
  \(\hat{\text{Pos}}_f = \text{Priestley}\)
**Definition** \((\mathcal{E}, \mathcal{M})\) is a **profinite** factorization system if \(\mathcal{E}\) is closed under cofiltered limits of quotients in \(\mathcal{D}_f\).

**Examples** with \(\mathcal{E}\) = surjective morphisms

- **Set** : \(\check{\text{Set}}_f = \text{Stone}\)
- **Pos** : with \(\mathcal{E}\) = surjective monotone maps
  \(\check{\text{Pos}}_f = \text{Priestley}\)
- \(\mathcal{D} \subseteq \Sigma\text{-Str}\) full subcategory closed under limits
  arbitrary operation symbols
  + finitely many relation symbols

\(\text{Pro} \mathcal{D}_f \subseteq \text{Stone} \mathcal{D}\)
\(\hat{\mathcal{E}} = \text{surjective continuous homomorphisms}\)
Profinite monad $\hat{T}$

- $\hat{T}$ is the codensity monad of the forgetful functor $D_f \to \hat{D}_f$

Example

$D_f = \text{Set}$, $TX = X^\ast$: the word monad

$\hat{T}$ is the monad of profinite words on $\hat{\text{Mon}}_f = \text{Stone Mon}$

$\hat{T}Y$ is the cofiltered limit of all finite $\hat{E}$-quotients of $Y$ carried by $T$-algebras

Example

For $TX = X^\ast$: a profinite word in a Stone monoid $Y$ is a compatible choice of a member of $A$ for every finite quotient monoid $A$ of $Y$.

Proposition

If $(E, M)$ is profinite then

1. $\hat{T}$ preserves $\hat{E}$ and
2. finite $T$-algebras coincide with finite $\hat{T}$-algebras
Profinite monad $\hat{T}$

- $\hat{T}$ is the codensity monad of the forgetful functor $D_f^T \to \hat{D}_f$

  **Example** $D = \text{Set}$, $TX = X^*$: the word monad
  $\hat{T}$ is the monad of **profinite words** on $\hat{\text{Mon}}_f = \text{Stone Mon}$

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Profinite monad $\hat{T}$

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**Example** $D = \text{Set}, \ TX = X^*$: the word monad

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**Proposition** If $(\mathcal{E}, \mathcal{M})$ is profinite then

1. $\hat{T}$ preserves $\hat{E}$
2. finite $T$-algebras coincide with finite $\hat{T}$-algebras
Profinite equation = equation in $\hat{D}_f$

$e: P \rightarrow Q$, $P$ projective
Generalized Reiterman’s Theorem

- Profinite equation = equation in $\hat{\mathcal{D}}_f$
  
  \[ e: P \to Q, \ P \text{ projective} \]

- A finite $T$-algebra satisfies $e$:
  
  it is $e$-injective
Generalized Reiterman’s Theorem

- Profinite equation = equation in $\hat{\mathcal{D}}_f$
  \[ e : P \rightarrow Q, \text{ } P \text{ projective} \]
- A finite $\mathcal{T}$-algebra satisfies $e$ :
  it is $e$-injective
- **Theorem** $\mathcal{A}$ a full subcategory of $\mathcal{D}_f^\mathcal{T}$ :
  $\mathcal{A}$ presentable by profinite equations $\iff$ a pseudovariety
Profinite equations in $\Sigma$-Str

**Example** $\mathcal{D} \subseteq \Sigma$-$\text{Str}$ closed under limits and subobjects

$\hat{\mathcal{D}}_f \subseteq \text{Stone } \Sigma$-$\text{Str}$

A **profinite equation**: $\alpha = \alpha'$ where $\alpha, \alpha' \in \hat{T}X$

$X$ projective in $\hat{\mathcal{D}}_f$

Given $e : (\hat{T}X, \mu_X) \to A$, take all $(\alpha, \alpha') \in \ker$
Profinite Equations in \( \Sigma\text{-Str} \)

- Back to Reiterman: \( U_f : (\Sigma\text{-Alg})_f \rightarrow \text{Set}_f \)

\[
\text{n.t. } \alpha : U_f^n \rightarrow U_f \quad \leftrightarrow \quad \text{elements of } \hat{T}n
\]

\[
\text{pseudoequations} \quad \leftrightarrow \quad \text{profinite equations}
\]
Profinite Equations in $\Sigma$-Str

- Back to Reiterman: $U_f : (\Sigma\text{-Alg})_f \rightarrow \text{Set}_f$

  n.t. $\alpha : U^n_f \rightarrow U_f \leftrightarrow$ elements of $\hat{T}n$
  pseudoequations $\leftrightarrow$ profinite equations

- Varieties of ordered algebras . . . inequalities $\alpha \leq \alpha'$ between terms

  $\mathcal{D} = \text{Pos} \quad \hat{\mathcal{D}}_f = \text{Priestley}$

  profinite equations $e : (\hat{T}X, \mu_X) \rightarrow A,$
  $X$ discretely ordered

  $\leftrightarrow$ inequalities

J. E. Pin & P. Weil (1996)