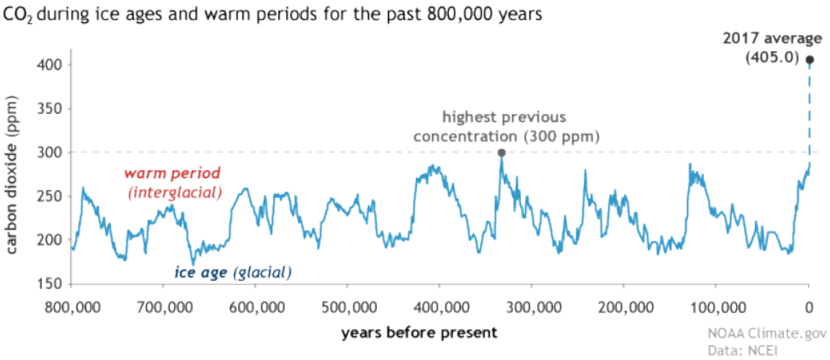
STRUCTURED COSPANS

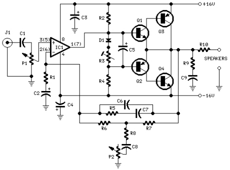
John Baez, Kenny Courser, Christina Vasilakopoulou

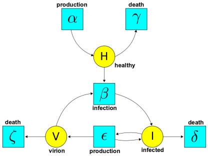
CT2019

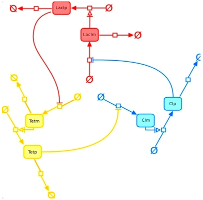
11 July 2019

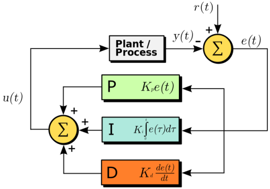

CO2 during ice ages and warm periods for the past 800,000 years
2017 average
(405.0)
400
350
300
250
200
150
carbon dioxide (ppm)
highest previous
concentration (300 ppm)
warm period
(interglacial)
ice age (glacial)
800,000
700,000
600,000
500,000
400,000
300,000
200,000
100,000
0
years before present
NOAA Climate.gov
Data: NCEI

Throughout science and engineering, people use *networks*, drawn as boxes connected by wires:



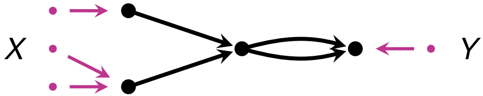






So, they're using categories! Which categories are these?

Networks of some particular kind, with specified inputs and outputs, can be seen as morphisms in some symmetric monoidal category:



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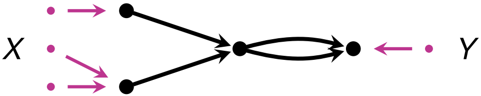
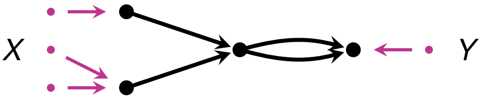


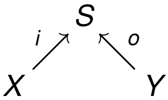
Such networks let us describe "open systems", meaning systems where:

- stuff can flow in or out;
- we can combine systems to form larger systems by composition and tensoring.

We can describe networks with inputs and outputs using cospans with extra structure. For example, this:



is really a cospan of finite sets:



where S is decorated with extra structure: edges making S into the vertices of a graph.

Fong invented 'decorated cospans' to make this precise:

- Brendan Fong, Decorated cospans, arXiv:1502.00872.

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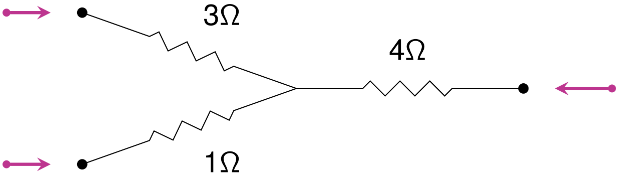
We’ve used them to study many kinds of networks.

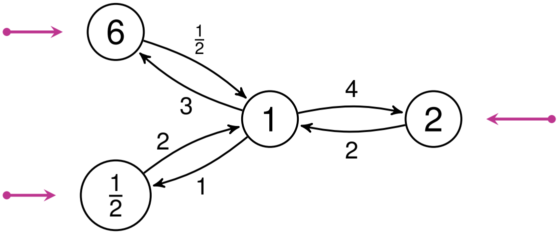
Electrical circuits:

- Brendan Fong, JB, A compositional framework for passive linear networks, arXiv:1504.05625.



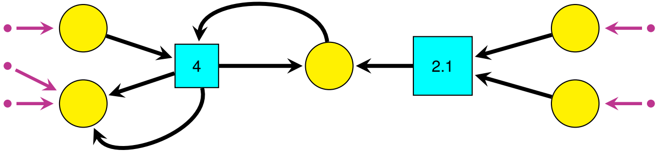
Markov processes:

- Brendan Fong, Blake Pollard, JB, A compositional framework for Markov processes, arXiv:1508.06448.



Petri nets with rates:

- Blake Pollard, JB, A compositional framework for reaction networks, arXiv:1704.02051.



Now Kenny Courser has developed a simpler formalism — ‘structured cospans’ — that avoids certain problems with decorated cospans.

Kenny has redone most of the previous work using structured cospans:

- Kenny Courser, *Open Systems: A Double Categorical Perspective*, https://tinyurl.com/courser-thesis.

Given a functor

$$L\colon \mathsf{A} \to \mathsf{X}$$

a **structured cospan** is a diagram

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

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a **structured cospan** is a diagram

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Think of A as a category of objects with 'less structure', and X as a category of objects with 'more structure'. L is often a left adjoint.

For example, a **Petri net with rates** is a diagram like this:

$$(0,\infty) \xleftarrow{\;r\;} T \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{N}[S]$$

where S and T are finite sets, and $\mathbb{N}[S]$ is the underlying set of the free commutative monoid on S.

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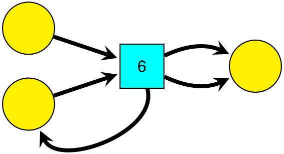
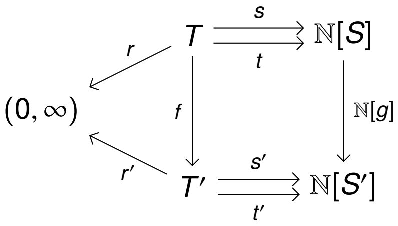


We call elements of S **species** ,
elements of T **transitions** ,
and $r(t)$ the **rate constant** of the transition $t \in T$.

There is a category Petri where morphisms are the obvious things:



where the square involving s and s' commutes, as does the square involving t and t'.

There is a functor $R \colon \mathrm{Petri} \to \mathrm{FinSet}$ sending any Petri net with rates to its underlying set of species.

This has a left adjoint $L \colon \mathrm{FinSet} \to \mathrm{Petri}$.

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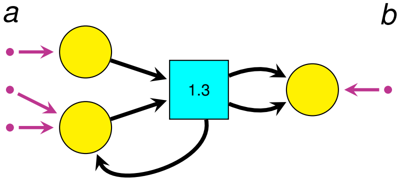
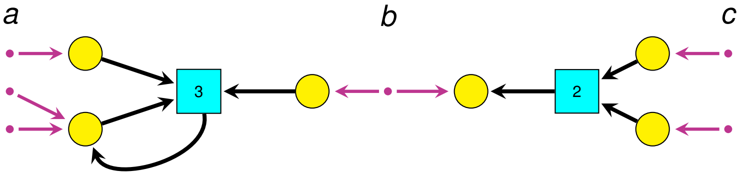
In this example, a structured cospan

$$L(a) \xrightarrow{i} x \xleftarrow{o} L(b)$$

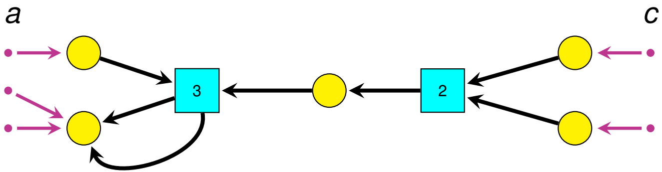
is called an **open Petri net with rates**:



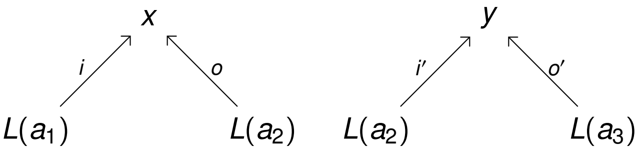
We can compose open Petri nets with rates:



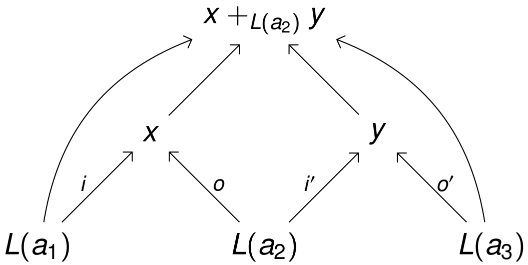
by identifying the outputs of the first with the inputs of the second:



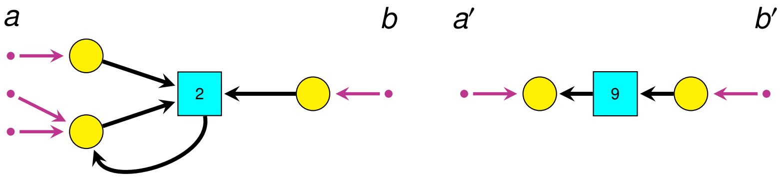
In other words, given open Petri nets with rates:



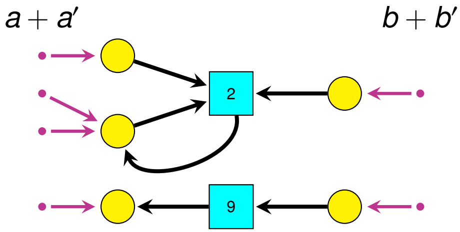
we compose them by taking a pushout in the category Petri:



To tensor open Petri nets with rates:



we set them side by side:



In other words, to tensor open Petri nets with rates:

$$\begin{array}{ccc} & x & \\ & {\scriptstyle i}\nearrow \quad \nwarrow{\scriptstyle o} & \\ L(a) & & L(b) \end{array} \qquad \begin{array}{ccc} & x' & \\ & {\scriptstyle i'}\nearrow \quad \nwarrow{\scriptstyle o'} & \\ L(a') & & L(b') \end{array}$$

we use coproducts in Set and Petri:

$$\begin{array}{ccccc} & & x + x' & & \\ & {\scriptstyle i+i'}\nearrow & & \nwarrow{\scriptstyle o+o'} & \\ & L(a) + L(a') & & L(b) + L(b') & \\ {\scriptstyle \cong}\nearrow & & & & \nwarrow{\scriptstyle \cong} \\ L(a + a') & & & & L(b + b') \end{array}$$

and the fact that $L\colon \mathrm{FinSet} \to \mathrm{Petri}$ preserves coproducts.

In general:

Theorem (Kenny Courser, JB)

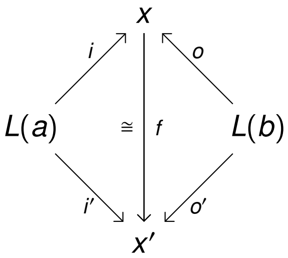
Let A *be a category with finite coproducts,*
X *a category with finite colimits, and*
$L\colon \mathsf{A} \to \mathsf{X}$ *a functor preserving finite coproducts.*

Then there is a symmetric monoidal category ${}_L\mathrm{Csp}(\mathsf{X})$ *where:*

- *an object is an object of* A
- *a morphism is an isomorphism class of structured cospans:*

$$L(a) \xrightarrow{\;i\;} x \xleftarrow{\;o\;} L(b)$$

Here two structured cospans are **isomorphic** if there is a commuting diagram of this form:



This theorem applies to many examples, giving structured cospan categories whose morphisms are:

- open electrical circuits
- open Markov processes
- open Petri nets
- open Petri nets with rates

etcetera.

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In all these examples A and X have finite colimits and $L\colon \mathsf{A} \to X$ is a left adjoint, so all the conditions of the theorems hold.

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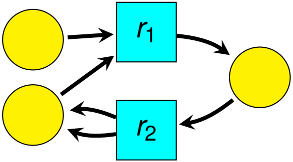
What can we do with structured cospan categories?

Given a Petri net with rates, we can write down a **rate equation** describing dynamics. For example, this Petri net with rates:



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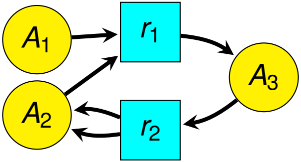


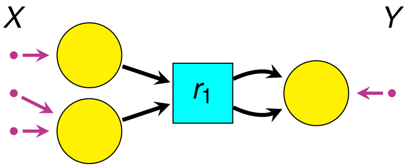
gives this rate equation:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2$$

$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + 2 r_2 A_3$$

$$\frac{dA_3}{dt} = r_1 A_1 A_2 - r_2 A_3$$

An open Petri net with rates $f\colon X \to Y$ gives an **open rate equation** involving flows in and out, which can be arbitrary smooth functions of time. For example this:



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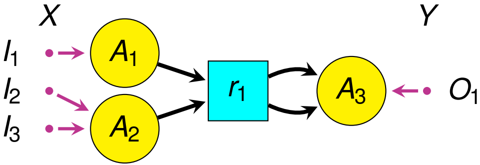

gives:

$$\frac{dA_1}{dt} = -r_1 A_1 A_2 + I_1(t)$$

$$\frac{dA_2}{dt} = -r_1 A_1 A_2 + I_2(t) + I_3(t)$$

$$\frac{dA_3}{dt} = 2r_1 A_1 A_2 - O_1(t)$$

Let Open(Petri) be the category with open Petri nets with rates as morphisms. The map sending open Petri nets to their open rate equations gives a symmetric monoidal functor

$$\square \colon \mathrm{Open(Petri)} \to \mathrm{Dynam}$$

where Dynam is a category of ‘open dynamical systems’.

So, we can describe dynamical systems *compositionally*, a piece at a time, using open Petri nets with rates.

Jonathan Lorand and I are using this to study questions from biochemistry.

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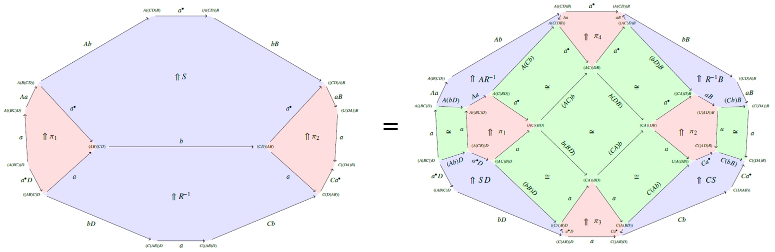
You might be thinking we should use a symmetric monoidal bicategory... and we *could*.



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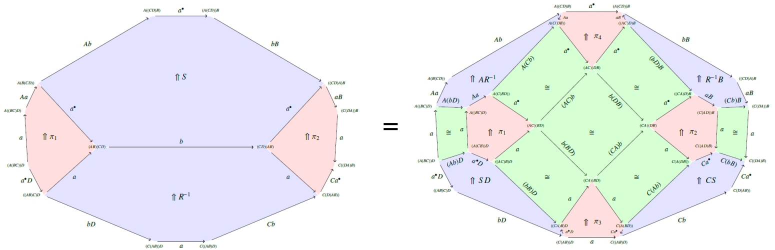

But Mike Shulman noticed that it’s easier to use a symmetric monoidal double category!

For us a **double category** is a weak category object in the 2-category **Cat**. It has a category of objects Ob and a category of morphisms Mor. Composition

$$\circ \colon \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor} \to \mathrm{Mor}$$

is associative and unital up to 2-isomorphisms obeying the usual equations.

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There is a 2-category **Dbl** of double categories, double functors, and transformations. **Dbl** has finite products.

In any 2-category with finite products we can define symmetric pseudomonoids. In **Cat** these are symmetric monoidal categories. In **Dbl** we call them **symmetric monoidal double categories**.

More concretely, a double category has figures like this:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

So, it has:

- **objects** such as A, B, C, D,

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So, it has:

- **objects** such as A, B, C, D,
- **vertical 1-morphisms** such as f and g,
- **horizontal 1-cells** such as M and N,
- **2-morphisms** such as α.

2-morphisms can be composed vertically and horizontally, and the interchange law holds:

$$\begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ D & \xrightarrow[N]{} & E \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{M'} & C \\ {\scriptstyle g}\downarrow & \Downarrow \beta & \downarrow{\scriptstyle h} \\ E & \xrightarrow[N']{} & F \end{array}$$

$$\begin{array}{ccc} D & \xrightarrow{N} & E \\ {\scriptstyle f'}\downarrow & \Downarrow \alpha' & \downarrow{\scriptstyle g'} \\ G & \xrightarrow[O]{} & H \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{N'} & F \\ {\scriptstyle g'}\downarrow & \Downarrow \beta' & \downarrow{\scriptstyle h'} \\ H & \xrightarrow[P]{} & I \end{array}$$

Vertical composition is strictly associative and unital, but horizontal composition is not.

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- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is a structured cospan* $L(a) \stackrel{i}{\rightarrow} x \stackrel{o}{\leftarrow} L(b)$
- *a 2-morphism is a commutative diagram*

$$
\begin{array}{ccccc}
L(a) & \xrightarrow{\;i\;} & x & \xleftarrow{\;o\;} & L(b) \\
{\scriptstyle L(f)}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle L(g)} \\
L(a') & \xrightarrow[\;i'\;]{} & x' & \xleftarrow[\;o'\;]{} & L(b')
\end{array}
$$

Horizontal composition is defined using pushouts in X; composing these:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x & \longleftarrow & L(b) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' & \longleftarrow & L(b') \end{array} \qquad \begin{array}{ccccc} L(b) & \longrightarrow & y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(b') & \longrightarrow & y' & \longleftarrow & L(c') \end{array}$$

gives this:

$$\begin{array}{ccccc} L(a) & \longrightarrow & x +_{L(b)} y & \longleftarrow & L(c) \\ \downarrow & & \downarrow & & \downarrow \\ L(a') & \longrightarrow & x' +_{L(b')} y' & \longleftarrow & L(c') \end{array}$$

Vertical composition is straightforward.

Tensoring uses binary coproducts in both A and X, and the fact that $L\colon \mathsf{A} \to \mathsf{X}$ preserves these:

$$\begin{array}{ccccc} L(a_1) & \longrightarrow & x_1 & \longleftarrow & L(b_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2) & \longrightarrow & x_2 & \longleftarrow & L(b_2) \end{array} \quad \otimes \quad \begin{array}{ccccc} L(a'_1) & \longrightarrow & x'_1 & \longleftarrow & L(b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a'_2) & \longrightarrow & x'_2 & \longleftarrow & L(b'_2) \end{array}$$

$$= \quad \begin{array}{ccccc} L(a_1 + a'_1) & \longrightarrow & x_1 + x'_1 & \longleftarrow & L(b_1 + b'_1) \\ \downarrow & & \downarrow & & \downarrow \\ L(a_2 + a'_2) & \longrightarrow & x_2 + x'_2 & \longleftarrow & L(b_2 + b'_2) \end{array}$$

How do structured cospans compare to *decorated* cospans?

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Given a suitable functor $F\colon \mathsf{A} \to \mathsf{Set}$, Fong defined an ***F*-decorated cospan** to be a pair

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad\qquad d \in F(c)$$

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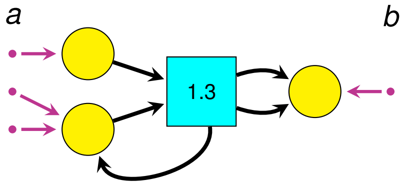
$$a \xrightarrow{i} c \xleftarrow{o} b \qquad d \in F(c)$$

For example, $F(c)$ could be the set of Petri nets with rates having c as their set of species.



The problem is that a functor $F\colon \mathsf{A} \to \mathsf{Set}$ corresponds to a *discrete* opfibration $R\colon \mathsf{X} \to \mathsf{A}$. These are not general enough!

For example: the functor $R\colon \mathsf{Petri} \to \mathsf{FinSet}$ sending any Petri net with rates to its underlying set of species is an opfibration, but not a discrete one.

The solution: use pseudofunctors $F\colon \mathsf{A} \to \mathbf{Cat}$.

Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

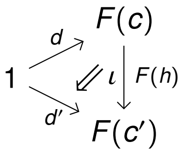
Given a finitely cocomplete category A *and a symmetric lax monoidal pseudofunctor* $F \colon \mathsf{A} \to \mathbf{Cat}$, *there is a symmetric monoidal double category* $F\mathbb{C}\mathbf{sp}$ *where:*

- *an object is an object of* A
- *a vertical 1-morphism is a morphism of* A
- *a horizontal 1-cell is an* F*-decorated cospan:*

$$a \xrightarrow{i} c \xleftarrow{o} b \qquad d \in F(c)$$

- *a 2-morphism is a commutative diagram and a triangle:*

$$\begin{array}{ccccc} a & \xrightarrow{i} & c & \xleftarrow{o} & b \\ {\scriptstyle f}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ a' & \xrightarrow[i']{} & c' & \xleftarrow[o']{} & b' \end{array}$$



Theorem (Kenny Courser, Christina Vasilakopoulou, JB)

Suppose A *is finitely cocomplete,* $F\colon \mathsf{A} \to \mathbf{Cat}$ *is a symmetric lax monoidal pseudofunctor, and* F *factors through the 2-category* **Rex** *of finitely cocomplete categories. Then the opfibration*

$$R\colon \textstyle\int F \to \mathsf{A}$$

has a left adjoint

$$L\colon \mathsf{A} \to \textstyle\int F$$

and there is an isomorphism of symmetric monoidal double categories

$${}_L\mathbb{C}\mathbf{sp}(\textstyle\int F) \cong F\mathbb{C}\mathbf{sp}.$$

So in this situation, which is common, structured cospans agree with the 'new improved' decorated cospans!