Hyperconnections

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What are hyperconnections?

**Answer:** hyperconnected functors over a fixed base

Recall from topos theory

the hyperconnected localic factorization

\[ \xrightarrow{F} \Rightarrow \overrightarrow{\rightarrow} \Rightarrow \text{hyper connections} \]

\[ \xrightarrow{F} \Rightarrow \overrightarrow{\rightarrow} \Rightarrow \text{hyper connected functors} \]

Algebraic direction

F\* preserves \(\Sigma\)

Algebraic direction
What are hyperconnections for restriction categories?

**Restriction category:**

with "restriction" combinator

\[
\begin{align*}
A & \xrightarrow{f} B \quad (1) \\
A & \xrightarrow{f} B
\end{align*}
\]

[1] \( f \circ f = f \)
[2] \( f \circ g = g \circ f \)
[3] \( f \circ \emptyset = f \emptyset \)
[4] \( h \circ f = h \circ f \circ h \)

\( \mathcal{O}(A) = \{ e : A \rightarrow A \mid e = e^2 \} \)

- the open sets of \( A \)
- or the admissible predicates

\( \text{Definition: } F : X \rightarrow \mathcal{Y} , \text{ a restriction functor, is hyperconnected in case for each } x \in X \)

\( F : \mathcal{O}(x) \rightarrow \mathcal{O}(F(x)) \)

is an isomorphism.

preserves and reflects admissible predicates

\[ F(\emptyset) = \emptyset \text{ for all } x \in X \]

\[ F(1) = 1 \text{ for all } x \in X \]
The localic/hyperconnected factorization for restriction functors:

**Def:** A restriction functor is **localic** iff
- it is bijective on objects
- for every map $F(x) \to F(y)$ the poset of maps $f$ with $g \leq F(f)$ is non-empty and downward directed.

**Proposition:** The category of restriction categories with restriction functors admits a factorization system of functors into localic followed by hyperconnected functors.

\[ \begin{array}{ccc}
X & \xrightarrow{F} & Y \\
\downarrow & & \downarrow \\
\text{localic} & \xrightarrow{L} & \text{hyperconnected} \\
\end{array} \]
What are hyperconnections for join restriction categories?

Answer: hyperconnected restriction functors... which are join restriction functors!

Do they induce a factorization system?

What does it look like?

A harder question to answer concretely!

What are join restriction categories?
Join restriction categories (join categories) are restriction categories with join of compatible set of parallel maps.

**Join functors** preserve:
- Restriction: \( F(f) = F(g) \)
- Joins: \( F(VF) = VF(g) \)

**Compatible**:
- \( f, g : A \to B \)
- \( f \cup g \iff f g = g f \)

**Compatible set**:
- \( F \subseteq X(A, B) \)
- Such that \( \forall f, g \in F. f \cup g \)

**Join of compatible set**:
- \( VF : A \to B \)
- \( f \in F \Rightarrow f \in VF \)
- \( \forall f, g \in F, f \cup g \Rightarrow VF \leq g \)

**Join restriction categories** (join categories) with join of compatible set of parallel maps.

**Least upper bound**

**O(A)** is a locale

\[ \bigwedge (VF) k = \bigvee \{ hfk \mid f \in F \} \]
Join categories have intersections of parallel maps

\[ f \land g = \bigvee \exists h \mid h \leq f, h \leq g \]

these are not very well-behaved

**Defn:** A join restriction functor is localic in case:

- It is bijective on objects
- For every \( y : Y_1 \to Y_2 \in Y \) there is a decomposition \( y = \bigvee y_i \) and maps \( x_i : F^{-1}(y_i) \to F^{-1}(y) \) such that \( F(x_i) \geq y_i \).
- \( F \) preserves intersections

**Proposition:** Localic join functors are orthogonal to hyper connected join functors

\[ \begin{array}{ccc}
X & \xrightarrow{L} & Y \\
F \downarrow & & G \\
A & \xrightarrow{H} & B
\end{array} \]

Hmmm...!
An (internal) \( I \)-partite category, \( B \), in a join restriction category \( X \) consists of:

- For each \( i \in I \) an object \( B(i) \)
- For each pair \( (ij) \in I \times I \) a span
- For each \( i,j,k \in I \) a composition \( \mu_{ijk} \)
- For each \( i \in I \) a unit \( e : B(i) \rightarrow B(i;i) \)

Satisfying the usual diagrams to make this an associative composition with units.
Externalizing $I$-partite categories:

Given an (internal) $I$-partite category $B$ in a join category $\mathbf{X}$, one can form a hyperconnection over $\mathbf{X}$:

- **Objects:** $i \in I$
- **Maps:** $a : i \to j$ are partial sections

Composition:

$$a : i \to j$$
$$b : j \to k$$

$$\langle a\overline{tb}, atb \rangle \mu : i \to j$$

Identity:

$$e : B(i) \to B(i)$$

Restriction:

$$\overline{ae} : i \to i$$

Join:

$$a \overline{e} : i \to j$$
Internalizing join functors over a base:

Let $\mathcal{X} \xrightarrow{F} \mathcal{B}$ be a join functor where we now assume $\mathcal{B}$ has gluings or is manifold complete. We construct a $\mathcal{X}_0$-partite category $\hat{F}$:

$$\hat{F}(x) = F(x)$$

$\hat{F}(x,y)$ is given by the atlas

- Charts: for each $f \in \mathcal{X}(x,y)$ a copy of $F(x)$
- Gluings: for each $f, g \in \mathcal{X}(x,y)$ a local atlas $F(fg) : F(x_f) \rightarrow F(x_g)$

Pullbacks of étale maps exist! This is an étale map or local homeomorphism

This gives a source étale $\mathcal{X}_0$-partite category
Theorem: There is a Galois adjunction between
join functors over $I\mathcal{B}$ (with glueings) and
partite categories with cofunctors within $\mathcal{B}$

\[
\begin{array}{ccc}
\text{Join} & \cong & \text{Part} \\
\text{In} & \cong & \text{Part} \ \bigcirc \\
\text{Ex} & \cong & \text{Ex}
\end{array}
\]

!! Amazingly!! the unit is a localic functor

\[
\begin{array}{ccc}
\text{localic} & \xrightarrow{\sim} & \text{hyper-connected}
\end{array}
\]

.... Which gives the factorization we sought!
Examples:

- $\text{Par}$ (sets and partial maps) has every map etale, so an internal category gives an external join monoid sitting over $\text{Par}$ by a hyperconnection.

\[ \begin{array}{c}
M \\
\downarrow \\
\text{Par}
\end{array} \]

(just a small category!)

$\mathcal{B}(\ast)$ is a complete atomic Boolean algebra.

Special example (due to [Lawson]):

$\mathbb{R}_n = \langle a_1, \ldots, a_n, a_1', \ldots, a_n' : a_i a_i = 1, a_i a_j = 0, i \neq j \rangle$ is a polycyclic monoid inverse monoid with 0.

$\mathbb{C}_n \rightarrow \mathbb{E}$ is a Cuntz-etalé monoid,

Total iocs $V_n, 1$ is the Thompson group.
• $\text{Id} : \text{Par} \rightarrow \text{Par}$ is a join functor

$$\text{Id}(x, y) = x \times y$$

More generally, for any étale join category (all maps local homeo) with gluings

$$\hat{\text{Id}}(x, y) = x \times y \circ \text{the product in the category of local homeomorphisms}$$

• $\text{Scheme} \xrightarrow{\text{Id}} \text{Scheme}$

For any $R$,

$$\hat{\text{Id}}(R, \mathbb{Z}[x])$$

The structure sheaf of $R$
- The Haefliger groupoid

\[ \text{Par Diffeo} (\mathbb{R}^n) \rightarrow \text{Open Par (Man)} \rightarrow \text{Int}(\mathbb{R}^n) \]

- Manifolds with open partial differentiable maps

- Internal étale groupoid

- The "full group"

\[ G \rightarrow \text{Set} \rightarrow \text{Grpd (Ex (Int (A)))} \rightarrow \text{full ...} \]

- "full topological group"

\[ G \rightarrow A \rightarrow \text{TopOpen} \rightarrow \text{action of group on topological space} \]
Join categories are internal partite categories in $\text{Par}(\text{locale})$ ...

Given a join category, $\mathcal{X}$, there is a fundamental functor

$$
\Theta : \mathcal{X} \rightarrow \text{Par}(\text{locale}) \; ; \; \downarrow f \mapsto \Theta(f) \uparrow f \uparrow \Theta(y) \rightarrow e
$$

This is always a hyperconnection ....

So a join inverse category corresponds precisely to a sourced étale internal partite category in $\text{Par}(\text{locale})$. 