FAST-GROWING CLONES

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**ABSTRACT CLONES**

the algebraist notion of Lawvere theory

**Def:**

\[ C = \{ C_n \in \text{Set} \mid \exists \text{ n such that} \} \]

\[ \mathcal{V} = \{ \mathcal{V}_i \in C_n \mid 0 \leq i \leq n \text{ such that} \} \]

\[ S = \{ s_{m,n} : C_m \times C_n^m \rightarrow C_n \mid m, n \text{ such that} \} \]

subject to axioms.
\[ \mathcal{C}(x) = \{ \mathcal{C}(x^n, x) \}_{n \in \mathbb{N}} \]

\[ \mathcal{\Pi} = \{ \mathcal{\Pi}_i^{(n)} \in \mathcal{C}(x^n, x) \}_{0 \leq i < n} \]

\[ \left\{ \mathcal{C}(x^m, x) \times \mathcal{C}(x^n, x)^m \right\}_{m, n \in \mathbb{N}} \]

\[ \mathcal{C}(x^m, x) \times \mathcal{C}(x^n, x^m) \rightarrow \mathcal{C}(x^n, x) \]
CONCRETE CLONES

\[ \mathcal{C}(\mathcal{X}) = \{ \mathcal{C}(x^n, x) \}_{n \in \mathbb{N}} \]

\[ \Pi = \{ \Pi_i^{(n)} \in \mathcal{C}(x^n, x) \}_{i \in \mathbb{N}, n \in \mathbb{N}} \]

\[ \left\{ \begin{array}{c}
\mathcal{C}(x^m, x) \times \mathcal{C}(x^n, x) \\
\Rightarrow \mathcal{C}(x^m, x) \times \mathcal{C}(x^n, x^m) \rightarrow \mathcal{C}(x^n, x) \end{array} \right\}_{m, n \in \mathbb{N}} \]

Universal Algebra Folklore.

Every abstract clone can be represented by a concrete clone on a set.
**ENRICHED CLONES**

**Def:** [Borceux & Day enriched theories]

Discrete finitary enriched abstract clones in a monoidal category with finite powers $S$,

$C = \{ C_n \in S \}_{n \in \mathbb{N}}$

$\mathcal{V} = \{ \langle \nu_i^n \rangle_i : I \to C_n^m \in S \}_{n,m \in \mathbb{N}}$

$S = \{ \{ S_{m,n} : C_n \otimes C_n^m \to C_n \in S \}_{m,n \in \mathbb{N}} \}$

satisfying axioms.
Def: [Power's discrete countable enriched Lawvere theories]

Discrete, countable enriched abstract clone in a monoidal category with countable powers $S$

$C = \{ C_n \in S \} \text{ new}$

$V = \{ \langle v_i^{(n)} \rangle_i : I \rightarrow C_n^m \text{ in } S \} \text{ new}$

$S = \{ S_{m,n} : C_m \otimes C_n^m \rightarrow C_n \text{ in } S \} \text{ new}$

satisfying axioms.

$\overline{\mathbb{N}} = \mathbb{N} \cup \mathbb{S} \cup \omega$
ENRICHED EMBEDDING

- Countable case.

Let $S$ be a monoidal closed category with countable powers.

\[ C_n \otimes C_\omega \xrightarrow{S_{n,\omega}} C_\omega \]

\[ C_n \rightarrow S(C_\omega^n, C_\omega) \]

provides a positive embedding.
ENRICHED EMBEDDING AND REPRESENTATION

• Countable case.

If $\mathcal{S}$ is a monoidal closed category with countable powers

$$\begin{align*}
\mathcal{C}_n \otimes \mathcal{C}_w^n & \xrightarrow{\mathcal{S}_{n,w}} \mathcal{C}_w \\
\mathcal{C}_n & \rightarrow \mathcal{S}(\mathcal{C}_w^n, \mathcal{C}_w)
\end{align*}$$

provides a positive embedding that in the presence of equalisers restricts to a positive representation.
ENRICHED EMBEDDING AND REPRESENTATION

- Finitary case

$S$ a monoidal biclosed category with countable powers, equalisers, and colimits of w-chains of sections preserved by finite powers.
ENRICHED EMBEDDING AND REPRESENTATION

- Finitary case.

\( S \) is a monoidal bi-closed category with countable powers, equalisers, and colimits of w-chains of sections preserved by finite powers.

\[
\begin{align*}
\underleftarrow{\text{Clw}(S)} & \quad \downarrow \quad \uparrow \quad \downarrow \\
- & \quad \uparrow \\
\text{Cl}(S) & \quad C
\end{align*}
\]

\[
C_w^\# = \text{colim}_n \quad C_0 \to C \to \cdots \quad \text{positively represents } C.
\]
Questions

- Are there embedding and representation theorems for general enriched abstract clones?

- Is infinitary structure necessary for embedding/representation theorems?

In particular, what about clone embeddings in the topos of finite sets?
CLONES IN FINITE SETS

- Abstract clones are monads.
- Concrete clones are double-dualisation (or continuation) monads

\[ K_R(x) = (x \Rightarrow R) \Rightarrow R \]
CLONES IN FINITE SETS

- Abstract clones are monads.
- Concrete clones are double-dualisation (or continuation) monads
  \[ K_R(X) = (X \Rightarrow R) \Rightarrow R \]

- Are there fast-growing clones?

  There is \( n \in \mathbb{N} \) with \( C_n > r(c^n) \)
THE SELECTION MONAD

\[ \text{[Escardo \& Oliva]} \]

\[ \mathcal{J}_R \mathcal{G} (x) = (x \Rightarrow R) \Rightarrow x \Rightarrow \mathcal{G} \]

\[ \mathcal{J}_R \mathcal{G} (A, B) = \mathcal{G}(R^B \times A, B) \]
THE SELECTION MONAD

[Escardo & Oliva]

\[ \mathcal{J}_R B \rightarrow K_R B \]

\[ \mathcal{J}_R (x) = (x \Rightarrow R) \Rightarrow x \Rightarrow R \]

\[ \mathcal{J}_R B(A,B) = \mathcal{C}(R^B \times A, B) \rightarrow \mathcal{C}(R^B, R^A) = K_R B(A, B) \]

(\& a ccc)
Lemma There is a distributive law

\[ T J_T \Rightarrow J_T T \]

for every strong monoid \( T \).
Lemma: There is a distributive law
\[ T J_T \Rightarrow J_T T \]
for every strong monad \( T \).

Then: The monad \( T_2 K_2 \) on finite sets is fast growing. Hence, there is no embedding theorem in the topos of finite sets.
Cor. There are monads on the topos of finite sets with free algebras that asymptotically grow faster than every iterated exponential, for any natural height, on their generators.
Cor. There are monads on the topos of finite sets with free algebras that asymptotically grow faster than every iterated exponential, for any natural height, on their generators.

**Questions**

- Algebraic/combinatorial constructions of fast-growing finite clones?
- Study of algebraic theories on finite sets.