

Edinburgh, CT2019

Adjunctions and limits for double and multiple categories

Marco Grandis
Genova

Based on a series of joint papers with R. Paré

1. Introduction

Why considering double (multiple) categories?

1. Introduction

Why considering double (multiple) categories?

- ▶ 1. **Adjunctions.** In a higher dimensional adjunction, the left adjoint is **colax** and the right one is **lax**

1. Introduction

Why considering double (multiple) categories?

- ▶ 1. **Adjunctions**. In a higher dimensional adjunction, the left adjoint is **colax** and the right one is **lax**
- ▶ - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category

1. Introduction

Why considering double (multiple) categories?

- ▶ **1. Adjunctions.** In a higher dimensional adjunction, the left adjoint is **colax** and the right one is **lax**
- ▶ - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category
- ▶ **2. Limits.** In dimension 2, many bicategories (of relations, spans, cospans, profunctors) have few limits and colimits but can be viewed as the vertical part of weak double categories **with all double limits and colimits**

1. Introduction

Why considering double (multiple) categories?

- ▶ **1. Adjunctions.** In a higher dimensional adjunction, the left adjoint is **colax** and the right one is **lax**
- ▶ - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category
- ▶ **2. Limits.** In dimension 2, many bicategories (of relations, spans, cospans, profunctors) have few limits and colimits but can be viewed as the vertical part of weak double categories **with all double limits and colimits**
- ▶ - in higher (also infinite) dimension, we have:
 - weak multiple categories of cubical spans or cospans,
 - chiral multiple categories of spans and cospans, etc.**with all multiple limits and colimits**

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F: \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \quad F(X) = X \otimes A, \quad G(Y) = \text{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F: \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \quad F(X) = X \otimes A, \quad G(Y) = \text{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

Extending F, G to **RelAb** (locally ordered 2-category) we get:

$$F' = \text{Rel}(F): \text{Rel}\mathbf{Ab} \rightarrow \text{Rel}\mathbf{Ab}, \quad F'(vu) \leq F'(v).F'(u) \quad (\text{colax})$$

$$G' = \text{Rel}(G): \text{Rel}\mathbf{Ab} \rightarrow \text{Rel}\mathbf{Ab}, \quad G'(v).G'(u) \leq G'(vu) \quad (\text{lax})$$

working on jointly-monic spans and jointly-epic cospans

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F: \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \quad F(X) = X \otimes A, \quad G(Y) = \text{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

Extending F, G to **RelAb** (locally ordered 2-category) we get:

$$F' = \text{Rel}(F): \text{Rel}\mathbf{Ab} \rightarrow \text{Rel}\mathbf{Ab}, \quad F'(vu) \leq F'(v).F'(u) \quad (\text{colax})$$

$$G' = \text{Rel}(G): \text{Rel}\mathbf{Ab} \rightarrow \text{Rel}\mathbf{Ab}, \quad G'(v).G'(u) \leq G'(vu) \quad (\text{lax})$$

working on jointly-monic spans and jointly-epic cospans

Extending the adjunction makes problems

- We cannot compose F' and G'
- What do we make of unit and counit?

3. The extension

We 'amalgamate' \mathbf{Ab} and \mathbf{RelAb} in the double category \mathbf{RelAb} :

3. The extension

We 'amalgamate' **Ab** and **RelAb** in the double category **RelAb**:

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow u & \leq & \downarrow v \\ B & \xrightarrow{g} & B' \end{array} \quad \begin{array}{l} f, g \text{ homomorphisms} \\ u, v \text{ relations} \\ gu \leq vf \quad (\text{flat cells}) \end{array}$$

3. The extension

We 'amalgamate' \mathbf{Ab} and \mathbf{RelAb} in the double category \mathbf{RelAb} :

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow u & \leq & \downarrow v \\ B & \xrightarrow{g} & B' \end{array} \quad \begin{array}{l} f, g \text{ homomorphisms} \\ u, v \text{ relations} \\ gu \leq vf \quad (\text{flat cells}) \end{array}$$

F, G can be extended to 'double functors' $\mathbf{RelAb} \rightarrow \mathbf{RelAb}$:

$$F'' = \mathbf{Rel}(F) \quad (\text{colax}), \quad G'' = \mathbf{Rel}(G) \quad (\text{lax})$$

3. The extension

We 'amalgamate' \mathbf{Ab} and \mathbf{RelAb} in the double category \mathbf{RelAb} :

$$\begin{array}{ccc}
 A & \xrightarrow{f} & A' \\
 \downarrow u & \leq & \downarrow v \\
 B & \xrightarrow{g} & B'
 \end{array}
 \quad
 \begin{array}{l}
 f, g \text{ homomorphisms} \\
 u, v \text{ relations} \\
 gu \leq vf \quad (\text{flat cells})
 \end{array}$$

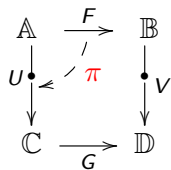
F, G can be extended to 'double functors' $\mathbf{RelAb} \rightarrow \mathbf{RelAb}$:

$$F'' = \mathbf{Rel}(F) \quad (\text{colax}), \quad G'' = \mathbf{Rel}(G) \quad (\text{lax})$$

F'' and G'' are orthogonal adjoints in \mathbf{Dbl} ($\mathbb{X} = \mathbb{A} = \mathbf{RelAb}$)

$$\begin{array}{ccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 \downarrow F'' & \eta & \parallel \\
 \mathbb{A} & \xrightarrow{G''} & \mathbb{X}
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbb{A} & \xrightarrow{G''} & \mathbb{X} \\
 \parallel & \varepsilon & \downarrow F'' \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A}
 \end{array}
 \quad
 \begin{array}{l}
 \frac{\eta}{\varepsilon} = 1_{F''} \\
 \varepsilon | \eta = e_{G''}
 \end{array}$$

4. The double category $\mathbb{D}bl$ of weak double categories



F, G lax functors

horizontal arrows

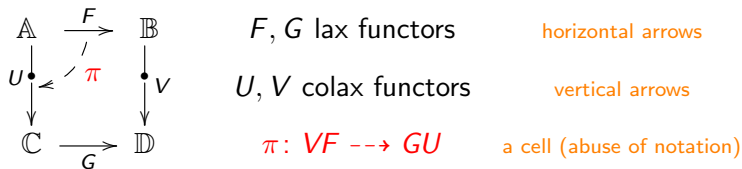
U, V colax functors

vertical arrows

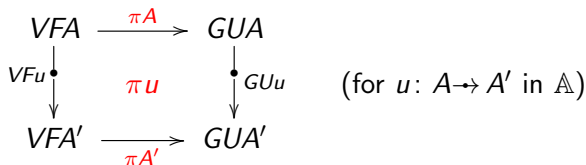
$\pi: VF \dashrightarrow GU$

a cell (abuse of notation)

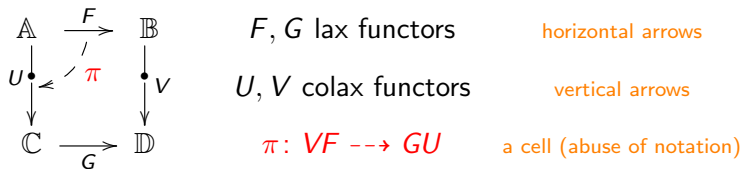
4. The double category $\mathbb{D}bl$ of weak double categories



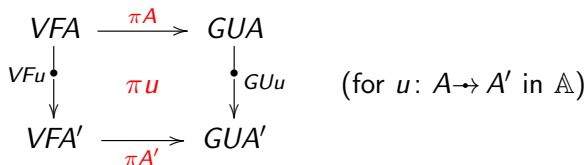
π has: horizontal maps $\pi A: VFA \rightarrow GUA$ and cells $\pi u: VFu \rightarrow GUu$ in \mathbb{D}



4. The double category $\mathbb{D}bl$ of weak double categories



π has: horizontal maps $\pi A: VFA \rightarrow GUA$ and cells $\pi u: VFu \rightarrow GUu$ in \mathbb{D}



Coherence conditions (besides naturality) for A and $w = u \otimes v$ in \mathbb{A} :

$$(\underline{VFA} | \pi e_A | \underline{GUA}) = (\underline{VFA} | e_{\pi A} | \underline{GUA})$$

$$(\underline{VF}(u, v) | \pi w | \underline{GU}(u, v)) = (\underline{V}(Fu, Fv) | (\pi u \otimes \pi v) | \underline{G}(Uu, Uv))$$

5. The second coherence condition (for vertical composition)

- based on the **laxity** comparisons \underline{F} , \underline{G} (of F , G) and the **colaxity** comparisons \underline{U} , \underline{V} (of U , V), for $w = u \otimes v$ in \mathbb{A}

$$(V\underline{F}(u, v) \mid \pi w \mid G\underline{U}(u, v)) = (\underline{V}(Fu, Fv) \mid (\pi u \otimes \pi v) \mid \underline{G}(Uu, Uv))$$

$$\begin{array}{ccccccc}
 VFA & \equiv & VFA & \longrightarrow & GUA & \equiv & GUA \\
 \downarrow V(Fu \otimes Fv) & & \downarrow VFw & \xrightarrow{\pi w} & \downarrow GUw & & \downarrow G(Uu \otimes Uv) \\
 & \underline{VF} & & & \underline{GU} & & \\
 VFA'' & \equiv & VFA'' & \longrightarrow & GUA'' & \equiv & GUA''
 \end{array}$$

$$\begin{array}{ccccccc}
 VFA & \equiv & VFA & \longrightarrow & GUA & \equiv & GUA \\
 \downarrow V(Fu \otimes Fv) & & \downarrow VFu & \xrightarrow{\pi u} & \downarrow GUu & & \downarrow G(Uu \otimes Uv) \\
 & \underline{VF} & VFA' & \longrightarrow & GUA' & \underline{GU} & \\
 & & \downarrow VFv & \xrightarrow{\pi v} & \downarrow GUv & & \\
 VFA'' & \equiv & VFA'' & \longrightarrow & GUA'' & \equiv & GUA''
 \end{array}$$

6. Double adjunctions and their composition

6. Double adjunctions and their composition

$(\eta, \varepsilon): F \dashv G$ orthogonal adjunction in \mathbb{Dbl}

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 F \downarrow \bullet & \swarrow \eta & \parallel \\
 A & \xrightarrow{G} & X
 \end{array}$$

$$\begin{array}{ccc}
 A & \xrightarrow{G} & X \\
 \parallel & \swarrow \varepsilon & \downarrow \bullet \\
 A & \xlongequal{\quad} & A
 \end{array}$$

$$\frac{\eta}{\varepsilon} = 1_F$$

$$\varepsilon | \eta = e_G$$

6. Double adjunctions and their composition

$(\eta, \varepsilon): F \dashv G$ orthogonal adjunction in \mathbb{Dbl}

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 \downarrow F & \swarrow \eta & \parallel \\
 \mathbb{A} & \xrightarrow{G} & \mathbb{X}
 \end{array} & &
 \begin{array}{ccc}
 \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\
 \parallel & \swarrow \varepsilon & \downarrow F \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A}
 \end{array}
 \end{array}
 \quad \begin{array}{l}
 \frac{\eta}{\varepsilon} = 1_F \\
 \varepsilon | \eta = e_G
 \end{array}$$

Composition with $(\eta', \varepsilon'): H \dashv K$ (pasting units and counits in \mathbb{Dbl})

6. Double adjunctions and their composition

$(\eta, \varepsilon): F \dashv G$ orthogonal adjunction in \mathbb{Dbl}

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 F \downarrow & \swarrow \eta & \parallel \\
 \mathbb{A} & \xrightarrow{G} & \mathbb{X}
 \end{array} & &
 \begin{array}{ccc}
 \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\
 \parallel & \swarrow \varepsilon & \downarrow F \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A}
 \end{array}
 \end{array}
 \quad \begin{array}{l}
 \eta \\
 \varepsilon
 \end{array} = 1_F$$

$$\varepsilon | \eta = e_G$$

Composition with $(\eta', \varepsilon'): H \dashv K$ (pasting units and counits in \mathbb{Dbl})

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 F \downarrow & \mathbf{1} & F \downarrow & \swarrow \eta & \parallel \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A} & \xrightarrow{-G} & \mathbb{X} \\
 H \downarrow & \swarrow \eta' & \parallel & \mathbf{e} & \parallel \\
 \mathbb{B} & \xrightarrow{K} & \mathbb{A} & \xrightarrow{G} & \mathbb{X}
 \end{array} & &
 \begin{array}{ccccc}
 \mathbb{B} & \xrightarrow{K} & \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\
 \parallel & \mathbf{e} & \parallel & \swarrow \varepsilon & \downarrow F \\
 \mathbb{B} & \xrightarrow{-K} & \mathbb{A} & \xlongequal{\quad} & \mathbb{A} \\
 \parallel & \swarrow \varepsilon' & \downarrow H & \mathbf{1} & \downarrow H \\
 \mathbb{B} & \xlongequal{\quad} & \mathbb{B} & \xlongequal{\quad} & \mathbb{B}
 \end{array}
 \end{array}$$

6. Double adjunctions and their composition

$(\eta, \varepsilon): F \dashv G$ orthogonal adjunction in \mathbb{Dbl}

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 F \downarrow & \swarrow \eta & \parallel \\
 \mathbb{A} & \xrightarrow{G} & \mathbb{X}
 \end{array} & &
 \begin{array}{ccc}
 \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\
 \parallel & \swarrow \varepsilon & \downarrow F \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A}
 \end{array}
 \end{array}
 \quad \begin{array}{l}
 \eta \\
 \varepsilon
 \end{array} = 1_F$$

$$\varepsilon | \eta = e_G$$

Composition with $(\eta', \varepsilon'): H \dashv K$ (pasting units and counits in \mathbb{Dbl})

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 \mathbb{X} & \xlongequal{\quad} & \mathbb{X} & \xlongequal{\quad} & \mathbb{X} \\
 F \downarrow & \mathbf{1} & F \downarrow & \swarrow \eta & \parallel \\
 \mathbb{A} & \xlongequal{\quad} & \mathbb{A} & \xrightarrow{-G} & \mathbb{X} \\
 H \downarrow & \swarrow \eta' & \parallel & \mathbf{e} & \parallel \\
 \mathbb{B} & \xrightarrow{K} & \mathbb{A} & \xrightarrow{G} & \mathbb{X}
 \end{array} & &
 \begin{array}{ccccc}
 \mathbb{B} & \xrightarrow{K} & \mathbb{A} & \xrightarrow{G} & \mathbb{X} \\
 \parallel & \mathbf{e} & \parallel & \swarrow \varepsilon & \downarrow F \\
 \mathbb{B} & \xrightarrow{-K} & \mathbb{A} & \xlongequal{\quad} & \mathbb{A} \\
 \parallel & \swarrow \varepsilon' & \downarrow H & \mathbf{1} & \downarrow H \\
 \mathbb{B} & \xlongequal{\quad} & \mathbb{B} & \xlongequal{\quad} & \mathbb{B}
 \end{array}
 \end{array}$$

(Pseudo-lax adjunction: in the 2-category $Lx\mathbb{Dbl} = \mathbf{Hor}\mathbb{Dbl}$)

(Colax-pseudo adjunction: in the 2-category $Cx\mathbb{Dbl} = \mathbf{Ver}^*\mathbb{Dbl}$)

7. Limits and colimits

The double category $\mathbb{R}el\mathbf{Ab}$ has all double limits and colimits
(the 2-category \mathbf{RelAb} even lacks products and a terminal object)

7. Limits and colimits

The double category $\mathbb{R}el\mathbf{Ab}$ has all double limits and colimits
(the 2-category \mathbf{RelAb} even lacks products and a terminal object)

A family of relations $u_i: A_i \rightarrow B_i$ ($i \in I$) has an obvious product

$$u: A \rightarrow B, \quad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

7. Limits and colimits

The double category \mathbf{RelAb} has all double limits and colimits (the 2-category \mathbf{RelAb} even lacks products and a terminal object)

A family of relations $u_i: A_i \rightarrow B_i$ ($i \in I$) has an obvious product

$$u: A \rightarrow B, \quad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{array}{ccc} A & \xrightarrow{p_i} & A_i \\ \downarrow u & \llcorner & \downarrow u_i \\ B & \xrightarrow{q_i} & B_i \end{array} \quad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

7. Limits and colimits

The double category \mathbf{RelAb} has all double limits and colimits (the 2-category \mathbf{RelAb} even lacks products and a terminal object)

A family of relations $u_i: A_i \rightarrow B_i$ ($i \in I$) has an obvious product

$$u: A \rightarrow B, \quad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{array}{ccc} A & \xrightarrow{p_i} & A_i \\ \downarrow u & \llcorner & \downarrow u_i \\ B & \xrightarrow{q_i} & B_i \end{array} \quad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

vertical arrows are viewed as higher-dimensional objects.

7. Limits and colimits

The double category \mathbf{RelAb} has all double limits and colimits (the 2-category \mathbf{RelAb} even lacks products and a terminal object)

A family of relations $u_i: A_i \rightarrow B_i$ ($i \in I$) has an obvious product

$$u: A \rightarrow B, \quad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{array}{ccc} A & \xrightarrow{p_i} & A_i \\ \downarrow u & \llcorner & \downarrow u_i \\ B & \xrightarrow{q_i} & B_i \end{array} \quad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

vertical arrows are viewed as higher-dimensional objects.

The existence of functorial double products in \mathbf{RelAb} means that:

- the category of horizontal arrows (i.e. \mathbf{Ab}) has products,
- the category of vertical arrows and double cells has products,
- these solutions agree w.r.t. vertical (co)domain and identities.

8. Another adjunction which only makes sense in $\mathbb{D}bl$

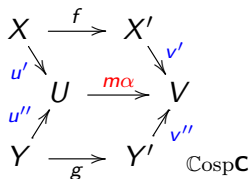
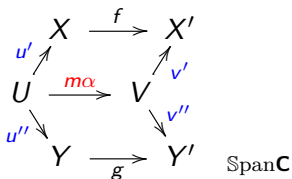
8. Another adjunction which only makes sense in $\mathbb{D}bl$

The weak double categories $\mathbf{Span}\mathbf{C}$ and $\mathbf{Cosp}\mathbf{C}$:

$$\alpha: u \rightarrow v: V \rightarrow \mathbf{C}$$

$$\alpha: u \rightarrow v: \wedge \rightarrow \mathbf{C}$$

on a category \mathbf{C} with pullbacks and pushouts



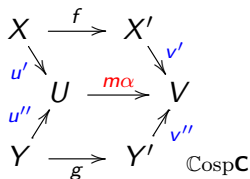
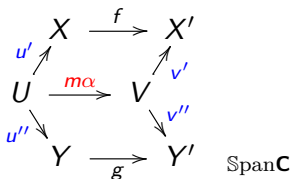
8. Another adjunction which only makes sense in $\mathbb{D}bl$

The weak double categories $\mathbf{Span}\mathbf{C}$ and $\mathbf{Cosp}\mathbf{C}$:

$$\alpha: u \rightarrow v: V \rightarrow \mathbf{C}$$

$$\alpha: u \rightarrow v: \wedge \rightarrow \mathbf{C}$$

on a category \mathbf{C} with pullbacks and pushouts



The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F: \mathbf{Span}\mathbf{C} \rightleftarrows \mathbf{Cosp}\mathbf{C} : G \quad (F \text{ colax}, G \text{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

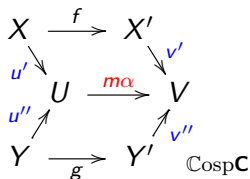
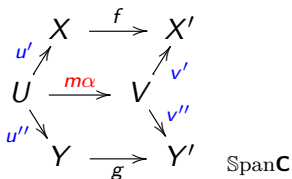
8. Another adjunction which only makes sense in $\mathbb{D}bl$

The weak double categories $\mathbf{Span}\mathbf{C}$ and $\mathbf{Cosp}\mathbf{C}$:

$$\alpha: u \rightarrow v: V \rightarrow \mathbf{C}$$

$$\alpha: u \rightarrow v: \wedge \rightarrow \mathbf{C}$$

on a category \mathbf{C} with pullbacks and pushouts



The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F: \mathbf{Span}\mathbf{C} \rightleftarrows \mathbf{Cosp}\mathbf{C} : G \quad (F \text{ colax}, G \text{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

We cannot compose them: we have an adjunction in $\mathbb{D}bl$

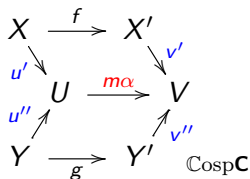
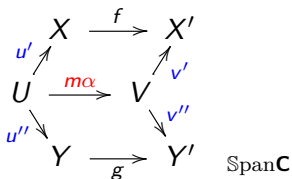
8. Another adjunction which only makes sense in $\mathbb{D}bl$

The weak double categories $\mathbf{Span}\mathbf{C}$ and $\mathbf{Cosp}\mathbf{C}$:

$$\alpha: u \rightarrow v: V \rightarrow \mathbf{C}$$

$$\alpha: u \rightarrow v: \wedge \rightarrow \mathbf{C}$$

on a category \mathbf{C} with pullbacks and pushouts



The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F: \mathbf{Span}\mathbf{C} \rightleftarrows \mathbf{Cosp}\mathbf{C} : G \quad (F \text{ colax}, G \text{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

We cannot compose them: we have an adjunction in $\mathbb{D}bl$

Restricting to the bicategories: still an adjunction in $\mathbb{D}bl$.

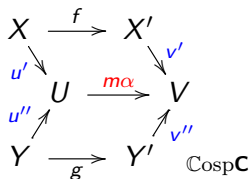
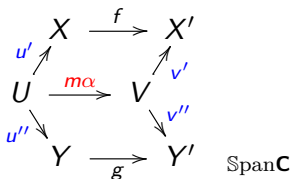
8. Another adjunction which only makes sense in $\mathbb{D}bl$

The weak double categories $\mathbf{Span}\mathbf{C}$ and $\mathbf{Cosp}\mathbf{C}$:

$$\alpha: u \rightarrow v: V \rightarrow \mathbf{C}$$

$$\alpha: u \rightarrow v: \wedge \rightarrow \mathbf{C}$$

on a category \mathbf{C} with pullbacks and pushouts



The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F: \mathbf{Span}\mathbf{C} \rightleftarrows \mathbf{Cosp}\mathbf{C} : G \quad (F \text{ colax}, G \text{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

We cannot compose them: we have an adjunction in $\mathbb{D}bl$

Restricting to the bicategories: still an adjunction in $\mathbb{D}bl$.

($\mathbf{Span}\mathbf{C}$ has all double (co)limits, if \mathbf{C} has (co)limits)

9. The pushout-pullback adjunction and abelian relations

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if \mathbf{C} is abelian, the po/pb adjunction is strong (left and right)

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if \mathbf{C} is abelian, the po/pb adjunction is strong (left and right)

→ idempotent lax monad T on $\text{Span}\mathbf{C}$: $\text{Alg}(T) = \text{Rel}\mathbf{C}$

→ idempotent colax comonad S on $\text{Cosp}\mathbf{C}$: $\text{Coalg}(S) = \text{Rel}\mathbf{C}$

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if \mathbf{C} is abelian, the po/pb adjunction is strong (left and right)
 - idempotent lax monad T on $\text{Span}\mathbf{C}$: $\text{Alg}(T) = \text{Rel}\mathbf{C}$
 - idempotent colax comonad S on $\text{Cosp}\mathbf{C}$: $\text{Coalg}(S) = \text{Rel}\mathbf{C}$
- if $\mathbf{C} = \mathbf{Set}$, the pushout-pullback adjunction is not strong

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

- (i) the comparison cells of F are made invertible by applying G
- (ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if \mathbf{C} is abelian, the po/pb adjunction is strong (left and right)

→ idempotent lax monad T on $\text{Span}\mathbf{C}$: $\text{Alg}(T) = \text{Rel}\mathbf{C}$

→ idempotent colax comonad S on $\text{Cosp}\mathbf{C}$: $\text{Coalg}(S) = \text{Rel}\mathbf{C}$

- if $\mathbf{C} = \mathbf{Set}$, the pushout-pullback adjunction is not strong

($\text{Rel}\mathbf{Set} = \text{Alg}(T')$ for the jointly-monic monad T' on $\text{Span}\mathbf{Set}$)

10. Weak multiple categories

10. Weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

10. Weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

- an n -dimensional cube is a functor $x: \mathcal{V}^n \rightarrow \mathbf{Set}$

with $2n$ faces: $\partial_i^\alpha x: \mathcal{V}^{n-1} \rightarrow \mathbf{Set}$ ($i = 1, \dots, n; \alpha = \pm$)

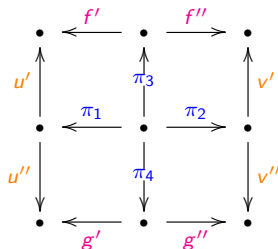
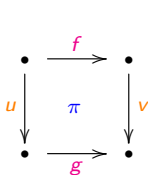
10. Weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

- an n -dimensional cube is a functor $x: \mathbb{V}^n \rightarrow \mathbf{Set}$

with $2n$ faces: $\partial_i^\alpha x: \mathbb{V}^{n-1} \rightarrow \mathbf{Set}$ ($i = 1, \dots, n; \alpha = \pm$)

- for instance $\pi: \mathbb{V}^2 \rightarrow \mathbf{Set}$ has four faces f, g, u, v (spans)



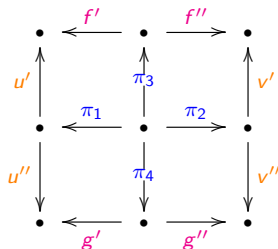
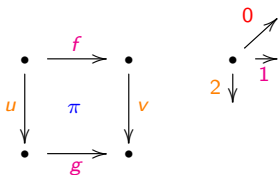
10. Weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

- an n -dimensional cube is a functor $x: \mathbb{V}^n \rightarrow \mathbf{Set}$

with $2n$ faces: $\partial_i^\alpha x: \mathbb{V}^{n-1} \rightarrow \mathbf{Set}$ ($i = 1, \dots, n; \alpha = \pm$)

- for instance $\pi: \mathbb{V}^2 \rightarrow \mathbf{Set}$ has four faces f, g, u, v (spans)



- transversal map $\varphi: x \rightarrow y: \mathbb{V}^n \rightarrow \mathbf{Set}$ (a natural transformation)

- they compose strictly (in direction 0, the transversal direction)

- and give the comparisons for the i -composition of n -cubes.

11. Extending adjunctions to weak multiple categories

11. Extending adjunctions to weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

11. Extending adjunctions to weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

The weak multiple category **CospSet** (of cubical type)

11. Extending adjunctions to weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

The weak multiple category **CospSet** (of cubical type)

- an n -dimensional cube is a functor $x: \Lambda^n \rightarrow \mathbf{Set}$
- a transversal map $\varphi: x \rightarrow y$ is a natural transformation

11. Extending adjunctions to weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

The weak multiple category **CospSet** (of cubical type)

- an n -dimensional cube is a functor $x: \Lambda^n \rightarrow \mathbf{Set}$

- a transversal map $\varphi: x \rightarrow y$ is a natural transformation

The colax-lax multiple adjunction, by pushouts and pullbacks:

$$F: \mathbf{SpanSet} \rightleftarrows \mathbf{CospSet} : G \quad (F \text{ colax}, G \text{ lax})$$

which lives in a double category of weak multiple categories.

11. Extending adjunctions to weak multiple categories

The weak multiple category **SpanSet** (of cubical type)

The weak multiple category **CospSet** (of cubical type)

- an n -dimensional cube is a functor $x: \wedge^n \rightarrow \mathbf{Set}$

- a transversal map $\varphi: x \rightarrow y$ is a natural transformation

The colax-lax multiple adjunction, by pushouts and pullbacks:

$$F: \mathbf{SpanSet} \rightleftarrows \mathbf{CospSet} : G \quad (F \text{ colax}, G \text{ lax})$$

which lives in a double category of weak multiple categories.

(**SpanSet** and **CospSet** have all multiple limits and colimits.)

12. Hints at **chiral** multiple categories (in dim. 3)

12. Hints at **chiral** multiple categories (in dim. 3)

The chiral triple category $SC(\mathbf{C})$ (not of cubical type)
for a category \mathbf{C} with pullbacks and pushouts

12. Hints at **chiral** multiple categories (in dim. 3)

The chiral triple category $SC(\mathbf{C})$ (not of cubical type)

for a category \mathbf{C} with pullbacks and pushouts

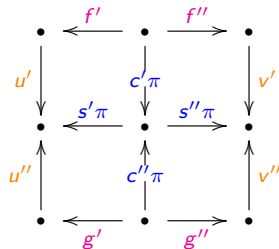
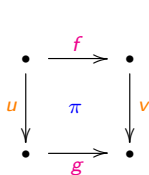
- a 1-cell $f: \vee \rightarrow \mathbf{C}$ is a span, a 2-cell $u: \wedge \rightarrow \mathbf{C}$ is a cospan,
- a 12-cell $\pi: \vee \times \wedge \rightarrow \mathbf{C}$ is a span of cospans (or a cospan of spans)

12. Hints at **chiral** multiple categories (in dim. 3)

The chiral triple category $\mathbf{SC}(\mathbf{C})$ (not of cubical type)

for a category \mathbf{C} with pullbacks and pushouts

- a 1-cell $f: \vee \rightarrow \mathbf{C}$ is a span, a 2-cell $u: \wedge \rightarrow \mathbf{C}$ is a cospan,
- a 12-cell $\pi: \vee \times \wedge \rightarrow \mathbf{C}$ is a span of cospans (or a cospan of spans)

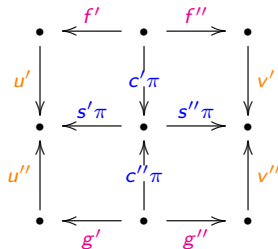
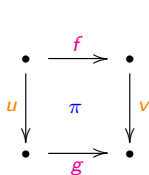


12. Hints at **chiral** multiple categories (in dim. 3)

The chiral triple category $SC(\mathbf{C})$ (not of cubical type)

for a category \mathbf{C} with pullbacks and pushouts

- a 1-cell $f: \mathbb{V} \rightarrow \mathbf{C}$ is a span, a 2-cell $u: \mathbb{\Lambda} \rightarrow \mathbf{C}$ is a cospan,
- a 12-cell $\pi: \mathbb{V} \times \mathbb{\Lambda} \rightarrow \mathbf{C}$ is a span of cospans (or a cospan of spans)



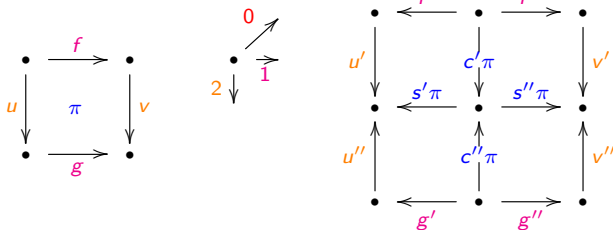
- a transversal map $\varphi: \pi \rightarrow \rho$ is a natural transformation (dim. 3)

12. Hints at **chiral** multiple categories (in dim. 3)

The chiral triple category $\mathbf{SC}(\mathbf{C})$ (not of cubical type)

for a category \mathbf{C} with pullbacks and pushouts

- a 1-cell $f: \mathbb{V} \rightarrow \mathbf{C}$ is a span, a 2-cell $u: \mathbb{\Lambda} \rightarrow \mathbf{C}$ is a cospan,
- a 12-cell $\pi: \mathbb{V} \times \mathbb{\Lambda} \rightarrow \mathbf{C}$ is a span of cospans (or a cospan of spans)



- a transversal map $\varphi: \pi \rightarrow \rho$ is a natural transformation (dim. 3)

Directed interchange for 1- and 2-directed compositions

$$\chi(\pi, \pi', \rho, \rho'): (\pi +_1 \pi') +_2 (\rho +_1 \rho') \rightarrow (\pi +_2 \rho) +_1 (\pi' +_2 \rho').$$

13. Extending adjunctions to chiral multiple categories

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F: \text{Span}_3(\mathbf{C}) \rightleftarrows \text{Cosp}_3(\mathbf{C}) : G \quad F \dashv G$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F: \text{Span}_3(\mathbf{C}) \rightleftarrows \text{Cosp}_3(\mathbf{C}) : G \quad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\text{Span}_3(\mathbf{C}) \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} \text{SC}(\mathbf{C}) \begin{array}{c} \xrightarrow{F''} \\ \xleftarrow{G''} \end{array} \text{Cosp}_3(\mathbf{C})$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F : \text{Span}_3(\mathbf{C}) \rightleftarrows \text{Cosp}_3(\mathbf{C}) : G \quad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\text{Span}_3(\mathbf{C}) \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} \text{SC}(\mathbf{C}) \begin{array}{c} \xrightarrow{F''} \\ \xleftarrow{G''} \end{array} \text{Cosp}_3(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $S_{-\infty}C_{\infty}(\mathbf{C})$

- spans of \mathbf{C} in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F: \text{Span}_3(\mathbf{C}) \rightleftarrows \text{Cosp}_3(\mathbf{C}) : G \quad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\text{Span}_3(\mathbf{C}) \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} \text{SC}(\mathbf{C}) \begin{array}{c} \xrightarrow{F''} \\ \xleftarrow{G''} \end{array} \text{Cosp}_3(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $S_{-\infty}C_{\infty}(\mathbf{C})$

- spans of \mathbf{C} in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

Multiple adjunctions of chiral multiple categories:

$$\text{Span}_{\mathbb{Z}}(\mathbf{C}) \rightleftarrows S_{-\infty}C_{\infty}(\mathbf{C}) \rightleftarrows \text{Cosp}_{\mathbb{Z}}(\mathbf{C})$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F: \text{Span}_3(\mathbf{C}) \rightleftarrows \text{Cosp}_3(\mathbf{C}) : G \quad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\text{Span}_3(\mathbf{C}) \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{G'} \end{array} \text{SC}(\mathbf{C}) \begin{array}{c} \xrightarrow{F''} \\ \xleftarrow{G''} \end{array} \text{Cosp}_3(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $S_{-\infty}C_{\infty}(\mathbf{C})$

- spans of \mathbf{C} in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

Multiple adjunctions of chiral multiple categories:

$$\text{Span}_{\mathbb{Z}}(\mathbf{C}) \rightleftarrows S_{-\infty}C_{\infty}(\mathbf{C}) \rightleftarrows \text{Cosp}_{\mathbb{Z}}(\mathbf{C})$$

in a double category of chiral categories, lax and colax functors.

14. Strict multiple categories (A. & C. Ehresmann)

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\mathbf{M}^{\text{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, \dots, i_n\} \subset \mathbb{N}$
- faces: $\partial_j^\alpha : A_{\mathbf{i}} \rightarrow A_{\mathbf{i}|j}$, for $i \in \mathbf{i}$, $\alpha = \pm$ $(\mathbf{i}|j = \mathbf{i} \setminus \{i\})$
- degeneracies: $e_j : A_{\mathbf{i}|j} \rightarrow A_{\mathbf{i}}$, for $i \in \mathbf{i}$

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\mathbf{M}^{\text{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, \dots, i_n\} \subset \mathbb{N}$
- faces: $\partial_i^\alpha : A_{\mathbf{i}} \rightarrow A_{\mathbf{i}|i}$, for $i \in \mathbf{i}$, $\alpha = \pm$ ($\mathbf{i}|i = \mathbf{i} \setminus \{i\}$)
- degeneracies: $e_j : A_{\mathbf{i}|j} \rightarrow A_{\mathbf{i}}$, for $i \in \mathbf{i}$

under the multiple relations ($i \neq j$):

$$\begin{aligned} \partial_i^\alpha \cdot \partial_j^\beta &= \partial_j^\beta \cdot \partial_i^\alpha, & e_i \cdot e_j &= e_j \cdot e_i, \\ \partial_i^\alpha \cdot e_j &= e_j \cdot \partial_i^\alpha, & \partial_i^\alpha \cdot e_i &= \text{id}. \end{aligned}$$

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\mathbf{M}^{\text{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, \dots, i_n\} \subset \mathbb{N}$
- faces: $\partial_i^\alpha : A_{\mathbf{i}} \rightarrow A_{\mathbf{i}|i}$, for $i \in \mathbf{i}$, $\alpha = \pm$ ($\mathbf{i}|i = \mathbf{i} \setminus \{i\}$)
- degeneracies: $e_i : A_{\mathbf{i}|i} \rightarrow A_{\mathbf{i}}$, for $i \in \mathbf{i}$

under the multiple relations ($i \neq j$):

$$\begin{aligned}\partial_i^\alpha \cdot \partial_j^\beta &= \partial_j^\beta \cdot \partial_i^\alpha, & e_i \cdot e_j &= e_j \cdot e_i, \\ \partial_i^\alpha \cdot e_j &= e_j \cdot \partial_i^\alpha, & \partial_i^\alpha \cdot e_i &= \text{id}.\end{aligned}$$

Strict multiple category A :

a multiple set with (strict) compositions and interchange

- i -composition: $x +_i y$, for $i \in \mathbf{i}$, $x, y \in A_{\mathbf{i}}$, $\partial_i^+ x = \partial_i^- y$.

15. Multiple categories, weak and lax

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i -composition in a *geometric* direction $i > 0$:
categorical up to transversal invertible comparisons

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i -composition in a *geometric direction* $i > 0$:
 - categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij -interchanger: a transversal **invertible** comparison ($0 < i < j$)

$$\chi_{ij}(\pi, \pi', \rho, \rho'): (\pi +_i \pi') +_j (\rho +_i \rho') \rightarrow (\pi +_j \rho) +_i (\pi' +_j \rho')$$

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i -composition in a *geometric direction* $i > 0$:
 - categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij -interchanger: a transversal **invertible** comparison ($0 < i < j$)
 $\chi_{ij}(\pi, \pi', \rho, \rho'): (\pi +_i \pi') +_j (\rho +_i \rho') \rightarrow (\pi +_j \rho) +_i (\pi' +_j \rho')$

Chiral multiple category (partially lax):

- ij -interchanger: a transversal **directed** comparison ($0 < i < j$)

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i -composition in a *geometric direction* $i > 0$:
categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij -interchanger: a transversal **invertible** comparison ($0 < i < j$)
 $\chi_{ij}(\pi, \pi', \rho, \rho'): (\pi +_i \pi') +_j (\rho +_i \rho') \rightarrow (\pi +_j \rho) +_i (\pi' +_j \rho')$

Chiral multiple category (partially lax):

- ij -interchanger: a transversal **directed** comparison ($0 < i < j$)

Intercategory (a laxer version):

- four ij -interchangers, for binary and zeroary compositions in directions $i < j$

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i -composition in a *geometric direction* $i > 0$:
 - categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij -interchanger: a transversal **invertible** comparison ($0 < i < j$)
 $\chi_{ij}(\pi, \pi', \rho, \rho'): (\pi +_i \pi') +_j (\rho +_i \rho') \rightarrow (\pi +_j \rho) +_i (\pi' +_j \rho')$

Chiral multiple category (partially lax):

- ij -interchanger: a transversal **directed** comparison ($0 < i < j$)

Intercategory (a laxer version):

- four ij -interchangers, for binary and zeroary compositions in directions $i < j$

in dimension 3, intercategories include:

duoidal categories, monoidal double categories, cubical bicategories, double bicategories, Gray categories

16. The multiple site M

16. The multiple site \underline{M}

The **multiple site** \underline{M} has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (elements: $t: \mathbf{i} \rightarrow 2$).

16. The multiple site \underline{M}

The **multiple site** \underline{M} has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (**elements**: $t: \mathbf{i} \rightarrow 2$).

The category $\underline{M} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

$$\text{faces:} \quad \partial_i^\alpha: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (\partial_i^\alpha t)(j) = t(j), \quad \partial_i^\alpha(t)(i) = \alpha$$

$$\text{degeneracies:} \quad e_j: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (e_j t)(j) = t(j)$$

16. The multiple site $\underline{\mathbf{M}}$

The **multiple site** $\underline{\mathbf{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (**elements**: $t: \mathbf{i} \rightarrow 2$).

The category $\underline{\mathbf{M}} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

$$\text{faces:} \quad \partial_i^\alpha: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (\partial_i^\alpha t)(j) = t(j), \quad \partial_i^\alpha(t)(i) = \alpha$$

$$\text{degeneracies:} \quad e_j: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (e_j t)(j) = t(j)$$

under the **comultiple relations** ($i \neq j$):

$$\partial_i^\alpha \cdot \partial_j^\beta = \partial_j^\beta \cdot \partial_i^\alpha, \quad e_i \cdot e_j = e_j \cdot e_i,$$

$$\partial_i^\alpha \cdot e_j = e_j \cdot \partial_i^\alpha, \quad e_i \cdot \partial_i^\alpha = \text{id}.$$

16. The multiple site $\underline{\mathbf{M}}$

The **multiple site** $\underline{\mathbf{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (**elements**: $t: \mathbf{i} \rightarrow 2$).

The category $\underline{\mathbf{M}} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

$$\text{faces:} \quad \partial_i^\alpha: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (\partial_i^\alpha t)(j) = t(j), \quad \partial_i^\alpha(t)(i) = \alpha$$

$$\text{degeneracies:} \quad e_j: 2^{\mathbf{i}|i} \rightarrow 2^{\mathbf{i}}, \quad (e_j t)(j) = t(j)$$

under the **comultiple relations** ($i \neq j$):

$$\partial_i^\alpha \cdot \partial_j^\beta = \partial_j^\beta \cdot \partial_i^\alpha, \quad e_i \cdot e_j = e_j \cdot e_i,$$

$$\partial_i^\alpha \cdot e_j = e_j \cdot \partial_i^\alpha, \quad e_i \cdot \partial_i^\alpha = \text{id}.$$

(The **cubical site** has objects 2^n ; **its indices must be normalised.**)

17. References for double and multiple categories

- [1] M. Grandis – R. Paré, Limits in double categories, Cah. Topol. Géom. Différ. Catég. 40 (1999), 162–220.
- [2] –, Adjoint for double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 193–240.
- [3] –, Intercategories: a framework for three dimensional category theory, J. Pure Appl. Algebra 221 (2017), 999–1054.
- [4] –, An introduction to multiple categories (On weak and lax multiple categories, I), Cah. Topol. Géom. Différ. Catég. 57 (2016), 103–159.
- [5] –, Limits in multiple categories (On weak and lax multiple categories, II), Cah. Topol. Géom. Différ. Catég. 57 (2016), 163–202.
- [6] –, Adjoints for multiple categories (On weak and lax multiple categories, III), Cah. Topol. Géom. Différ. Catég. 58 (2017), 3–48.
- [7] –, A multiple category of lax multiple categories, Cah. Topol. Géom. Différ. Catég. 58 (2017), 195–212.