Edinburgh, CT2019

Adjunctions and limits for double and multiple categories

Marco Grandis
Genova

Based on a series of joint papers with R. Paré

1. Introduction

Why considering double (multiple) categories?

1. Introduction

Why considering double (multiple) categories?

- 1. Adjunctions. In a higher dimensional adjunction, the left adjoint is colax and the right one is lax

1. Introduction

Why considering double (multiple) categories?

- 1. Adjunctions. In a higher dimensional adjunction, the left adjoint is colax and the right one is lax
- - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category

1. Introduction

Why considering double (multiple) categories?

- 1. Adjunctions. In a higher dimensional adjunction, the left adjoint is colax and the right one is lax
- - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category

- 2. Limits. In dimension 2, many bicategories (of relations, spans, cospans, profunctors) have few limits and colimits but can be viewed as the vertical part of weak double categories with all double limits and colimits

1. Introduction

Why considering double (multiple) categories?

- 1. Adjunctions. In a higher dimensional adjunction, the left adjoint is colax and the right one is lax
- - they cannot be composed, but can be viewed as a vertical or horizontal arrow, in a double category

- 2. Limits. In dimension 2, many bicategories (of relations, spans, cospans, profunctors) have few limits and colimits but can be viewed as the vertical part of weak double categories with all double limits and colimits
- - in higher (also infinite) dimension, we have:

  – weak multiple categories of cubical spans or cospans,

  – chiral multiple categories of spans and cospans, etc.

  with all multiple limits and colimits

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F \colon \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \qquad F(X) = X \otimes A, \quad G(Y) = \mathrm{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F\colon \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \qquad F(X) = X \otimes A, \quad G(Y) = \mathrm{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

Extending F, G to $\mathrm{Rel}\mathbf{Ab}$ (locally ordered 2-category) we get:

$$F' = \mathrm{Rel}(F)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad F'(vu) \leqslant F'(v).F'(u) \quad (\textit{colax})$$

$$G' = \mathrm{Rel}(G)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad G'(v).G'(u) \leqslant G'(vu) \quad (\textit{lax})$$

working on jointly-monic spans and jointly-epic cospans

2. A problem with adjunctions

Exponential law $F \dashv G$ in **Ab**, for a fixed abelian group A

$$F\colon \mathbf{Ab} \rightleftarrows \mathbf{Ab} : G, \qquad F(X) = X \otimes A, \quad G(Y) = \mathrm{Hom}(A, Y)$$

or any adjunction $F \dashv G$ between abelian categories

Extending F, G to $\mathrm{Rel}\mathbf{Ab}$ (locally ordered 2-category) we get:

$$F' = \mathrm{Rel}(F)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad F'(vu) \leqslant F'(v).F'(u) \quad (\textit{colax})$$

$$G' = \mathrm{Rel}(G)\colon \mathrm{Rel}\mathbf{Ab} \to \mathrm{Rel}\mathbf{Ab}, \quad G'(v).G'(u) \leqslant G'(vu) \quad (\textit{lax})$$

working on jointly-monic spans and jointly-epic cospans

Extending the adjunction makes problems

- We cannot compose F' and G'
- What do we make of unit and counit?

3. The extension

We ‘amalgamate’ **Ab** and $\mathrm{Rel}\mathbf{Ab}$ in the double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$:

3. The extension

We 'amalgamate' **Ab** and $\mathrm{Rel}\mathbf{Ab}$ in the double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"', "\bullet" marking] \arrow[dr, phantom, "\leqslant"] & A' \arrow[d, "v", "\bullet" marking] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

f, g homomorphisms

u, v relations

$gu \leqslant vf$ (*flat cells*)

3. The extension

We 'amalgamate' **Ab** and $\mathbb{R}\mathrm{el}$**Ab** in the double category $\mathbb{R}\mathrm{el}$**Ab**:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\leqslant"] & A' \arrow[d, "v"] \\ B \arrow[r, "g"'] & B' \end{tikzcd} \qquad \begin{array}{c} f, g \text{ homomorphisms} \\ u, v \text{ relations} \\ gu \leqslant vf \quad (\textit{flat cells}) \end{array}$$

F, G can be extended to 'double functors' $\mathbb{R}\mathrm{el}\mathbf{Ab} \to \mathbb{R}\mathrm{el}\mathbf{Ab}$:

$$F'' = \mathbb{R}\mathrm{el}(F) \quad (\textit{colax}), \qquad G'' = \mathbb{R}\mathrm{el}(G) \quad (\textit{lax})$$

3. The extension

We 'amalgamate' **Ab** and $\mathrm{Rel}\mathbf{Ab}$ in the double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\leqslant"] & A' \arrow[d, "v"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

f, g homomorphisms

u, v relations

$gu \leqslant vf$ (*flat cells*)

F, G can be extended to 'double functors' $\mathbb{R}\mathrm{el}\mathbf{Ab} \to \mathbb{R}\mathrm{el}\mathbf{Ab}$:

$$F'' = \mathbb{R}\mathrm{el}(F) \quad (\mathit{colax}), \qquad G'' = \mathbb{R}\mathrm{el}(G) \quad (\mathit{lax})$$

F'' and G'' are *orthogonal adjoints* in $\mathbb{D}\mathrm{bl}$ $(\mathbb{X} = \mathbb{A} = \mathbb{R}\mathrm{el}\mathbf{Ab})$

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "F''"'] \arrow[dr, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{A} \arrow[r, "G''"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "G''"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathbb{X} \arrow[d, "F''"] \\ \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd}$$

$$\frac{\eta}{\varepsilon} = 1_{F''}$$

$$\varepsilon \,|\, \eta = e_{G''}$$

4. The double category $\mathbb{D}\mathrm{bl}$ of weak double categories

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "F"] \arrow[d, "U"'] \arrow[dr, phantom, "\pi"] & \mathbb{B} \arrow[d, "V"] \\
\mathbb{C} \arrow[r, "G"'] & \mathbb{D}
\end{tikzcd}
$$

F, G lax functors	horizontal arrows
U, V colax functors	vertical arrows
$\pi \colon VF \dashrightarrow GU$	a cell (abuse of notation)

4. The double category $\mathbb{D}\mathrm{bl}$ of weak double categories

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "U"'] & \mathbb{B} \arrow[d, "V"] \arrow[dl, phantom, "\color{red}\pi"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{D} \end{tikzcd}$$

F, G lax functors — horizontal arrows

U, V colax functors — vertical arrows

$\color{red}\pi\colon VF \dashrightarrow GU$ — a cell (abuse of notation)

$\color{red}\pi$ has: horizontal maps $\color{red}\pi A$: $VFA \to GUA$ and cells $\color{red}\pi u$: $VFu \to GUu$ in $\mathbb{D}$

$$\begin{tikzcd} VFA \arrow[r, "\color{red}\pi A"] \arrow[d, "VFu"'] \arrow[dr, phantom, "\color{red}\pi u"] & GUA \arrow[d, "GUu"] \\ VFA' \arrow[r, "\color{red}\pi A'"'] & GUA' \end{tikzcd}$$

(for $u\colon A \to A'$ in $\mathbb{A}$)

4. The double category $\mathbb{D}\mathrm{bl}$ of weak double categories

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "U"'] \arrow[dr, phantom, "\pi"] & \mathbb{B} \arrow[d, "V"] \\ \mathbb{C} \arrow[r, "G"'] & \mathbb{D} \end{tikzcd}$$

F, G lax functors — horizontal arrows

U, V colax functors — vertical arrows

$\pi \colon VF \dashrightarrow GU$ — a cell (abuse of notation)

π has: horizontal maps $\pi A \colon VFA \to GUA$ and cells $\pi u \colon VFu \to GUu$ in $\mathbb{D}$

$$\begin{tikzcd} VFA \arrow[r, "\pi A"] \arrow[d, "VFu"'] \arrow[dr, phantom, "\pi u"] & GUA \arrow[d, "GUu"] \\ VFA' \arrow[r, "\pi A'"'] & GUA' \end{tikzcd}$$

(for $u \colon A \to\!\!\!\!\bullet\; A'$ in $\mathbb{A}$)

Coherence conditions (*besides naturality*) for A and $w = u \otimes v$ in $\mathbb{A}$:

$$(V\underline{F}A \mid \pi e_A \mid G\underline{U}A) = (\underline{V}FA \mid e_{\pi A} \mid \underline{G}UA)$$

$$(V\underline{F}(u, v) \mid \pi w \mid G\underline{U}(u, v)) = (\underline{V}(Fu, Fv) \mid (\pi u \otimes \pi v) \mid \underline{G}(Uu, Uv))$$

5. The second coherence condition (for vertical composition)

- based on the laxity comparisons $\underline{F}$, $\underline{G}$ (of F, G) and the colaxity comparisons $\underline{U}$, $\underline{V}$ (of U, V), for $w = u \otimes v$ in $\mathbb{A}$

$$(V\underline{F}(u,v) \,|\, \pi w \,|\, G\underline{U}(u,v)) = (\underline{V}(Fu, Fv) \,|\, (\pi u \otimes \pi v) \,|\, \underline{G}(Uu, Uv))$$

$$\begin{tikzcd}
VFA \arrow[r, equal] \arrow[d, "V(Fu\otimes Fv)"'] \arrow[dr, phantom, "V\underline{F}"] & VFA \arrow[r] \arrow[d, "VFw"] \arrow[dr, phantom, "\pi w"] & GUA \arrow[r, equal] \arrow[d, "GUw"'] \arrow[dr, phantom, "G\underline{U}"] & GUA \arrow[d, "G(Uu\otimes Uv)"] \\
VFA'' \arrow[r, equal] & VFA'' \arrow[r] & GUA'' \arrow[r, equal] & GUA''
\end{tikzcd}$$

$$\begin{tikzcd}
VFA \arrow[r, equal] \arrow[dd, "V(Fu\otimes Fv)"'] & VFA \arrow[r] \arrow[d, "VFu"] \arrow[dr, phantom, "\pi u"] & GUA \arrow[r, equal] \arrow[d, "GUu"'] & GUA \arrow[dd, "G(Uu\otimes Uv)"] \\
{} \arrow[r, phantom, "\underline{V}F"] & VFA' \arrow[r] \arrow[d, "VFv"] \arrow[dr, phantom, "\pi v"] & GUA' \arrow[d, "GUv"'] \arrow[r, phantom, "\underline{G}U"] & {} \\
VFA'' \arrow[r, equal] & VFA'' \arrow[r] & GUA'' \arrow[r, equal] & GUA''
\end{tikzcd}$$

6. Double adjunctions and their composition

6. Double adjunctions and their composition

(η, ε): $F \dashv G$ *orthogonal adjunction* in $\mathbb{D}\mathrm{bl}$

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{A} \arrow[r, "G"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathbb{X} \arrow[d, "F"] \\ \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd}$$

$$\frac{\eta}{\varepsilon} = 1_F$$

$$\varepsilon \mid \eta = e_G$$

6. Double adjunctions and their composition

$(\eta, \varepsilon)\colon F \dashv G$ *orthogonal adjunction* in $\mathbb{D}\mathrm{bl}$

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{A} \arrow[r, "G"'] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathbb{X} \arrow[d, "F"] \\
\mathbb{A} \arrow[r, equal] & \mathbb{A}
\end{tikzcd}
\qquad
\begin{array}{c}
\dfrac{\eta}{\varepsilon} = 1_F \\[2ex]
\varepsilon \,|\, \eta = e_G
\end{array}
$$

Composition with $(\eta', \varepsilon')\colon H \dashv K$ (pasting units and counits in $\mathbb{D}\mathrm{bl}$)

6. Double adjunctions and their composition

$(\eta, \varepsilon) \colon F \dashv G$ *orthogonal adjunction* in $\mathbb{D}\mathrm{bl}$

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{A} \arrow[r, "G"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathbb{X} \arrow[d, "F"] \\ \mathbb{A} \arrow[r, equal] & \mathbb{A} \end{tikzcd} \qquad \frac{\eta}{\varepsilon} = 1_F \qquad \varepsilon \,|\, \eta = e_G$$

Composition with $(\eta', \varepsilon') \colon H \dashv K$ (pasting units and counits in $\mathbb{D}\mathrm{bl}$)

$$\begin{tikzcd} \mathbb{X} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "1"] & \mathbb{X} \arrow[r, equal] \arrow[d, "F"'] \arrow[dr, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{A} \arrow[r, equal] \arrow[d, "H"'] \arrow[dr, phantom, "\eta'"] & \mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "e"] & \mathbb{X} \arrow[d, equal] \\ \mathbb{B} \arrow[r, "K"'] & \mathbb{A} \arrow[r, "G"'] & \mathbb{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[d, equal] \arrow[dr, phantom, "e"] & \mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon"] & \mathbb{X} \arrow[d, "F"] \\ \mathbb{B} \arrow[r, "K"] \arrow[d, equal] \arrow[dr, phantom, "\varepsilon'"] & \mathbb{A} \arrow[r, equal] \arrow[d, "H"] \arrow[dr, phantom, "1"] & \mathbb{A} \arrow[d, "H"] \\ \mathbb{B} \arrow[r, equal] & \mathbb{B} \arrow[r, equal] & \mathbb{B} \end{tikzcd}$$

6. Double adjunctions and their composition

$(\eta, \varepsilon)\colon F \dashv G$ *orthogonal adjunction* in $\mathbb{D}\mathrm{bl}$

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "F"', "\bullet" marking] & \mathbb{X} \arrow[d, equal] \\
\mathbb{A} \arrow[r, "G"'] \arrow[ur, phantom, "\eta"] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d, equal] & \mathbb{X} \arrow[d, "F", "\bullet" marking] \\
\mathbb{A} \arrow[r, equal] \arrow[ur, phantom, "\varepsilon"] & \mathbb{A}
\end{tikzcd}
\qquad
\frac{\eta}{\varepsilon} = 1_F \qquad \varepsilon \,|\, \eta = e_G
$$

Composition with $(\eta', \varepsilon')\colon H \dashv K$ (pasting units and counits in $\mathbb{D}\mathrm{bl}$)

$$
\begin{tikzcd}
\mathbb{X} \arrow[r, equal] \arrow[d, "F"', "\bullet" marking] & \mathbb{X} \arrow[r, equal] \arrow[d, "F"', "\bullet" marking] \arrow[dl, phantom, "1"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{A} \arrow[r, equal] \arrow[d, "H"', "\bullet" marking] & \mathbb{A} \arrow[r, dashed, "G"] \arrow[d, equal] \arrow[ur, phantom, "\eta"] & \mathbb{X} \arrow[d, equal] \\
\mathbb{B} \arrow[r, "K"'] \arrow[ur, phantom, "\eta'"] & \mathbb{A} \arrow[r, "G"'] \arrow[ur, phantom, "e"] & \mathbb{X}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{B} \arrow[r, "K"] \arrow[d, equal] & \mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dl, phantom, "e"] & \mathbb{X} \arrow[d, "F", "\bullet" marking] \\
\mathbb{B} \arrow[r, dashed, "K"] \arrow[d, equal] & \mathbb{A} \arrow[r, equal] \arrow[d, "H", "\bullet" marking] \arrow[ur, phantom, "\varepsilon"] & \mathbb{A} \arrow[d, "H", "\bullet" marking] \\
\mathbb{B} \arrow[r, equal] \arrow[ur, phantom, "\varepsilon'"] & \mathbb{B} \arrow[r, equal] \arrow[ur, phantom, "1"] & \mathbb{B}
\end{tikzcd}
$$

(Pseudo-lax adjunction: in the 2-category $\mathrm{Lx}\mathbf{Dbl} = \mathbf{Hor}\mathbb{D}\mathrm{bl}$)

(Colax-pseudo adjunction: in the 2-category $\mathrm{Cx}\mathbf{Dbl} = \mathbf{Ver}^*\mathbb{D}\mathrm{bl}$)

7. Limits and colimits

The double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ has all double limits and colimits
(the 2-category $\mathrm{Rel}\mathbf{Ab}$ even lacks products and a terminal object)

7. Limits and colimits

The double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ has all double limits and colimits

(the 2-category $\mathrm{Rel}\mathbf{Ab}$ even lacks products and a terminal object)

A family of relations $u_i \colon A_i \nrightarrow B_i$ $(i \in I)$ has an obvious product

$$u \colon A \nrightarrow B, \qquad u = \{((a_i),(b_i)) \mid (a_i,b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

7. Limits and colimits

The double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ has all double limits and colimits

(the 2-category $\mathrm{Rel}\mathbf{Ab}$ even lacks products and a terminal object)

A family of relations $u_i \colon A_i \to B_i$ $(i \in I)$ has an obvious product

$$u \colon A \to B, \qquad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{tikzcd} A \arrow[r, "p_i"] \arrow[d, "u"'] \arrow[dr, phantom, "\leqslant"] & A_i \arrow[d, "u_i"] \\ B \arrow[r, "q_i"'] & B_i \end{tikzcd} \qquad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

7. Limits and colimits

The double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ has all double limits and colimits
(the 2-category $\mathrm{Rel}\mathbf{Ab}$ even lacks products and a terminal object)

A family of relations $u_i \colon A_i \to\!\!\!\!\!\bullet\; B_i$ $(i \in I)$ has an obvious product

$$u \colon A \to\!\!\!\!\!\bullet\; B, \qquad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{tikzcd} A \arrow[r, "p_i"] \arrow[d, "u"'] \arrow[dr, phantom, "\leqslant"] & A_i \arrow[d, "u_i"] \\ B \arrow[r, "q_i"'] & B_i \end{tikzcd} \qquad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

vertical arrows are viewed as higher-dimensional objects.

7. Limits and colimits

The double category $\mathbb{R}\mathrm{el}\mathbf{Ab}$ has all double limits and colimits
(the 2-category $\mathrm{Rel}\mathbf{Ab}$ even lacks products and a terminal object)

A family of relations $u_i \colon A_i \to B_i$ $(i \in I)$ has an obvious product

$$u \colon A \to B, \qquad u = \{((a_i), (b_i)) \mid (a_i, b_i) \in u_i, \text{ for } i \in I\}$$

with projection cells π_i that satisfy the obvious universal property

$$\begin{tikzcd} A \arrow[r, "p_i"] \arrow[d, "u"'] \arrow[dr, phantom, "\leqslant"] & A_i \arrow[d, "u_i"] \\ B \arrow[r, "q_i"'] & B_i \end{tikzcd} \qquad \begin{array}{l} A = \prod_i A_i \\ \\ B = \prod_i B_i \end{array}$$

vertical arrows are viewed as higher-dimensional objects.

The existence of functorial double products in $\mathbb{R}\mathrm{el}\mathbf{Ab}$ means that:
- the category of horizontal arrows (i.e. $\mathbf{Ab}$) has products,
- the category of vertical arrows and double cells has products,
- these solutions agree w.r.t. vertical (co)domain and identities.

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

The weak double categories $\mathbb{S}\mathrm{pan}\mathbf{C}$ and $\mathbb{C}\mathrm{osp}\mathbf{C}$:

$\alpha \colon u \to v \colon \vee \to \mathbf{C}$ $\qquad\qquad$ $\alpha \colon u \to v \colon \wedge \to \mathbf{C}$

on a category $\mathbf{C}$ with pullbacks and pushouts

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & X' \\
U \arrow[ur, "u'"] \arrow[dr, "u''"'] \arrow[rr, "m\alpha"] & & V \arrow[ur, "v'"'] \arrow[dr, "v''"] & \\
 & Y \arrow[rr, "g"'] & & Y'
\end{tikzcd} \quad \mathbb{S}\mathrm{pan}\mathbf{C}$$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u'"'] & & X' \arrow[dr, "v'"] & \\
 & U \arrow[rr, "m\alpha"] & & V \\
Y \arrow[rr, "g"'] \arrow[ur, "u''"] & & Y' \arrow[ur, "v''"'] &
\end{tikzcd} \quad \mathbb{C}\mathrm{osp}\mathbf{C}$$

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

The weak double categories $\mathbb{S}\mathrm{pan}\mathbf{C}$ and $\mathbb{C}\mathrm{osp}\mathbf{C}$:

$\alpha \colon u \to v \colon \vee \to \mathbf{C}$ $\qquad\qquad$ $\alpha \colon u \to v \colon \wedge \to \mathbf{C}$

on a category $\mathbf{C}$ with pullbacks and pushouts

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & X' \\
U \arrow[ur, "u'"] \arrow[dr, "u''"'] \arrow[rr, "m\alpha"] & & V \arrow[ur, "v'"'] \arrow[dr, "v''"] & \\
 & Y \arrow[rr, "g"'] & & Y'
\end{tikzcd} \quad \mathbb{S}\mathrm{pan}\mathbf{C}$$

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u'"'] & & X' \arrow[dr, "v'"] & \\
 & U \arrow[rr, "m\alpha"] & & V \\
Y \arrow[rr, "g"'] \arrow[ur, "u''"] & & Y' \arrow[ur, "v''"'] &
\end{tikzcd} \quad \mathbb{C}\mathrm{osp}\mathbf{C}$$

The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F \colon \mathbb{S}\mathrm{pan}\mathbf{C} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{C} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

The weak double categories $\mathbb{S}\mathrm{pan}\mathbf{C}$ and $\mathbb{C}\mathrm{osp}\mathbf{C}$:

$\alpha\colon u \to v\colon \vee \to \mathbf{C}$ $\qquad\qquad$ $\alpha\colon u \to v\colon \wedge \to \mathbf{C}$

on a category $\mathbf{C}$ with pullbacks and pushouts

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & X' \\
U \arrow[ur, "u'"] \arrow[dr, "u''"'] \arrow[rr, "m\alpha"] & & V \arrow[ur, "v'"'] \arrow[dr, "v''"] & \\
 & Y \arrow[rr, "g"'] & & Y'
\end{tikzcd}
\qquad \mathbb{S}\mathrm{pan}\mathbf{C}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u'"'] & & X' \arrow[dr, "v'"] & \\
 & U \arrow[rr, "m\alpha"] & & V \\
Y \arrow[ur, "u''"] \arrow[rr, "g"'] & & Y' \arrow[ur, "v''"'] &
\end{tikzcd}
\qquad \mathbb{C}\mathrm{osp}\mathbf{C}$$

The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F\colon \mathbb{S}\mathrm{pan}\mathbf{C} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{C} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback

We cannot compose them: we have an adjunction in $\mathbb{D}\mathrm{bl}$

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

The weak double categories $\mathbb{S}\mathrm{pan}\mathbf{C}$ and $\mathbb{C}\mathrm{osp}\mathbf{C}$:

$\alpha\colon u \to v\colon \vee \to \mathbf{C}$ $\qquad\qquad$ $\alpha\colon u \to v\colon \wedge \to \mathbf{C}$

on a category $\mathbf{C}$ with pullbacks and pushouts

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & X' \\
U \arrow[ur, "u'"] \arrow[dr, "u''"'] \arrow[rr, "m\alpha"] & & V \arrow[ur, "v'"'] \arrow[dr, "v''"] & \\
 & Y \arrow[rr, "g"'] & & Y'
\end{tikzcd} \quad \mathbb{S}\mathrm{pan}\mathbf{C}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u'"'] & & X' \arrow[dr, "v'"] & \\
 & U \arrow[rr, "m\alpha"] & & V \\
Y \arrow[rr, "g"'] \arrow[ur, "u''"] & & Y' \arrow[ur, "v''"'] &
\end{tikzcd} \quad \mathbb{C}\mathrm{osp}\mathbf{C}$$

The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F\colon \mathbb{S}\mathrm{pan}\mathbf{C} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{C} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback
We cannot compose them: we have an adjunction in $\mathbb{D}\mathrm{bl}$
Restricting to the bicategories: still an adjunction in $\mathbb{D}\mathrm{bl}$.

8. Another adjunction which only makes sense in $\mathbb{D}\mathrm{bl}$

The weak double categories $\mathbb{S}\mathrm{pan}\mathbf{C}$ and $\mathbb{C}\mathrm{osp}\mathbf{C}$:

$\alpha\colon u \to v\colon \vee \to \mathbf{C}$ $\qquad\qquad$ $\alpha\colon u \to v\colon \wedge \to \mathbf{C}$

on a category $\mathbf{C}$ with pullbacks and pushouts

$$\begin{tikzcd}
 & X \arrow[rr, "f"] & & X' \\
U \arrow[ur, "u'"] \arrow[dr, "u''"'] \arrow[rr, "m\alpha"] & & V \arrow[ur, "v'"'] \arrow[dr, "v''"] & \\
 & Y \arrow[rr, "g"'] & & Y'
\end{tikzcd} \quad \mathbb{S}\mathrm{pan}\mathbf{C}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "u'"'] & & X' \arrow[dr, "v'"] & \\
 & U \arrow[rr, "m\alpha"] & & V \\
Y \arrow[ur, "u''"] \arrow[rr, "g"'] & & Y' \arrow[ur, "v''"'] &
\end{tikzcd} \quad \mathbb{C}\mathrm{osp}\mathbf{C}$$

The *pushout-pullback* adjunction, trivial on objects and hor. arr.:

$$F\colon \mathbb{S}\mathrm{pan}\mathbf{C} \rightleftarrows \mathbb{C}\mathrm{osp}\mathbf{C} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

F acts on spans by pushout, G acts on cospans by pullback
We cannot compose them: we have an adjunction in $\mathbb{D}\mathrm{bl}$
Restricting to the bicategories: still an adjunction in $\mathbb{D}\mathrm{bl}$.

($\mathbb{S}\mathrm{pan}\mathbf{C}$ has all double (co)limits, if $\mathbf{C}$ has (co)limits)

9. The pushout-pullback adjunction and abelian relations

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if $\mathbf{C}$ is abelian, the po/pb adjunction is strong (left and right)

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if $\mathbf{C}$ is abelian, the po/pb adjunction is strong (left and right)
$\rightarrow$ idempotent lax monad T on $\mathbb{S}\mathrm{pan}\mathbf{C}$: $\mathbb{A}\mathrm{lg}(T) = \mathbb{R}\mathrm{el}\mathbf{C}$
$\rightarrow$ idempotent colax comonad S on $\mathbb{C}\mathrm{osp}\mathbf{C}$: $\mathbb{C}\mathrm{oalg}(S) = \mathbb{R}\mathrm{el}\mathbf{C}$

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if $\mathbf{C}$ is abelian, the po/pb adjunction is strong (left and right)
$\to$ idempotent lax monad T on $\mathbb{S}\mathrm{pan}\mathbf{C}$: $\mathbb{A}\mathrm{lg}(T) = \mathbb{R}\mathrm{el}\mathbf{C}$
$\to$ idempotent colax comonad S on $\mathbb{C}\mathrm{osp}\mathbf{C}$: $\mathbb{C}\mathrm{oalg}(S) = \mathbb{R}\mathrm{el}\mathbf{C}$

- if $\mathbf{C} = \mathbf{Set}$, the pushout-pullback adjunction is not strong

9. The pushout-pullback adjunction and abelian relations

Extending pseudo-lax adjunctions:

A double adjunction $F \dashv G$ is *left strong* if:

(i) the comparison cells of F are made invertible by applying G

(ii) the 'sesqui-functor' $T = GF$ becomes a lax functor
(with the composed comparisons)

Theorem. This produces a lax monad T on the domain of F

- if $\mathbf{C}$ is abelian, the po/pb adjunction is strong (left and right)
$\to$ idempotent lax monad T on $\mathbb{S}\mathrm{pan}\mathbf{C}$: $\mathbb{A}\mathrm{lg}(T) = \mathbb{R}\mathrm{el}\mathbf{C}$
$\to$ idempotent colax comonad S on $\mathbb{C}\mathrm{osp}\mathbf{C}$: $\mathbb{C}\mathrm{oalg}(S) = \mathbb{R}\mathrm{el}\mathbf{C}$

- if $\mathbf{C} = \mathbf{Set}$, the pushout-pullback adjunction is not strong
($\mathbb{R}\mathrm{el}\mathbf{Set} = \mathbb{A}\mathrm{lg}(T')$ for the jointly-monic monad T' on $\mathbb{S}\mathrm{pan}\mathbf{Set}$)

10. Weak multiple categories

10. Weak multiple categories

The weak multiple category Span**Set** (of cubical type)

10. Weak multiple categories

The weak multiple category Span**Set** (of cubical type)

- an n-dimensional cube is a functor $x\colon \vee^n \to \mathbf{Set}$

with $2n$ faces: $\partial_i^\alpha x\colon \vee^{n-1} \to \mathbf{Set} \quad (i = 1, ..., n;\ \alpha = \pm)$

10. Weak multiple categories

The weak multiple category Span$\mathbf{Set}$ (of cubical type)

- an n-dimensional cube is a functor $x\colon \vee^n \to \mathbf{Set}$
with $2n$ faces: $\partial_i^\alpha x\colon \vee^{n-1} \to \mathbf{Set} \quad (i = 1, ..., n;\ \alpha = \pm)$
- for instance $\pi\colon \vee^2 \to \mathbf{Set}$ has four faces f, g, u, v (spans)

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
& {} \\
\bullet \arrow[ur, "0"] \arrow[r, "1"'] \arrow[d, "2"'] & {} \\
{} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "f'"'] \arrow[r, "f''"] & \bullet \\
\bullet \arrow[u, "u'"] \arrow[d, "u''"'] & \bullet \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "\pi_3"] \arrow[d, "\pi_4"] & \bullet \arrow[u, "v'"'] \arrow[d, "v''"] \\
\bullet & \bullet \arrow[l, "g'"] \arrow[r, "g''"'] & \bullet
\end{tikzcd}$$

10. Weak multiple categories

The weak multiple category Span$\mathbf{Set}$ (of cubical type)

- an n-dimensional cube is a functor $x \colon \vee^n \to \mathbf{Set}$

with $2n$ faces: $\partial_i^\alpha x \colon \vee^{n-1} \to \mathbf{Set} \quad (i = 1, ..., n;\ \alpha = \pm)$

- for instance $\pi \colon \vee^2 \to \mathbf{Set}$ has four faces f, g, u, v (spans)

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & 0 \\
\bullet \arrow[urr] \arrow[r, "1"'] \arrow[d, "2"'] & & \\
 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "f'"'] \arrow[r, "f''"] & \bullet \\
\bullet \arrow[u, "u'"] \arrow[d, "u''"'] & \bullet \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, "\pi_3"] \arrow[d, "\pi_4"] & \bullet \arrow[u, "v'"'] \arrow[d, "v''"] \\
\bullet & \bullet \arrow[l, "g'"] \arrow[r, "g''"'] & \bullet
\end{tikzcd}$$

- transversal map $\varphi \colon x \to y \colon \vee^n \to \mathbf{Set}$ (a natural transformation)
- they compose strictly (in direction 0, the *transversal* direction)
- and give the comparisons for the i-composition of n-cubes.

11. Extending adjunctions to weak multiple categories

11. Extending adjunctions to weak multiple categories

The weak multiple category Span**Set** (of cubical type)

11. Extending adjunctions to weak multiple categories

The weak multiple category Span**Set** (of cubical type)

The weak multiple category Cosp**Set** (of cubical type)

11. Extending adjunctions to weak multiple categories

The weak multiple category Span**Set** (of cubical type)

The weak multiple category Cosp**Set** (of cubical type)
- an n-dimensional cube is a functor $x\colon \wedge^n \to \mathbf{Set}$
- a transversal map $\varphi\colon x \to y$ is a natural transformation

11. Extending adjunctions to weak multiple categories

The weak multiple category Span**Set** (of cubical type)

The weak multiple category Cosp**Set** (of cubical type)
- an n-dimensional cube is a functor $x \colon \wedge^n \to \mathbf{Set}$
- a transversal map $\varphi \colon x \to y$ is a natural transformation

The colax-lax multiple adjunction, by pushouts and pullbacks:

$$F \colon \mathrm{Span}\mathbf{Set} \rightleftarrows \mathrm{Cosp}\mathbf{Set} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

which lives in a double category of weak multiple categories.

11. Extending adjunctions to weak multiple categories

The weak multiple category Span**Set** (of cubical type)

The weak multiple category Cosp**Set** (of cubical type)
- an n-dimensional cube is a functor $x \colon \wedge^n \to \mathbf{Set}$
- a transversal map $\varphi \colon x \to y$ is a natural transformation

The colax-lax multiple adjunction, by pushouts and pullbacks:

$$F \colon \mathrm{Span}\mathbf{Set} \rightleftarrows \mathrm{Cosp}\mathbf{Set} : G \qquad (F \textit{ colax}, G \textit{ lax})$$

which lives in a double category of weak multiple categories.

(Span**Set** and Cosp**Set** have all multiple limits and colimits.)

12. Hints at chiral multiple categories (in dim. 3)

12. Hints at chiral multiple categories (in dim. 3)

The chiral triple category $\mathrm{SC}(\mathbf{C})$ (not of cubical type)
for a category $\mathbf{C}$ with pullbacks and pushouts

12. Hints at chiral multiple categories (in dim. 3)

The chiral triple category $\mathrm{SC}(\mathbf{C})$ (not of cubical type)
for a category $\mathbf{C}$ with pullbacks and pushouts

- a 1-cell $f\colon \vee \to \mathbf{C}$ is a span, a 2-cell $u\colon \wedge \to \mathbf{C}$ is a cospan,
- a 12-cell $\pi\colon \vee \times \wedge \to \mathbf{C}$ is a span of cospans (or a cospan of spans)

12. Hints at chiral multiple categories (in dim. 3)

The chiral triple category SC(C) (not of cubical type)

for a category **C** with pullbacks and pushouts

- a 1-cell $f\colon \vee \to \mathbf{C}$ is a span, a 2-cell $u\colon \wedge \to \mathbf{C}$ is a cospan,
- a 12-cell $\pi\colon \vee \times \wedge \to \mathbf{C}$ is a span of cospans (or a cospan of spans)

$$
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & 0 \\
\bullet \arrow[ur] \arrow[r, "1"'] \arrow[d, "2"'] & {} \\
{} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "u'"'] & \bullet \arrow[l, "f'"'] \arrow[r, "f''"] \arrow[d, "c'\pi"] & \bullet \arrow[d, "v'"] \\
\bullet & \bullet \arrow[l, "s'\pi"'] \arrow[r, "s''\pi"] & \bullet \\
\bullet \arrow[u, "u''"] & \bullet \arrow[l, "g'"] \arrow[r, "g''"'] \arrow[u, "c''\pi"'] & \bullet \arrow[u, "v''"']
\end{tikzcd}
$$

12. Hints at chiral multiple categories (in dim. 3)

The chiral triple category $\mathrm{SC}(\mathbf{C})$ (not of cubical type)
for a category $\mathbf{C}$ with pullbacks and pushouts

- a 1-cell $f\colon \vee \to \mathbf{C}$ is a span, a 2-cell $u\colon \wedge \to \mathbf{C}$ is a cospan,
- a 12-cell $\pi\colon \vee \times \wedge \to \mathbf{C}$ is a span of cospans (or a cospan of spans)

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[d, "v"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd} \qquad \begin{tikzcd} & {} \\ \bullet \arrow[ur, "0"] \arrow[r, "1"'] \arrow[d, "2"'] & {} \\ {} & \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d, "u'"'] & \bullet \arrow[l, "f'"'] \arrow[r, "f''"] \arrow[d, "c'\pi"] & \bullet \arrow[d, "v'"] \\ \bullet & \bullet \arrow[l, "s'\pi"'] \arrow[r, "s''\pi"] & \bullet \\ \bullet \arrow[u, "u''"] & \bullet \arrow[l, "g'"] \arrow[r, "g''"'] \arrow[u, "c''\pi"'] & \bullet \arrow[u, "v''"'] \end{tikzcd}$$

- a transversal map $\varphi\colon \pi \to \rho$ is a natural transformation (dim. 3)

12. Hints at chiral multiple categories (in dim. 3)

The chiral triple category SC(**C**) (not of cubical type)
for a category **C** with pullbacks and pushouts

- a 1-cell $f\colon \vee \to \mathbf{C}$ is a span, a 2-cell $u\colon \wedge \to \mathbf{C}$ is a cospan,
- a 12-cell $\pi\colon \vee \times \wedge \to \mathbf{C}$ is a span of cospans (or a cospan of spans)

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\pi"] & \bullet \arrow[d, "v"] \\
\bullet \arrow[r, "g"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {} \\
\bullet \arrow[ur, "0"] \arrow[r, "1"'] \arrow[d, "2"'] & {} \\
{} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d, "u'"'] & \bullet \arrow[l, "f'"'] \arrow[r, "f''"] \arrow[d, "c'\pi"] & \bullet \arrow[d, "v'"] \\
\bullet & \bullet \arrow[l, "s'\pi"'] \arrow[r, "s''\pi"] & \bullet \\
\bullet \arrow[u, "u''"] & \bullet \arrow[l, "g'"] \arrow[r, "g''"'] \arrow[u, "c''\pi"'] & \bullet \arrow[u, "v''"']
\end{tikzcd}$$

- a transversal map $\varphi\colon \pi \to \rho$ is a natural transformation (dim. 3)

Directed interchange for 1- and 2-directed compositions

$$\chi(\pi, \pi', \rho, \rho')\colon (\pi +_1 \pi') +_2 (\rho +_1 \rho') \to (\pi +_2 \rho) +_1 (\pi' +_2 \rho').$$

13. Extending adjunctions to chiral multiple categories

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F\colon \mathrm{Span}_{\mathbf{3}}(\mathbf{C}) \rightleftarrows \mathrm{Cosp}_{\mathbf{3}}(\mathbf{C}) : G \qquad F \dashv G$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F \colon \mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C}) : G \qquad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathsf{SC}(\mathbf{C}) \underset{G''}{\overset{F''}{\rightleftarrows}} \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C})$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F\colon \mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C}) : G \qquad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathsf{SC}(\mathbf{C}) \underset{G''}{\overset{F''}{\rightleftarrows}} \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $\mathsf{S}_{-\infty}C_{\infty}(\mathbf{C})$

- spans of **C** in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F\colon \mathrm{Span}_{\mathbf{3}}(\mathbf{C}) \rightleftarrows \mathrm{Cosp}_{\mathbf{3}}(\mathbf{C}) : G \qquad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\mathrm{Span}_{\mathbf{3}}(\mathbf{C}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathrm{SC}(\mathbf{C}) \underset{G''}{\overset{F''}{\rightleftarrows}} \mathrm{Cosp}_{\mathbf{3}}(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $\mathrm{S}_{-\infty}C_{\infty}(\mathbf{C})$

- spans of **C** in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

Multiple adjunctions of chiral multiple categories:

$$\mathrm{Span}_{\mathbb{Z}}(\mathbf{C}) \rightleftarrows \mathrm{S}_{-\infty}\mathrm{C}_{\infty}(\mathbf{C}) \rightleftarrows \mathrm{Cosp}_{\mathbb{Z}}(\mathbf{C})$$

13. Extending adjunctions to chiral multiple categories

In dim. 3, the colax-lax adjunction of weak triple categories

$$F\colon \mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C}) : G \qquad F \dashv G$$

factorises by colax-lax adjunctions of chiral triple categories

$$\mathsf{Span}_{\mathbf{3}}(\mathbf{C}) \underset{G'}{\overset{F'}{\rightleftarrows}} \mathsf{SC}(\mathbf{C}) \underset{G''}{\overset{F''}{\rightleftarrows}} \mathsf{Cosp}_{\mathbf{3}}(\mathbf{C})$$

In infinite dimension: the chiral 'unbounded' category $\mathsf{S}_{-\infty}C_{\infty}(\mathbf{C})$

- spans of **C** in each negative direction
- ordinary maps in direction 0
- cospans in positive directions.

Multiple adjunctions of chiral multiple categories:

$$\mathsf{Span}_{\mathbb{Z}}(\mathbf{C}) \rightleftarrows \mathsf{S}_{-\infty}\mathsf{C}_{\infty}(\mathbf{C}) \rightleftarrows \mathsf{Cosp}_{\mathbb{Z}}(\mathbf{C})$$

in a double category of chiral categories, lax and colax functors.

14. Strict multiple categories (A. & C. Ehresmann)

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\underline{\mathbb{M}}^{\mathrm{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}$
- faces: $\partial_i^\alpha : A_{\mathbf{i}} \to A_{\mathbf{i}|i}$, for $i \in \mathbf{i}$, $\alpha = \pm$ $\quad (\mathbf{i}|i = \mathbf{i} \setminus \{i\})$
- degeneracies: $e_i : A_{\mathbf{i}|i} \to A_{\mathbf{i}}$, for $i \in \mathbf{i}$

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\underline{\mathbb{M}}^{\mathrm{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}$
- faces: $\partial_i^\alpha \colon A_{\mathbf{i}} \to A_{\mathbf{i}|i}$, for $i \in \mathbf{i}$, $\alpha = \pm$ $\quad$ $(\mathbf{i}|i = \mathbf{i} \setminus \{i\})$
- degeneracies: $e_i \colon A_{\mathbf{i}|i} \to A_{\mathbf{i}}$, for $i \in \mathbf{i}$

under the multiple relations $(i \neq j)$:

$$\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha, \qquad e_i.e_j = e_j.e_i,$$
$$\partial_i^\alpha.e_j = e_j.\partial_i^\alpha, \qquad \partial_i^\alpha.e_i = \mathrm{id}.$$

14. Strict multiple categories (A. & C. Ehresmann)

Multiple set A (in $\mathbf{Set}^{\underline{\mathbb{M}}^{\mathrm{op}}}$):

- a set $A_{\mathbf{i}}$, for every (finite) multi-index $\mathbf{i} = \{i_1, ..., i_n\} \subset \mathbb{N}$
- faces: $\partial_i^\alpha \colon A_{\mathbf{i}} \to A_{\mathbf{i}|i}$, for $i \in \mathbf{i}$, $\alpha = \pm$ $\quad(\mathbf{i}|i = \mathbf{i} \setminus \{i\})$
- degeneracies: $e_i \colon A_{\mathbf{i}|i} \to A_{\mathbf{i}}$, for $i \in \mathbf{i}$

under the multiple relations $(i \neq j)$:

$$\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha, \qquad e_i.e_j = e_j.e_i,$$
$$\partial_i^\alpha.e_j = e_j.\partial_i^\alpha, \qquad \partial_i^\alpha.e_i = \mathrm{id}.$$

Strict multiple category A:

a multiple set with (strict) compositions and interchange

- i-composition: $x +_i y$, for $i \in \mathbf{i}$, $x, y \in A_{\mathbf{i}}$, $\partial_i^+ x = \partial_i^- y$.

15. Multiple categories, weak and lax

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i-composition in a *geometric* direction $i > 0$:

  categorical up to transversal invertible comparisons

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i-composition in a *geometric* direction $i > 0$:

  categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij-interchanger: a transversal invertible comparison $(0 < i < j)$

$$\chi_{ij}(\pi, \pi', \rho, \rho')\colon (\pi +_i \pi') +_j (\rho +_i \rho') \to (\pi +_j \rho) +_i (\pi' +_j \rho')$$

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i-composition in a *geometric* direction $i > 0$:
  categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij-interchanger: a transversal invertible comparison $(0 < i < j)$

$$\chi_{ij}(\pi, \pi', \rho, \rho') \colon (\pi +_i \pi') +_j (\rho +_i \rho') \to (\pi +_j \rho) +_i (\pi' +_j \rho')$$

Chiral multiple category (partially lax):

- ij-interchanger: a transversal directed comparison $(0 < i < j)$

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i-composition in a *geometric* direction $i > 0$:
  categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij-interchanger: a transversal invertible comparison $(0 < i < j)$

$$\chi_{ij}(\pi, \pi', \rho, \rho'): (\pi +_i \pi') +_j (\rho +_i \rho') \to (\pi +_j \rho) +_i (\pi' +_j \rho')$$

Chiral multiple category (partially lax):

- ij-interchanger: a transversal directed comparison $(0 < i < j)$

Intercategory (a laxer version):

- four ij-interchangers, for binary and zeroary compositions in directions $i < j$

15. Multiple categories, weak and lax

Weak multiple category:

- the 0-composition is categorical (the transversal direction)
- i-composition in a *geometric* direction $i > 0$:
  categorical up to transversal invertible comparisons
- strict interchange of the transversal composition with any other
- ij-interchanger: a transversal invertible comparison $(0 < i < j)$

$$\chi_{ij}(\pi, \pi', \rho, \rho') \colon (\pi +_i \pi') +_j (\rho +_i \rho') \to (\pi +_j \rho) +_i (\pi' +_j \rho')$$

Chiral multiple category (partially lax):

- ij-interchanger: a transversal directed comparison $(0 < i < j)$

Intercategory (a laxer version):

- four ij-interchangers, for binary and zeroary compositions in directions $i < j$

in dimension 3, intercategories include:
duoidal categories, monoidal double categories, cubical bicategories, double bicategories, Gray categories

16. The multiple site $\underline{\mathrm{M}}$

16. The multiple site $\underline{\mathbb{M}}$

The multiple site $\underline{\mathbb{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (elements: $t \colon \mathbf{i} \to 2$).

16. The multiple site $\underline{\mathbb{M}}$

The multiple site $\underline{\mathbb{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (elements: $t \colon \mathbf{i} \to 2$).

The category $\underline{\mathbb{M}} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

faces: $\partial_i^\alpha \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}$, $\quad (\partial_i^\alpha t)(j) = t(j),\ \partial_i^\alpha(t)(i) = \alpha$

degeneracies: $e_i \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}$, $\quad (e_i t)(j) = t(j)$

16. The multiple site $\underline{\mathbb{M}}$

The multiple site $\underline{\mathbb{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (elements: $t \colon \mathbf{i} \to 2$).

The category $\underline{\mathbb{M}} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

$$\begin{array}{lll} \textit{faces}: & \partial_i^\alpha \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}, & (\partial_i^\alpha t)(j) = t(j),\ \partial_i^\alpha(t)(i) = \alpha \\ \textit{degeneracies}: & e_i \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}, & (e_i t)(j) = t(j) \end{array}$$

under the comultiple relations $(i \neq j)$:

$$\begin{array}{ll} \partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha, & e_i.e_j = e_j.e_i, \\ \partial_i^\alpha.e_j = e_j.\partial_i^\alpha, & e_i.\partial_i^\alpha = \mathrm{id}. \end{array}$$

16. The multiple site $\underline{\mathbb{M}}$

The multiple site $\underline{\mathbb{M}}$ has an object $2^{\mathbf{i}} = \mathbf{Set}(\mathbf{i}, 2)$
for every (finite) multi-index $\mathbf{i} \subset \mathbb{N}$ (elements: $t \colon \mathbf{i} \to 2$).

The category $\underline{\mathbb{M}} \subset \mathbf{Set}$ is generated by the following mappings

$$(i \in \mathbf{i}, \quad i \neq j, \quad \alpha \in 2, \quad \mathbf{i}|i = \mathbf{i} \setminus \{i\}, \quad j \in \mathbf{i}|i)$$

$$\begin{array}{lll} \textit{faces}: & \partial_i^\alpha \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}, & (\partial_i^\alpha t)(j) = t(j), \ \partial_i^\alpha(t)(i) = \alpha \\ \textit{degeneracies}: & e_i \colon 2^{\mathbf{i}|i} \to 2^{\mathbf{i}}, & (e_i t)(j) = t(j) \end{array}$$

under the comultiple relations $(i \neq j)$:

$$\begin{array}{ll} \partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_i^\alpha, & e_i.e_j = e_j.e_i, \\ \partial_i^\alpha.e_j = e_j.\partial_i^\alpha, & e_i.\partial_i^\alpha = \mathrm{id}. \end{array}$$

(The cubical site has objects 2^n; its indices must be normalised.)

17. References for double and multiple categories

[1] M. Grandis – R. Paré, Limits in double categories, Cah. Topol. Géom. Différ. Catég. 40 (1999), 162–220.

[2] –, Adjoint for double categories, Cah. Topol. Géom. Différ. Catég. 45 (2004), 193–240.

[3] –, Intercategories: a framework for three dimensional category theory, J. Pure Appl. Algebra 221 (2017), 999–1054.

[4] –, An introduction to multiple categories (On weak and lax multiple categories, I), Cah. Topol. Géom. Différ. Catég. 57 (2016), 103–159.

[5] –, Limits in multiple categories (On weak and lax multiple categories, II), Cah. Topol. Géom. Différ. Catég. 57 (2016), 163–202.

[6] –, Adjoints for multiple categories (On weak and lax multiple categories, III), Cah. Topol. Géom. Différ. Catég. 58 (2017), 3–48.

[7] –, A multiple category of lax multiple categories, Cah. Topol. Géom. Différ. Catég. 58 (2017), 195–212.