Graphical Linear Algebra
(a CT2019 tutorial)

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(joint work with Filippo Bonchi, Brendan Fong, Dusko Pavlovic, Robin Piedeleu, Josh Holland, Jens Seeber, Fabio Zanasi)
Mathematics of Open Systems

- components with open terminals
- arrows of some (symmetric) monoidal category
- monoidal functor Syntax $\rightarrow$ Semantics
- relational semantics as opposed to functional semantics
Plan

- composing props
- interacting Hopf algebras
- graphical linear algebra in action
- cartesian bicategories and Frobenius theories
- generating functions and signal flow graphs
- graphical affine algebra and non-passive electrical circuits
Props
(Mac Lane 1965, Lack 2004)

- symmetric strict monoidal categories where
  1. objects are natural numbers and
  2. $m \otimes n := m + n$

- morphisms of props = identity-on-objects symmetric strict monoidal functors

- examples
  - $\mathbf{P}$ - arrows $m$ to $n$ are permutations from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$ (empty if $m \neq n$)
  - $\mathbf{F}$ - arrows $m$ to $n$ are functions from $\{1,\ldots,m\}$ to $\{1,\ldots,n\}$
  - any Lawvere theory
  - $\mathbf{Rel}_X$ - arrows $m$ to $n$ are relations from $X^m$ to $X^n$
  - $\mathbf{LinRel}_k$ - arrows $n$ to $n$ are linear relations from $k^m$ to $k^n$
Presentations of Props

(Lack 2004)

- props can be used as coat hangers for algebraic structure

- example: the prop of commutative monoids $\mathbf{Cm}$

- observation: $\mathbf{Cm} \cong \mathbf{F}$, to give a string diagram $m \rightarrow n$ in $\mathbf{Cm}$ is to give a function $\{1, \ldots m\} \rightarrow \{1, \ldots, n\}$
Composing props - Intuition

Green prop P

Purple prop Q

When can we understand $P;Q$ as a prop?

$\lambda$

$P \Lambda Q$
Composing Props - A Rough Sketch

(Lack 2004)

• recall (Street 1972): monads as arrows of a 2-category

• *mental gymnastics*: category = monad in \text{Span}(	extbf{Set})

• prop = monad in \text{Prof}(	extbf{Mon}) on object \( P \)

• now, given two props \( R, S \), we can compose them

• to make sense of the composite as a monad (i.e. a prop) we need a **distributive law**
Example - Composing with

- i.e. we need to turn a cospan of functions
  
  \[
  \begin{array}{c}
  m_1 \\
  \downarrow \\
  n \\
  \downarrow \\
  m_2
  \end{array}
  \]

  into a span of functions
  
  \[
  \begin{array}{c}
  m_1 \\
  \downarrow \\
  n \\
  \downarrow \\
  m_2
  \end{array}
  \]

  in a way that satisfies the requirements of distributive laws

- taking the pullback in \( F \) works!
other pullbacks responsible for:

- \[ \begin{array}{c}
\text{other pullbacks responsible for:} \\
\end{array} \]

i.e. the theory of commutative bialgebra
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Span(\(F\)) = commutative bialgebra = matrices of natural numbers Mat\(_N\)

Sugar:

\[
\begin{align*}
0 & := \begin{array}{c}
\text{small black dot} \\
\text{solid circular node}
\end{array}, \\
2^{k+1} & := \begin{array}{c}
\text{small black dot} \\
\text{solid circular node}
\end{array}.
\end{align*}
\]

Lemma

\[
\begin{align*}
\begin{array}{c}
\text{small black dot} \\
\text{solid circular node}
\end{array} + \begin{array}{c}
\text{small black dot} \\
\text{solid circular node}
\end{array} & = \begin{array}{c}
\text{small black dot} \\
\text{solid circular node}
\end{array}.
\end{align*}
\]

Proof

\[
\begin{align*}
\begin{array}{c}
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commutative Hopf algebra = matrices of integers

- Add an antipode and equations:

- Mat\(\mathbb{Z}\) has both pullbacks and pushouts

- a slight generalisation of Lack’s notion of composing props allows us to derive presentations for

  - \(\text{IH}^{\text{Span}}\) - A presentation of \(\text{Span}(\text{Mat}\mathbb{Z})\)

  - \(\text{IH}^{\text{Span}}\) - presentation of \(\text{Cospan}(\text{Mat}\mathbb{Z})\)
Presentation of Span(MatZ)

\[ \begin{align*}
\text{IH}^{\text{Span}} & \\
\text{IH}^\text{op} & \\
\end{align*} \]

\[ \begin{align*}
\begin{array}{c}
p \rightarrow p \\
\end{array} & = \\
\begin{array}{c}
\longrightarrow \\
\end{array} & (p \neq 0) \\
\end{align*} \]

\[ \begin{align*}
\begin{array}{c}
\longrightarrow \\
\end{array} & = \\
\begin{array}{c}
\longrightarrow \\
\end{array} & = id_0 \\
\end{align*} \]

Presentation of Cospan(MatZ)

\[ \begin{align*}
\text{IH}^{\text{Cospan}} & \\
\text{IH}^\text{op} & \\
\end{align*} \]

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Glueing Spans and Cospans

\[ \text{IH} \cong \text{LinRel}_Q \]

\[ \text{Cospan} \left( \text{Mat}_\mathbb{Z} \right) \]
GLA: a presentation of LinRel_\mathcal{Q}

e.s. Frobenius

\begin{align*}
\text{c. bialgebra} & \quad \text{c. bialgebra} \\
\text{c. monoid} & \quad \text{c. comonoid} \\
\text{c. comonoid} & \quad \text{c. monoid}
\end{align*}

\begin{align*}
\text{e.s. Frobenius} & \quad (p \neq 0) \\
\text{e.s. Frobenius} & \quad (p \neq 0)
\end{align*}
Plan

- composing props
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- **graphical linear algebra in action**
- cartesian bicategories and Frobenius theories
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• Colour
  • black and white satisfy exactly the same equations in the equational theory
  • so every proof is in fact a proof of two theorems: invert the colours!

• Left-Right
  • every fact is still a fact when viewed in the mirror
Basic concepts, diagrammatically

- transpose
- combine colour and mirror image symmetries
- kernel (nullspace)
- cokernel (left nullspace)
- image (columnspace)
- coimage (rowspace)

**Fact.** Given a linear subspace $R:0\to k$ in $\text{LinRel}$, its orthogonal complement $R^\perp$ is its colour inverted diagram.

**Corollary.** The “fundamental theorem of linear algebra”

$$\ker A = \text{im}(A^T)^\perp$$

$$\ker A^T = \text{im}(A)^\perp$$
Diagrammatic reasoning in action

**Fact.** $A$ is injective iff $\begin{array}{c} \text{A} \ \text{A} \end{array} = \begin{array}{c} \text{A} \ \text{A} \end{array}$

**Theorem.** $A$ is injective iff $\ker A = 0$
Span vs Cospan

• every linear relation can be written in span form, or in cospan form

• span form = choose a basis

• cospan form = choose a set of equations

\[ \begin{align*}
\text{Span: } & \quad \begin{array}{c}
\text{Choose a basis: } a[1, -1, 0] + b[0, 1, 2] \\
\text{Cospan: } & \quad \begin{array}{c}
\text{Choose a set of equations: } x + y = 0, 2y - z = 0 \\
\end{array}
\end{array}
\end{align*} \]
Fun Stuff - Rediscovering Fraction Arithmetic

\[ \frac{p}{q} \] := \[ \frac{p}{q} \]

\[ p \quad q \] = \[ r \quad s \]

\( \Leftrightarrow \)

\( sp = qr \)

\[ p \quad q \quad r \quad s \] = \[ p \quad r \quad q \quad s \]

= \[ rp \quad sq \]

= \[ sp + qr \quad sq \]
Fun Stuff - Dividing by Zero

- LinRel_1,1

- Projective arithmetic with two additional elements

  - The unique 0-dimensional subspace \( \perp = \{ (0,0) \} \)

  - The unique 2-dimensional subspace \( \top = \{ (x,y) \mid x,y \in \mathbb{Q} \} \)

### Table 1: Projective Arithmetic Operators

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>r/s</th>
<th>( \infty )</th>
<th>( \top )</th>
<th>( \perp )</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>r/s</td>
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<td>( \top )</td>
<td>( \perp )</td>
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<td>( \frac{(sp+qr)}{qs} )</td>
<td>( \infty )</td>
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</table>

### Table 2: Linear Relational Operators

<table>
<thead>
<tr>
<th></th>
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Cartesian categories

(Fox 1976)

cartesian categories are those sym. mon. cats. where every object has a commutative comonoid structure

and everything commutes with the structure

Example: $\text{Set}_x$
Cartesian bicategories
(Carboni, Walters 1987)

special Frobenius structure where monoid is right adjoint to comonoid

and everything laxly commutes with the structure

Example: $\text{Rel}_x$
LinRel is a cartesian bicategory

- LinRel is a cartesian bicategory
  - In fact, it is an abelian bicategory

- To obtain a presentation we add just one inequality

  (Bonchi, Holland, Pavlovic, S. 2017)

\[\begin{array}{c}
\circ \leq \bullet
\end{array}\]

- This breaks the symmetry between white and black!
Lawvere theories

- recipe for Lawvere-theories-as-props
  1. add a cocommutative comonoid structure
  2. make all generators commute with it
  3. add your other equations (which may make use of the comonoid structure)

\[ x \cdot x^{-1} = e \]
Frobenius theories
(Bonchi, Pavlovic, S. 2017)

• recipe for Frobenius-theories-as-locally-ordered-prop

• add a Frobenius bimonoid structure where monoid is right adjoint to comonoid

• make all your generators laxly commute with it

• add your other equations (which may make use of the Frobenius structure)

\[ \text{e.g. } \text{id}_0 \leq \bullet \bullet \]
Functorial semantics

• For Lawvere theories
  • models = cartesian functors
  • homomorphisms = natural transformations

• For Frobenius theories
  • models = morphisms of cartesian bicategories
  • homomorphisms = lax natural transformations

• Rel models of GLA = Vect_\mathbb{Q}
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- *generating functions and signal flow graphs*
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Generalising GLA

- The cube construction works for $\text{Mat}_R$ whenever $R$ is a PID.

\[
\begin{align*}
&\text{IH}^\text{Cospan} \rightarrow \text{IH}^\text{Span} \\
&\text{IH}^\text{Cospan} \rightarrow \text{IH} \rightarrow \text{IH}^\text{Span} \\
&\text{IH} \rightarrow \text{Span}(\text{Mat}_Z) \\
&\text{Cospan}(\text{Mat}_Z) \rightarrow \text{LinRel}
\end{align*}
\]
Generating Functions and Laurent series

equational Theory

polynomial fractions

Laurent series

isomorphisms

faithful homomorphisms
Example

As linear relation over $\mathbb{Q}(x)$ is the space generated by

\[(1, x/(1-x-x^2))\]

As linear relation over $\mathbb{Q}((x))$ is the space generated by

\[(1,0,0,\ldots, 0,1,1,2,3,5,8,\ldots)\]
Signal flow graphs
(Shannon 1942)

- directed circuits with
  - addition gates
  - junctions
  - “registers”
  - act as integrators in the continuous semantics
  - act as one place buffers in the discrete semantics
- guarded feedback
Example - Fibonacci

(Bonchi, S., Zanasi 2015)
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**Definition.** Given a field $k$, a $k$-affine relation $k \rightarrow l$ is a set $R \subseteq k^k \times k^l$ which is either empty, or s.t. there is a $k$-linear relation $C$ and a vector $(a,b)$ s.t. $R = (a,b) + C$

$GLA + \text{ above} \cong \text{AffRel}_k$
Example: Non-passive electrical circuits

\[ I(\text{[ ]}) = \] 

\[ I(\text{[ ]}) = \] 

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\[ I(\text{[ ]}) = \]
Resistors in parallel

\[ I \left( \frac{1}{a} + \frac{1}{b} \right) = \frac{ab}{a+b} \]

What if \( a=b=0? \)
Current sources in parallel are additive
Voltage sources in parallel are "illegal"

• F. Bonchi, P. Sobocinski, F. Zanasi. *Interacting Bialgebras are Frobenius*. FoSSaCS 2014


• S. Lack. *Composing PROPs*. TAC 13:147—164, 2004


See you in Tallinn!

PhD projects in open game theory, CT in programming, Frobenius theories, string diagrams in database theory and logic, …

Visitors welcome!