Graphical Linear Algebra

(a CT2019 tutorial)

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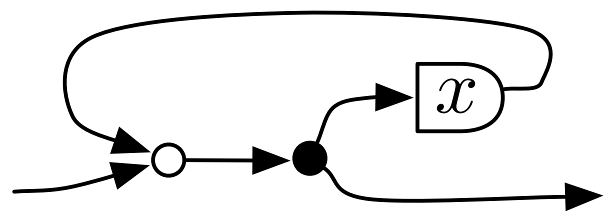
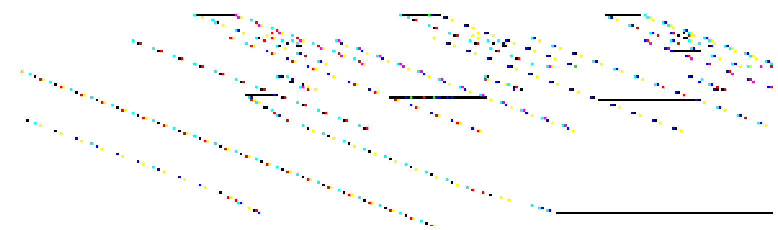
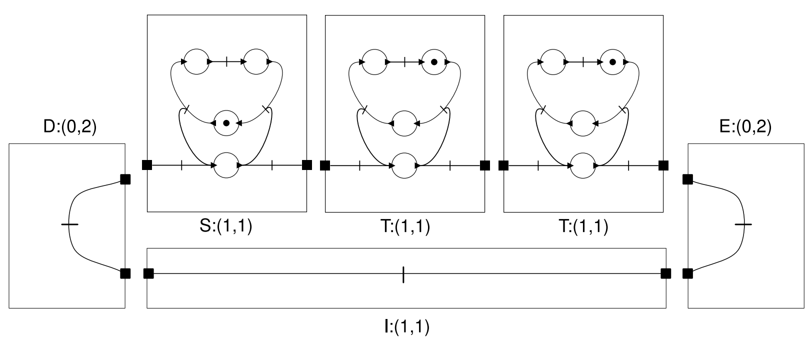
Mathematics of Open Systems



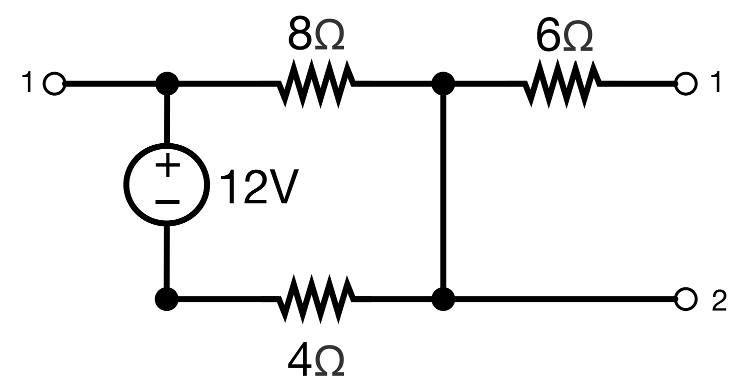
(Shannon 1942; Baez and Erbele 2014;
Bonchi, S., Zanasi 2014)



(Katis, Sabadini, Walters 1996)



(S 2010, Bonchi, Holland, Piedeleu, S. Zanasi 2019)



(Baez, Coya, Rebro 2017;
Coya 2018; Bonchi, Piedeleu, S., Zanasi 2019)

- components with open terminals
- arrows of some (symmetric) monoidal category
- monoidal functor **Syntax** → **Semantics**
- *relational* semantics as opposed to *functional* semantics

Plan

- **composing props**
- interacting Hopf algebras
- graphical linear algebra in action
- cartesian bicategories and Frobenius theories
- *generating functions and signal flow graphs*
- *graphical affine algebra and non-passive electrical circuits*

Props

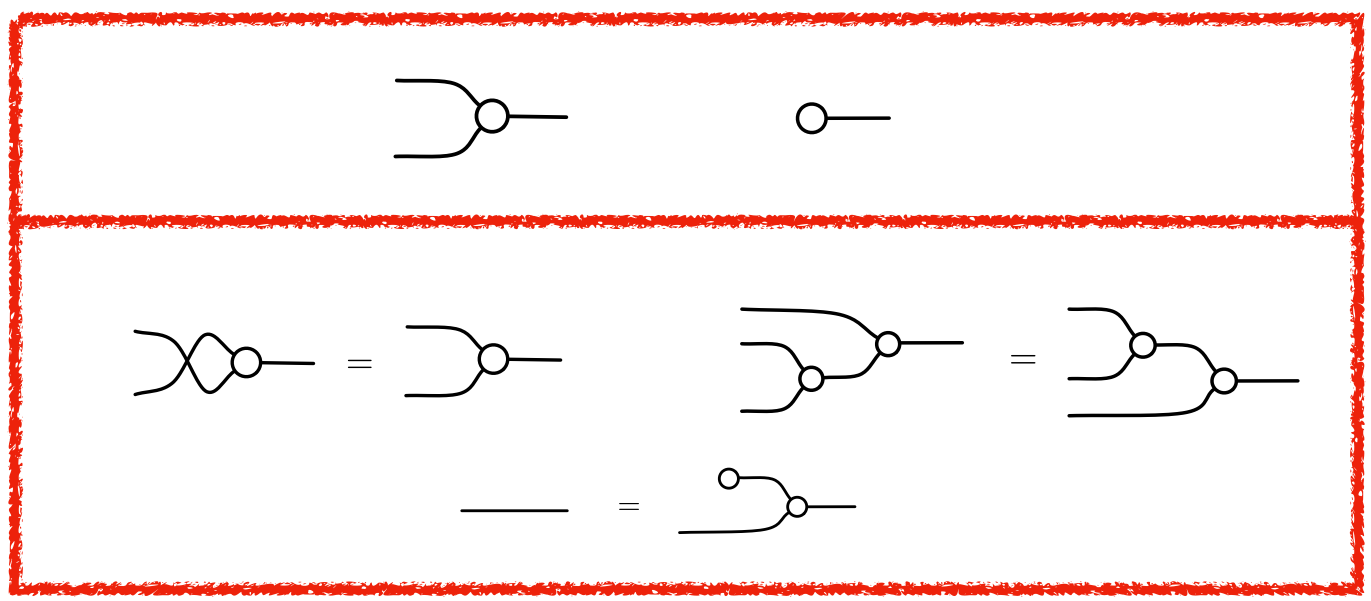
(Mac Lane 1965, Lack 2004)

- symmetric strict monoidal categories where
  1. objects are natural numbers and
  2. $m \otimes n := m+n$
- morphisms of props = identity-on-objects symmetric strict monoidal functors
- *examples*
  - **P** - arrows m to n are permutations from {1,…,m} to {1,…,n} (empty if $m \neq n$)
  - **F** - arrows m to n are functions from {1,…,m} to {1,…,n}
  - any Lawvere theory
  - $\mathbf{Rel}_X$ - arrows m to n are relations from X^m to X^n
  - $\mathbf{LinRel}_k$ - arrows n to n are linear relations from k^m to k^n

Presentations of Props

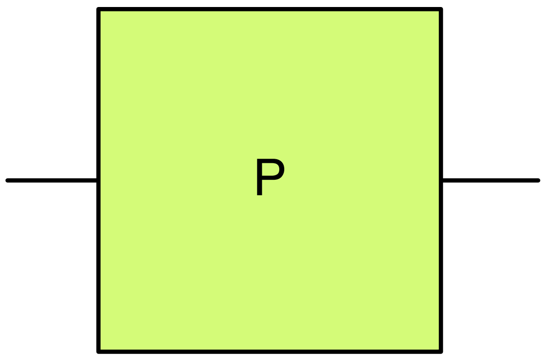
(Lack 2004)

- props can be used as coat hangers for algebraic structure
- *example*: the prop of commutative monoids **Cm**

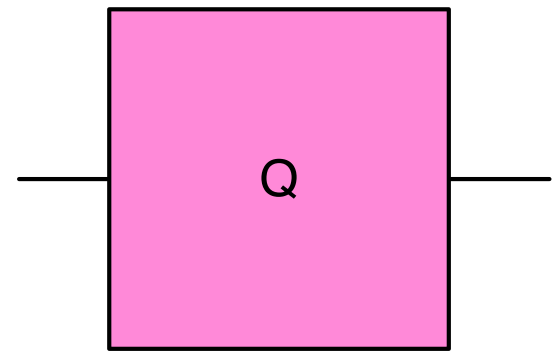


- *observation*: **Cm** ≅ **F**, to give a string diagram m→n in **Cm** is to give a function {1,…m}→{1,…,n}

Composing props - Intuition

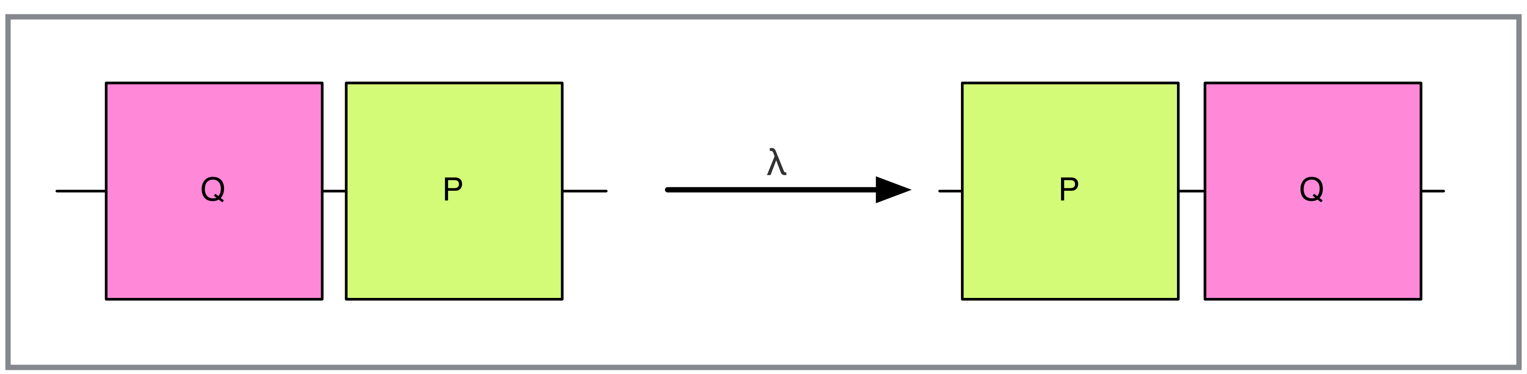


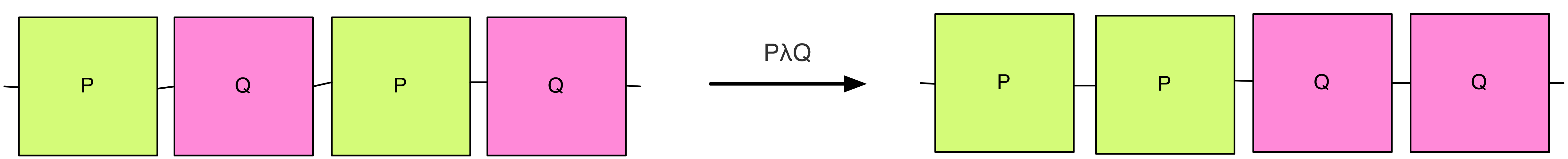
Green prop P



Purple prop Q

When can we understand P;Q as a prop?

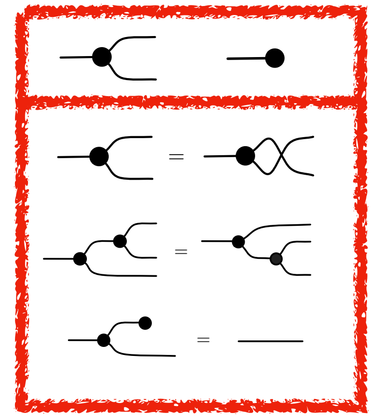
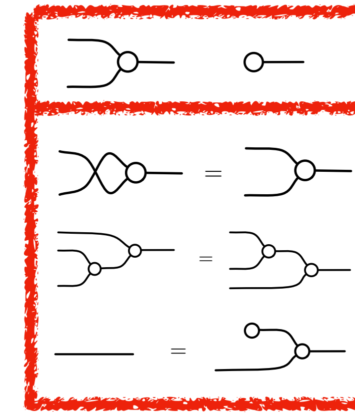


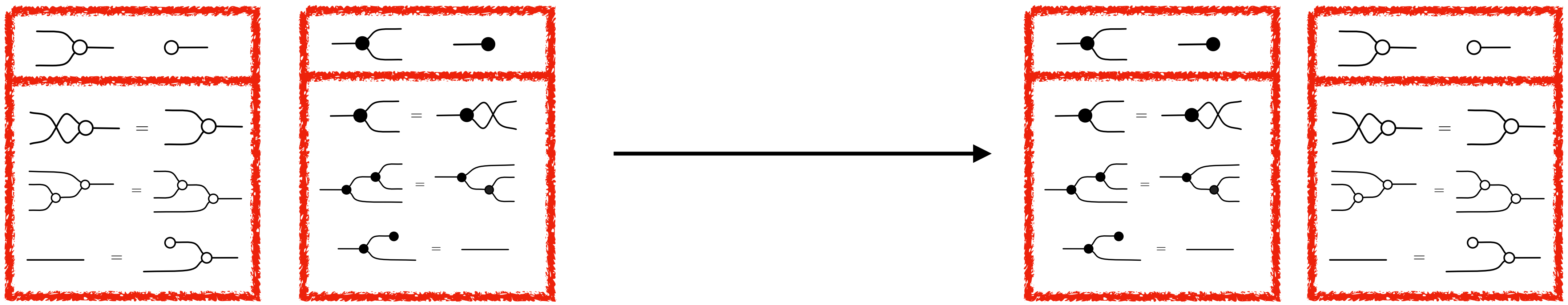


Composing Props - A Rough Sketch

(Lack 2004)

- recall (Street 1972): monads as arrows of a 2-category
- *mental gymnastics*: category = monad in Span(**Set**)
- prop = monad in Prof(**Mon**) on object **P**
- now, given two props **R**, **S**, we can **compose** them
- to make sense of the composite as a monad (i.e. a prop) we need a **distributive law**

Example - Composing  with 



- ie. we need to turn a cospan of functions

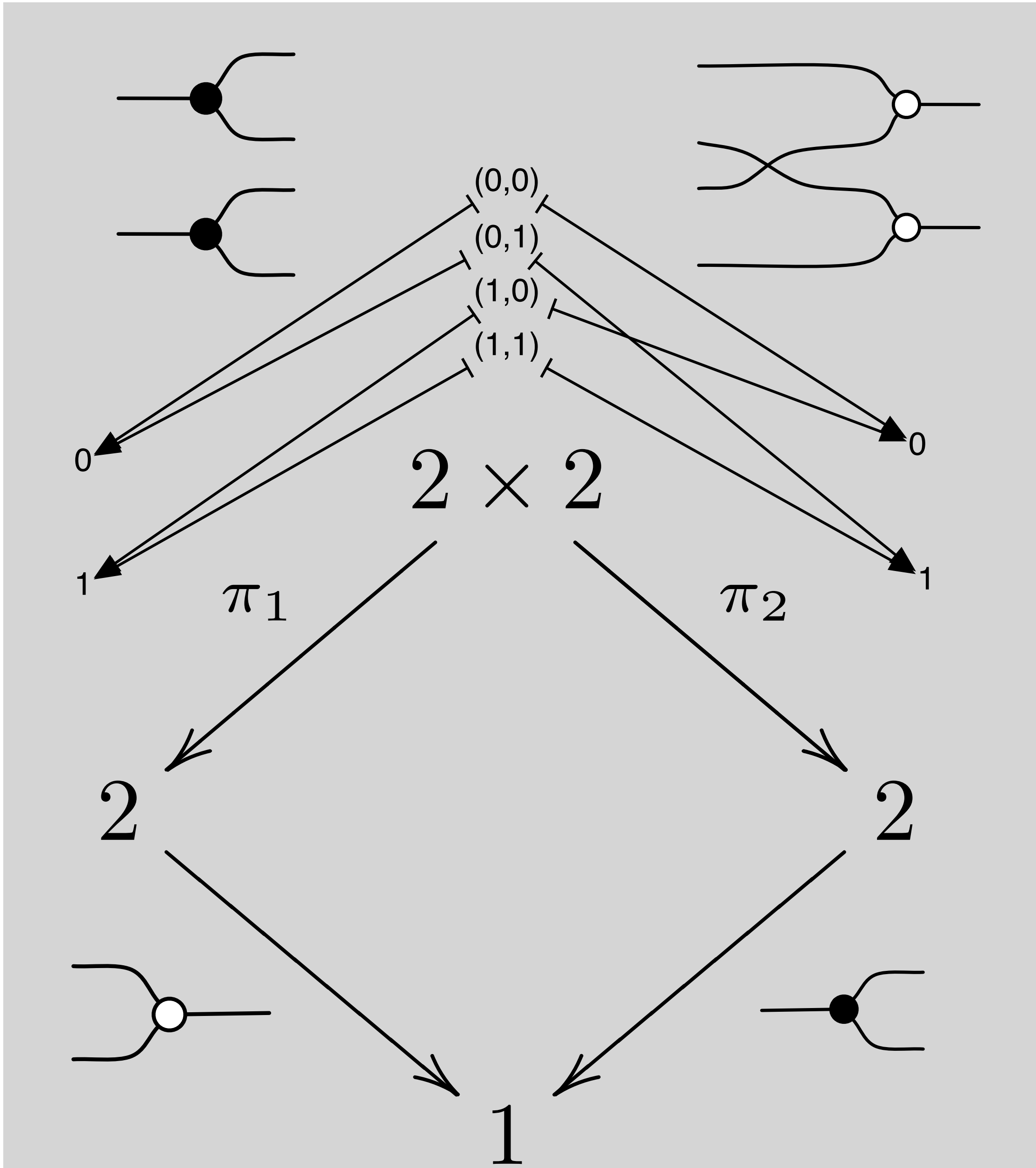
$$m_1 \longrightarrow n \longleftarrow m_2$$

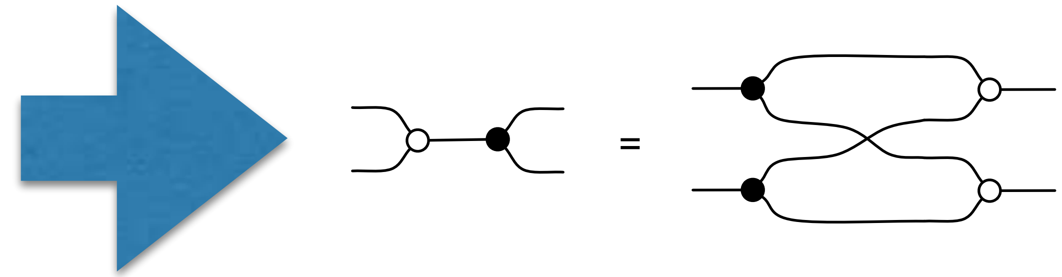
into a span of functions

$$m_1 \longleftarrow n \longrightarrow m_2$$

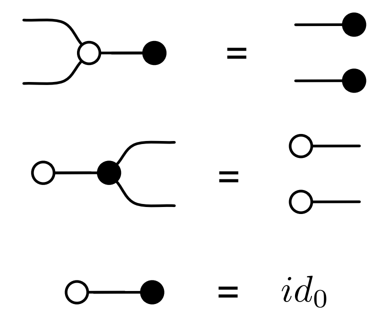
in a way that satisfies the requirements of distributive laws

- taking the pullback in **F** works!



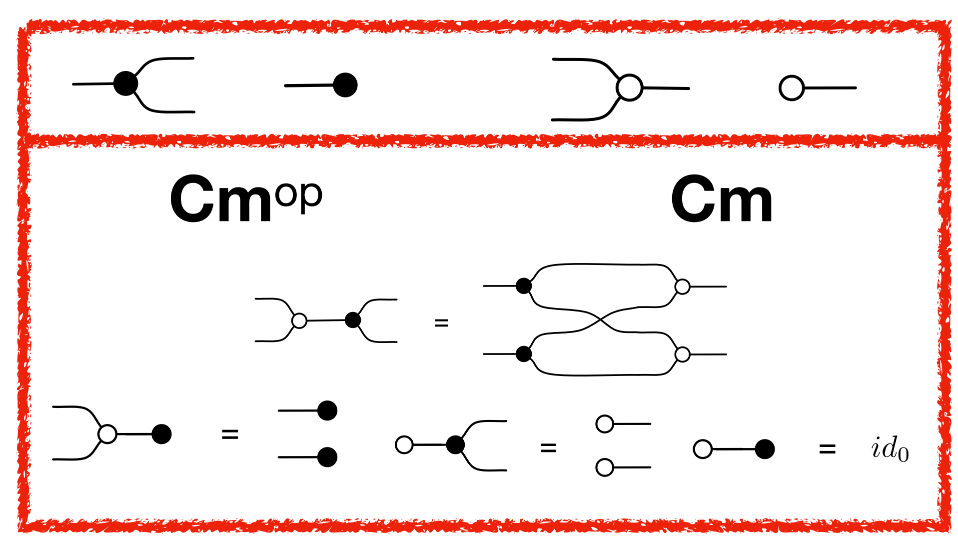


other pullbacks responsible for:



i.e. the theory of commutative bialgebra

Span **F** $\cong$



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- **interacting Hopf algebras**
- graphical linear algebra in action
- cartesian bicategories and Frobenius theories
- *generating functions and signal flow graphs*
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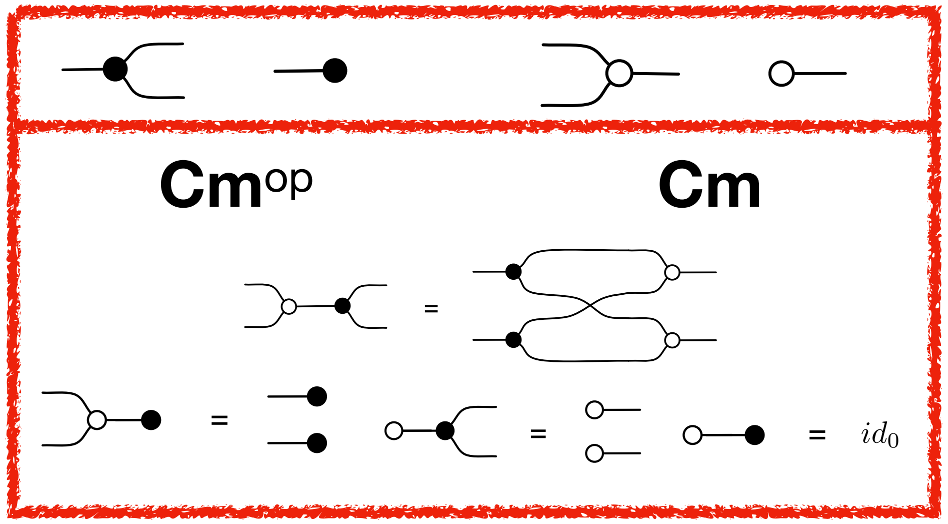
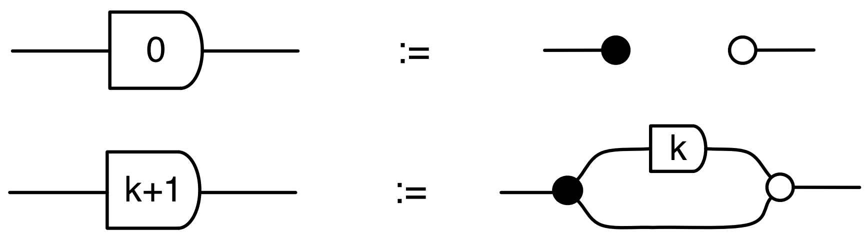
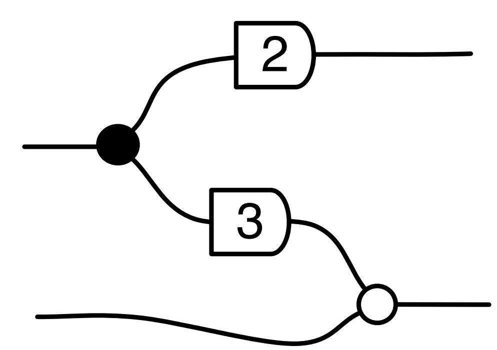


Span(F) = commutative bialgebra = matrices of natural numbers $\mathrm{Mat}_{\mathbb{N}}$

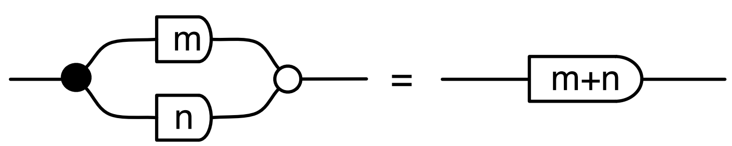
Sugar:



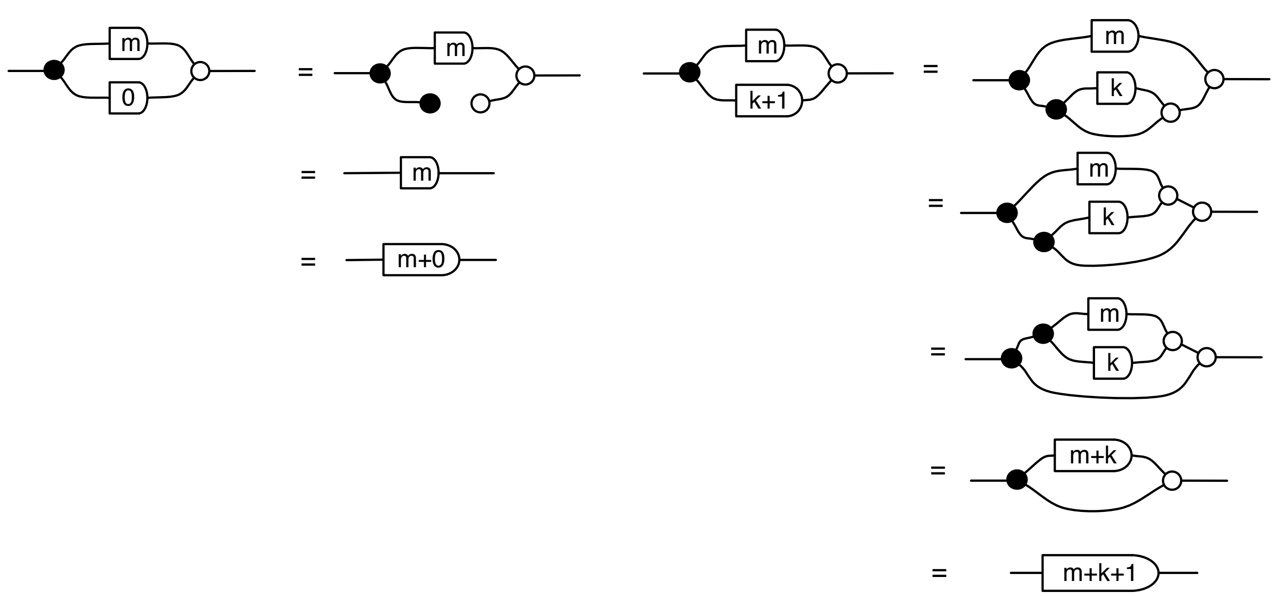
$$\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$$



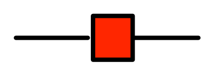
Lemma

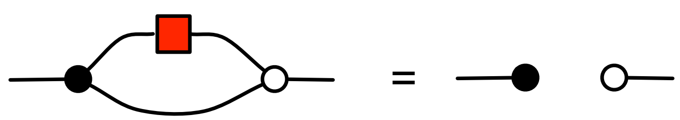


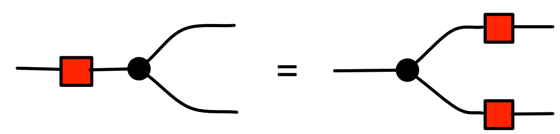
Proof

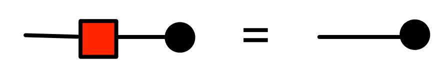


commutative Hopf algebra = matrices of integers

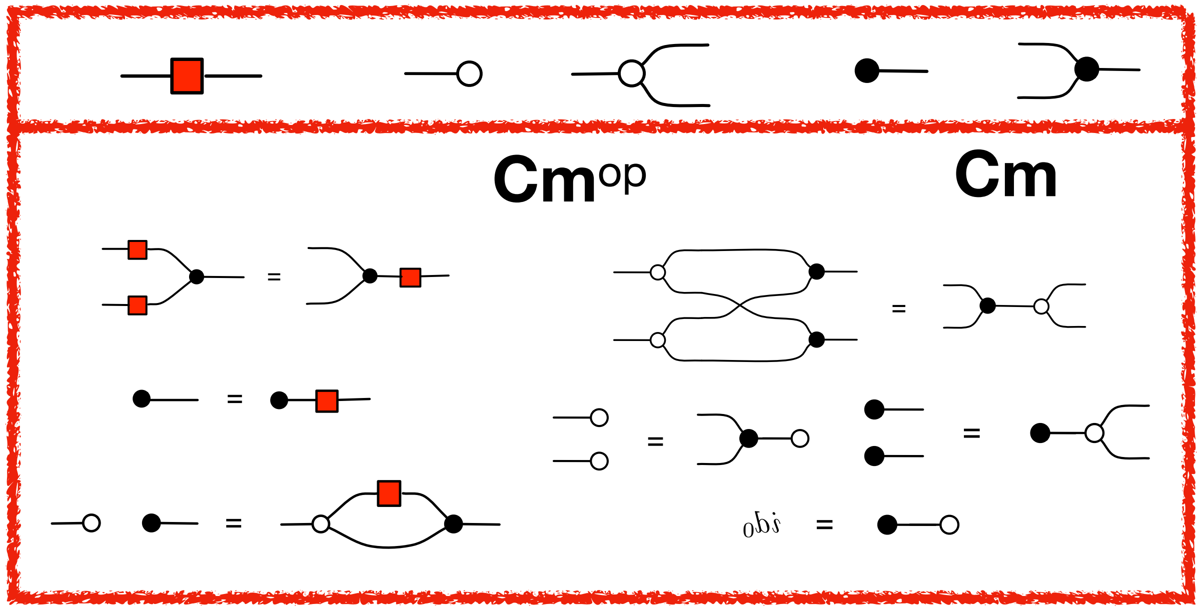
- Add an antipode  and equations:

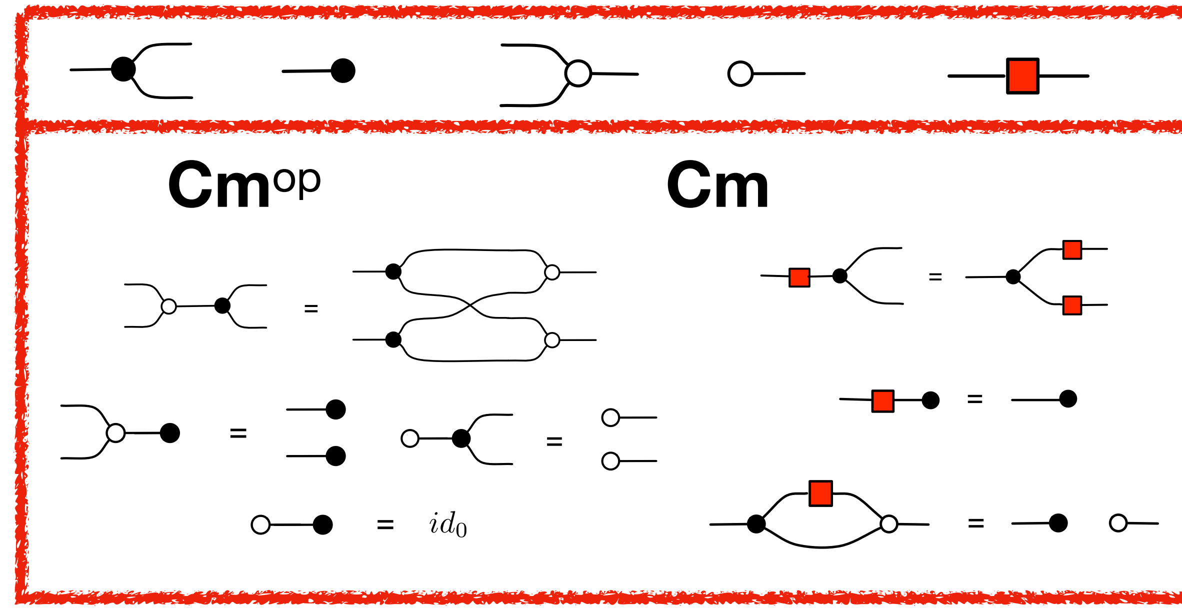


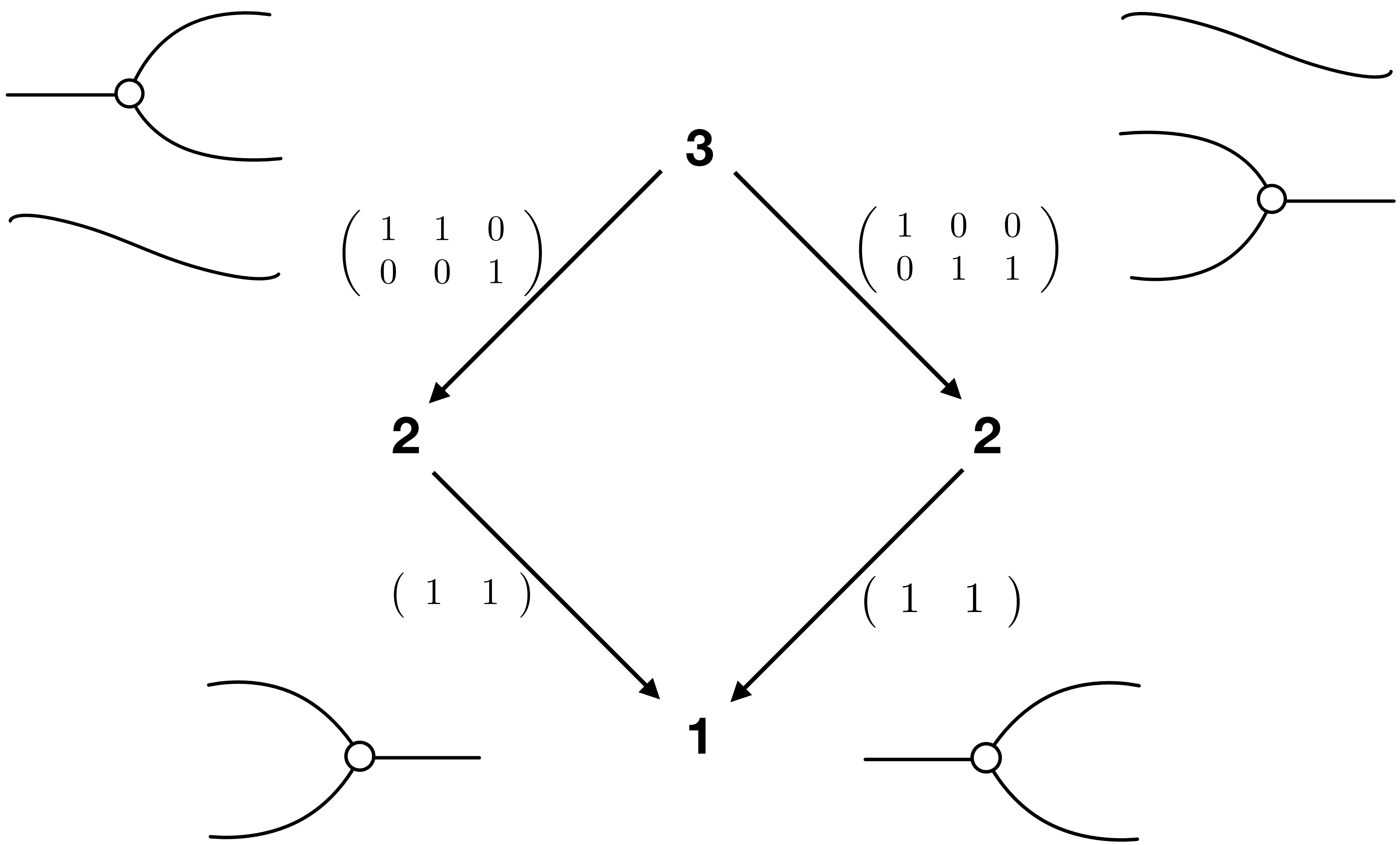




- Mat$_{\mathbb{Z}}$ has both pullbacks and pushouts
- a slight generalisation of Lack's notion of composing props allows us to derive presentations for
  - **IH**$^{\mathrm{Span}}$ - A presentation of Span(Mat$_{\mathbb{Z}}$)
  - **IH**$^{\mathrm{Span}}$ - presentation of Cospan(Mat$_{\mathbb{Z}}$)

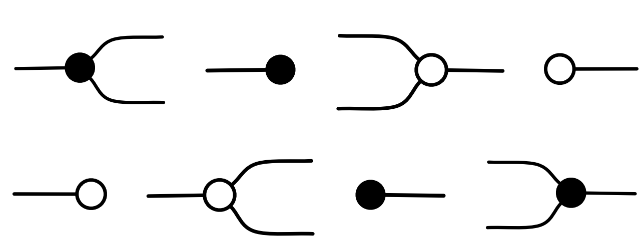

Cmop
Cm

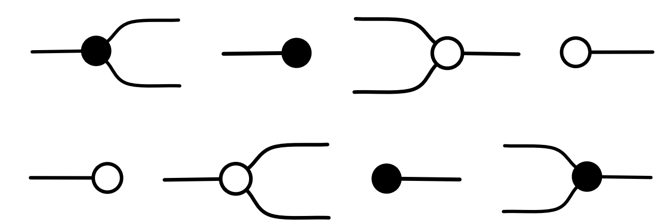

Cmop
Cm
id_0


3
2
2
1
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 \end{pmatrix}$

Presentation of Span($Mat_{\mathbb{Z}}$)

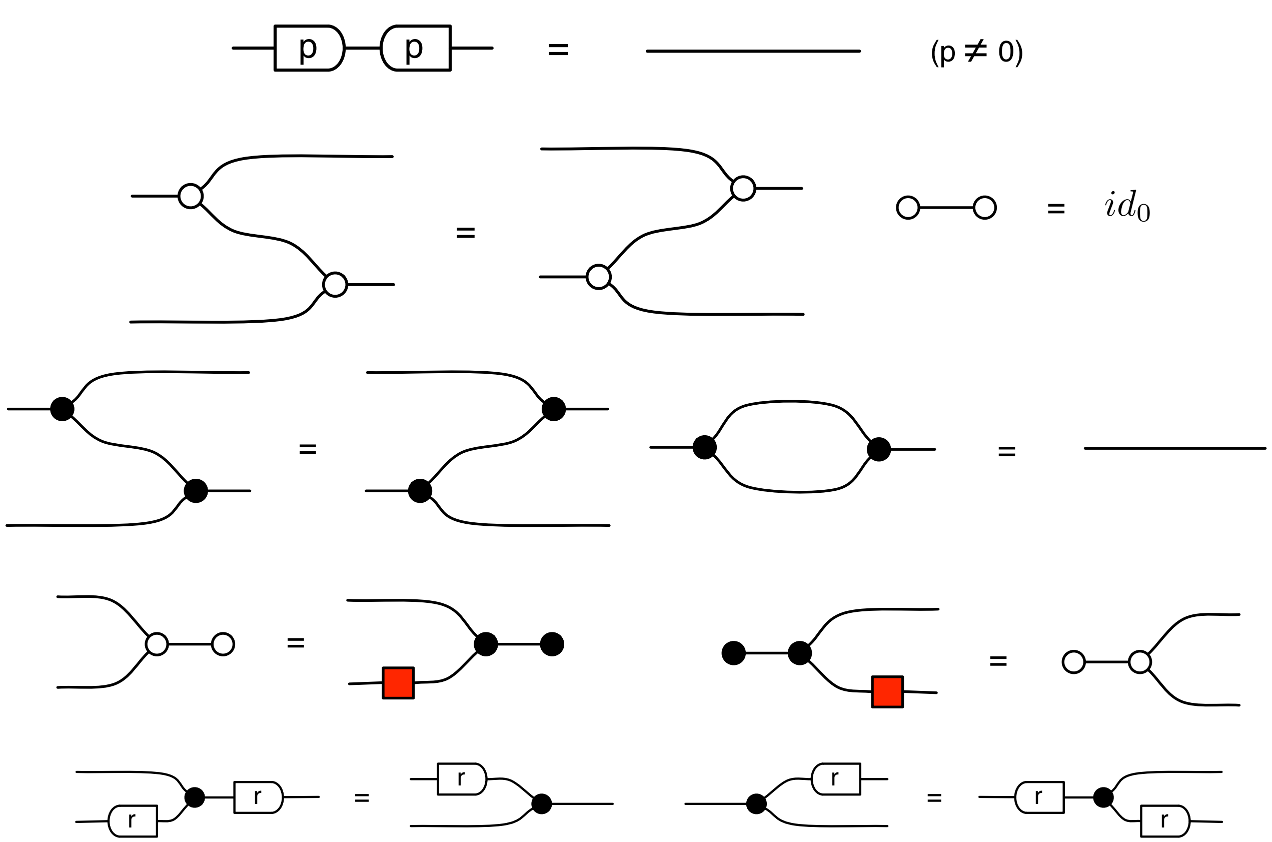
Presentation of Cospan($Mat_{\mathbb{Z}}$)





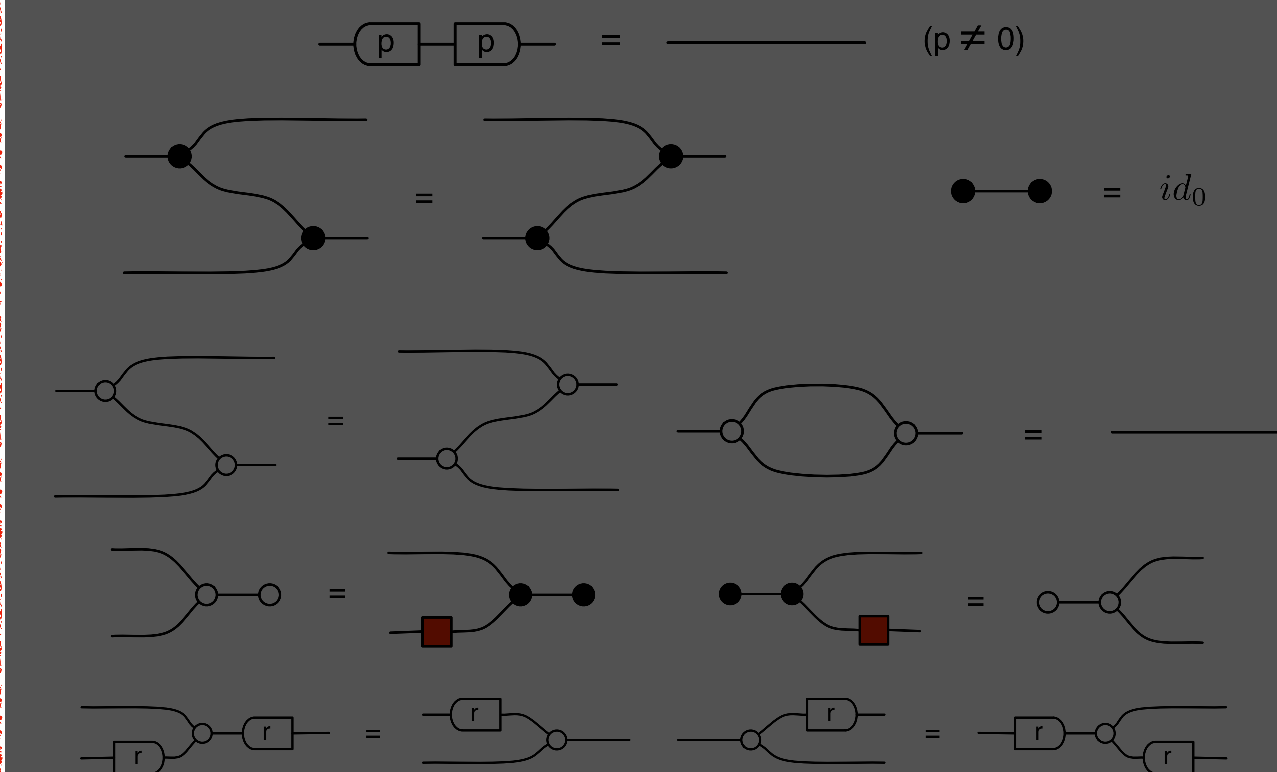
H

H^{op}



H

H^{op}



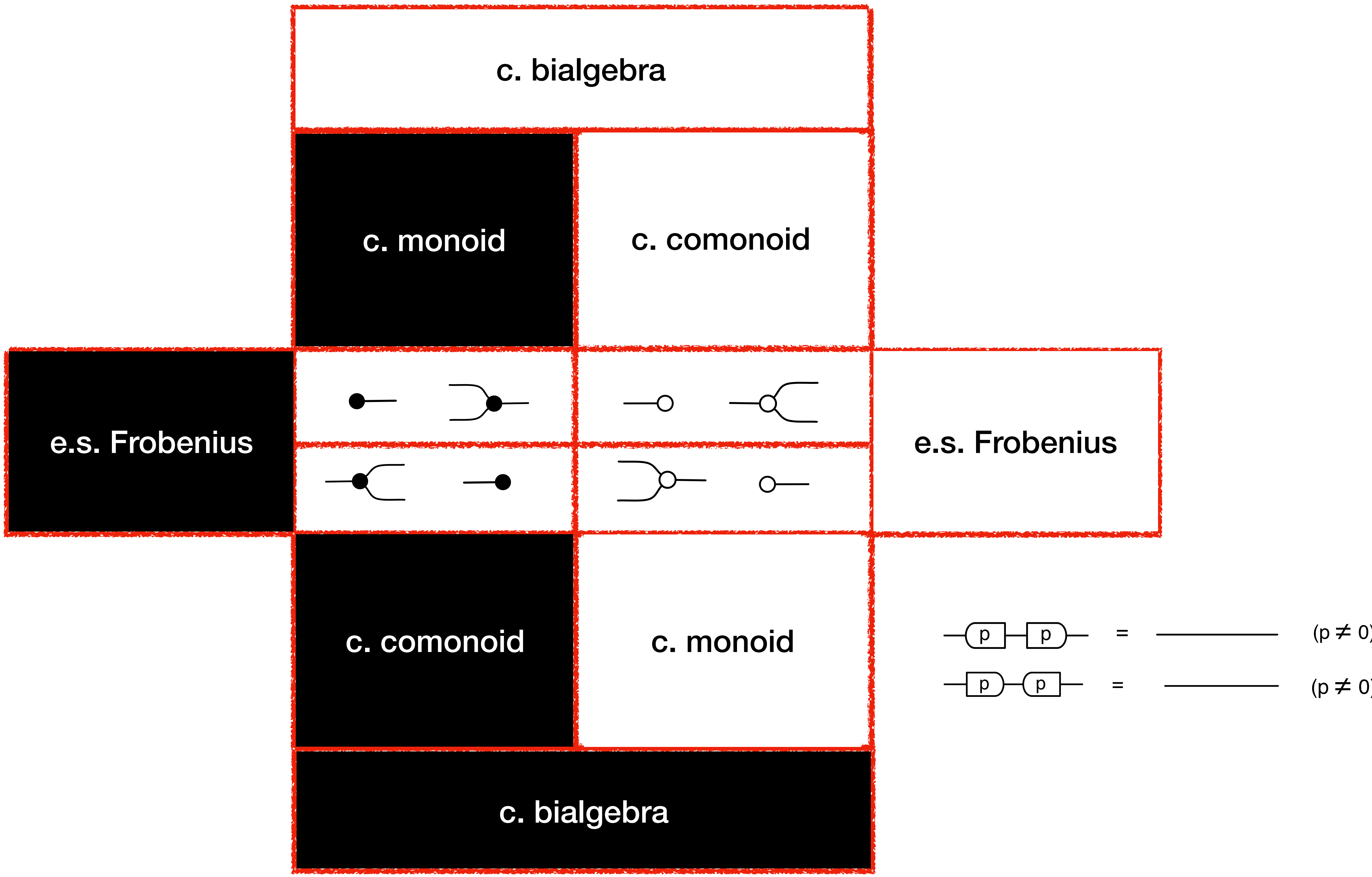
IH^{Span}

IH^{Cospan}

Glueing Spans and Cospans

$$
\begin{tikzcd}
 & \mathbf{H} + \mathbf{H}^{op} \arrow[rr] \arrow[dl] \arrow[dd, "\cong"'] & & \mathbf{IH}^{\mathrm{Span}} \arrow[dl] \arrow[dd, "\cong"] \\
\mathbf{IH}^{\mathrm{Cospan}} \arrow[rr, crossing over] \arrow[dd, "\cong"'] & & \mathbf{IH} & \\
 & \mathbf{Mat}_{\mathbf{Z}} + \mathbf{Mat}_{\mathbf{Z}}{}^{op} \arrow[rr] \arrow[dl] & & \mathrm{Span}(\mathbf{Mat}_{\mathbf{Z}}) \arrow[dl] \\
\mathrm{Cospan}(\mathbf{Mat}_{\mathbf{Z}}) \arrow[rr] & & \mathbf{LinRel}_{\mathbf{Q}} \arrow[from=uu, crossing over] &
\end{tikzcd}
$$

GLA: a presentation of LinRel$_Q$



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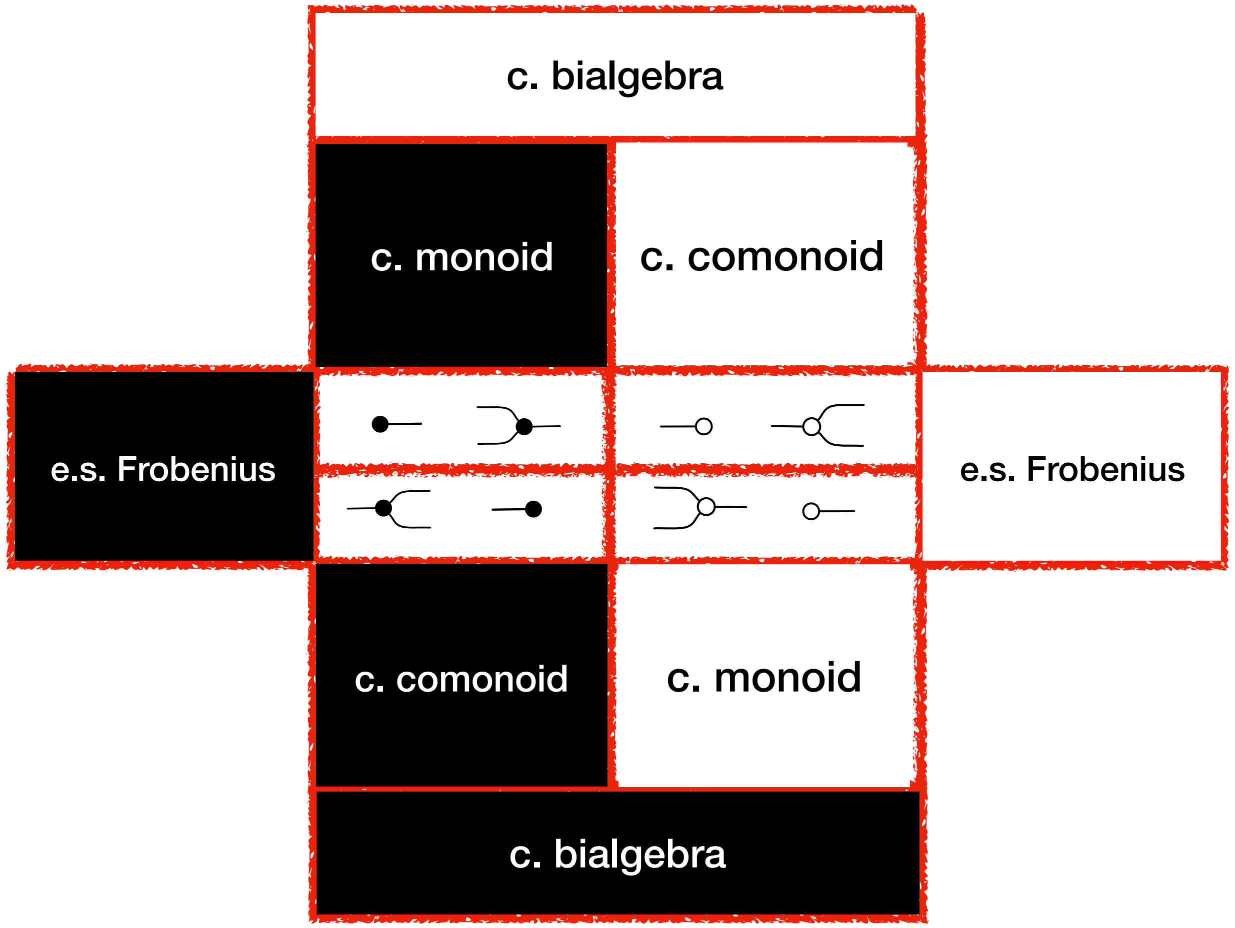
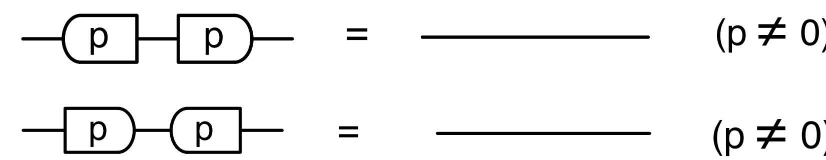
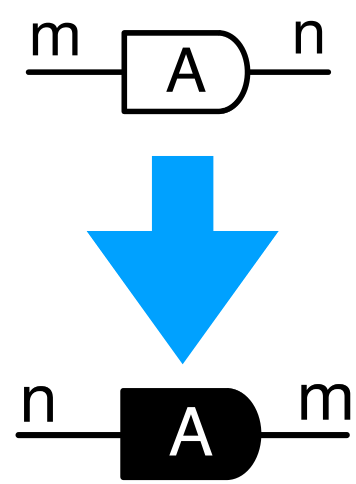




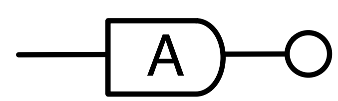
- **Colour**
  - black and white satisfy **exactly the same** equations in the equational theory
  - so every proof is in fact a proof of two theorems: invert the colours!
- **Left-Right**
  - every fact is still a fact when viewed in the mirror

Basic concepts, diagrammatically

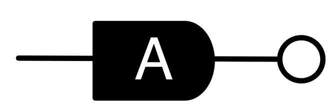
- transpose
    - combine colour and mirror image symmetries



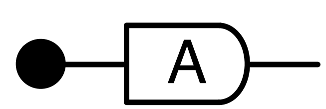
- kernel (nullspace)



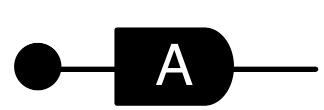
- cokernel (left nullspace)



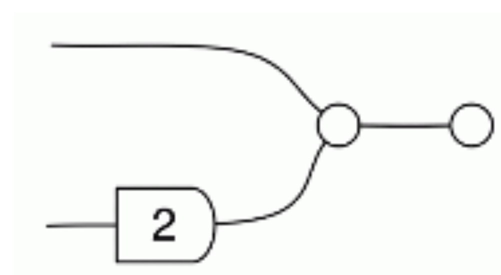
- image (columnspace)



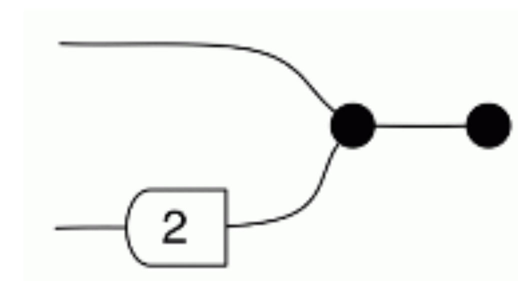
- coimage (rowspace)



Fact. Given a linear subspace R:0->k in **LinRel**, its orthogonal complement $R^{\perp}$ is its colour inverted diagram



$$\begin{pmatrix} x \\ y \end{pmatrix} \mid x + 2y = 0$$



$$\begin{pmatrix} x \\ 2x \end{pmatrix}$$

Corollary. The “fundamental theorem of linear algera”

$$\ker A = \mathrm{im}(A^T)^{\perp}$$

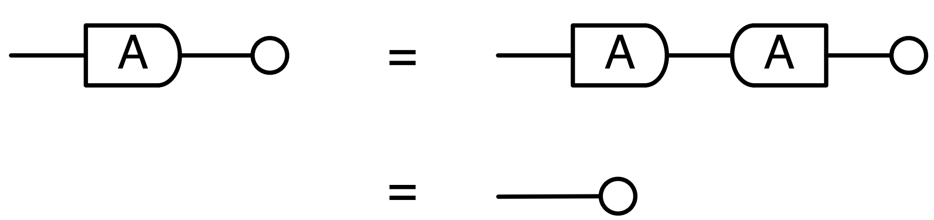
$$\ker A^T = \mathrm{im}(A)^{\perp}$$

Diagrammatic reasoning in action

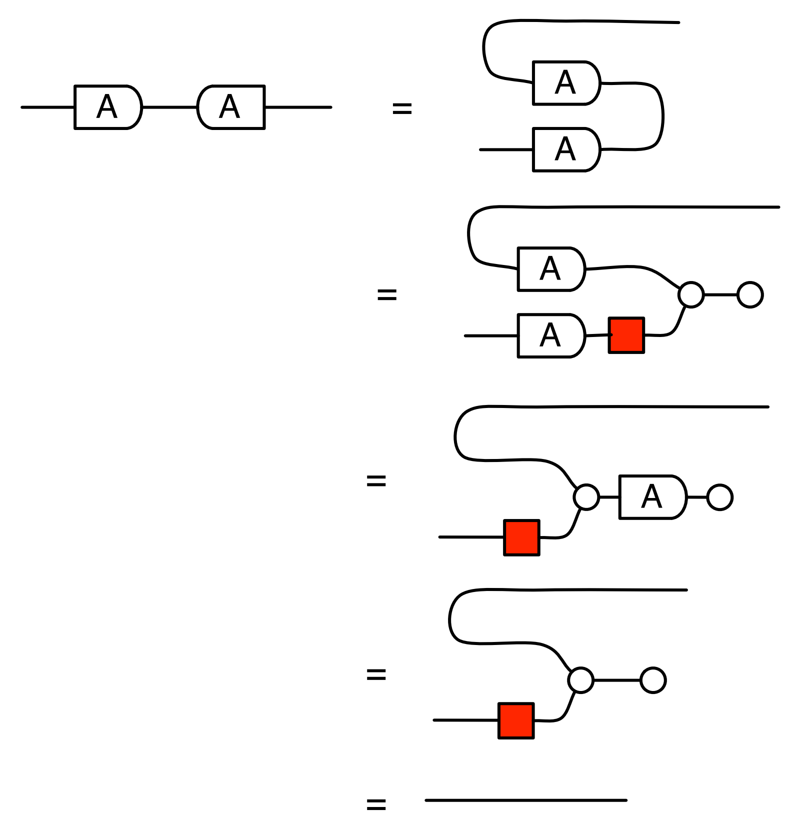
Fact. A is injective iff

Theorem. A is injective iff ker A = 0

$\Rightarrow$

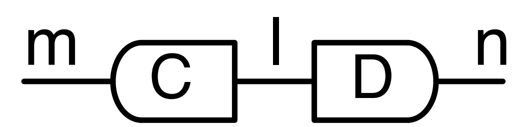


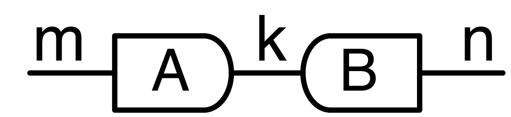
$\Leftarrow$

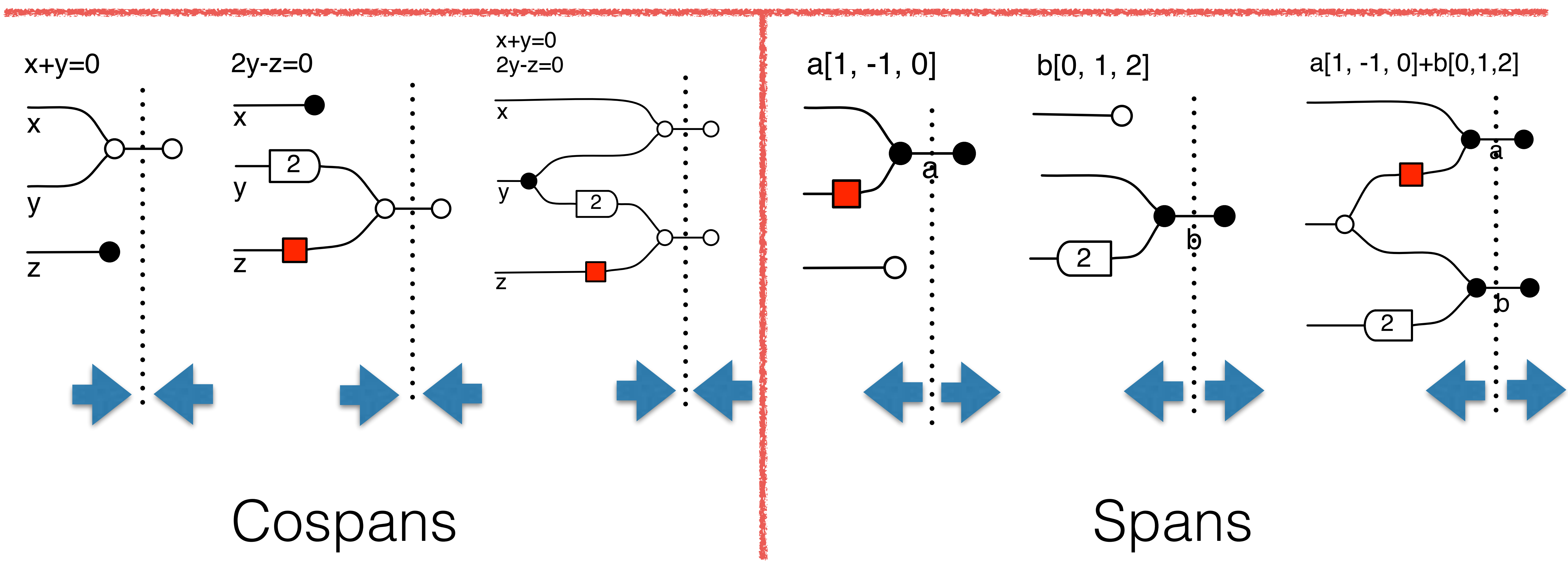


Span vs Cospan

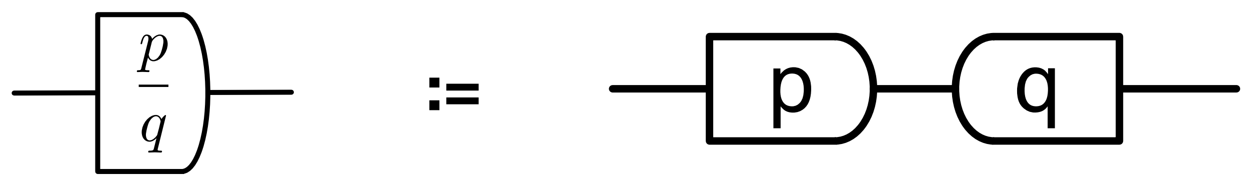
- every linear relation can be written in *span form*, or in *cospan form*

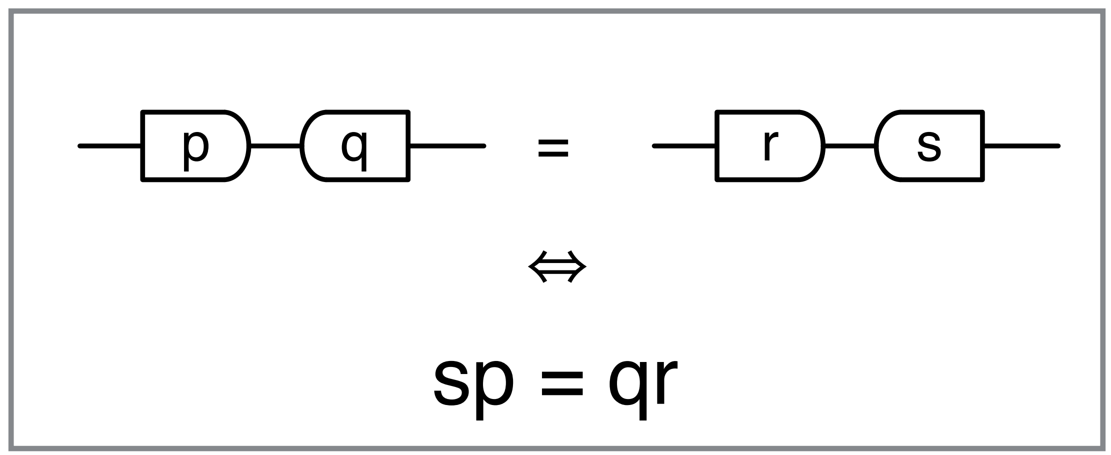
- span form = choose a basis 

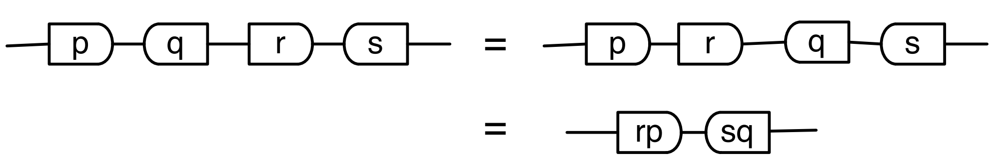
- cospan form = choose a set of equations 

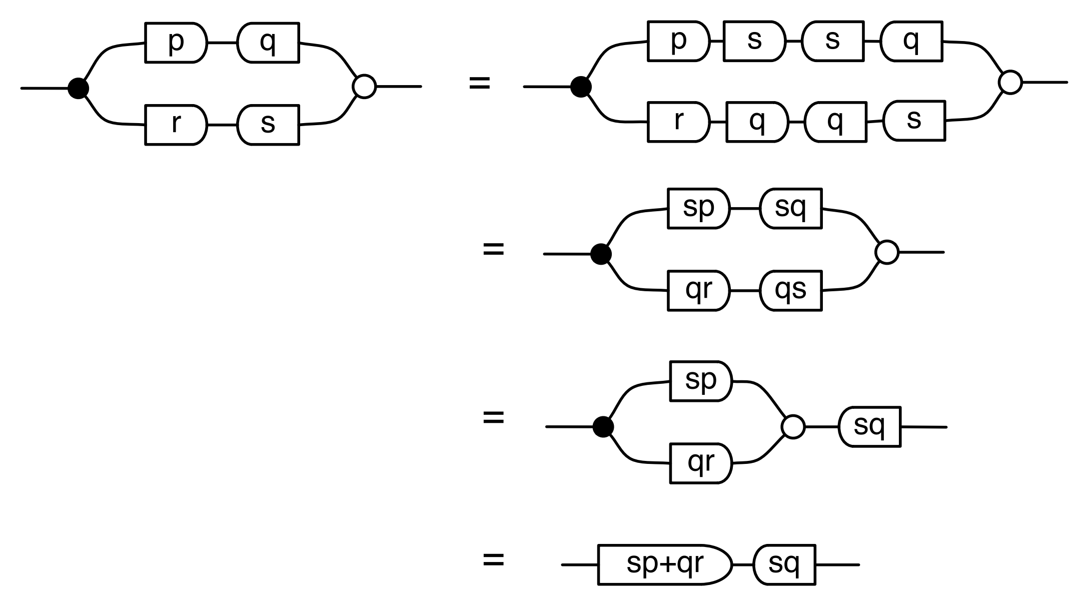


Fun Stuff - Rediscovering Fraction Arithmetic



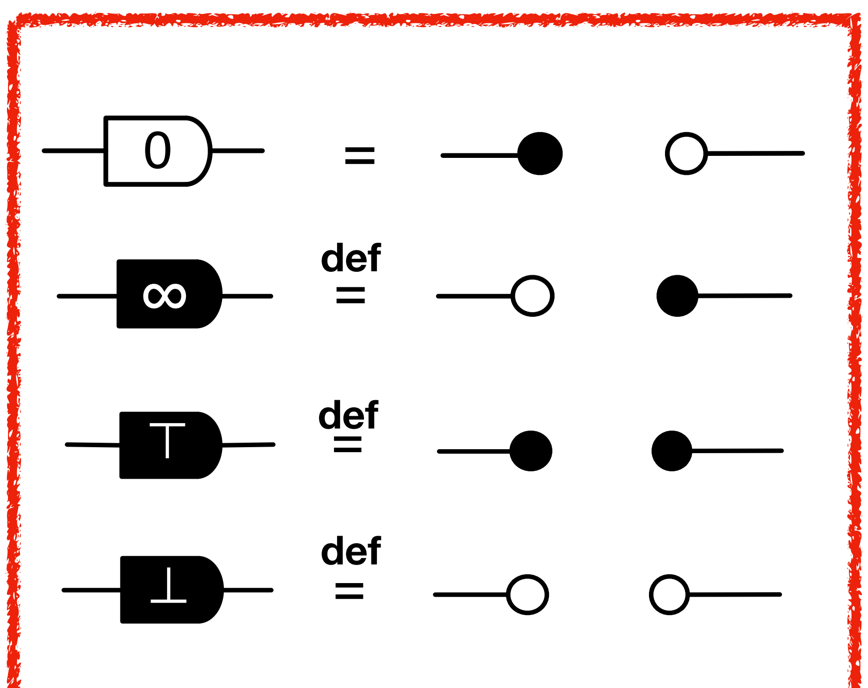






Fun Stuff - Dividing by Zero

- $\mathrm{LinRel}_{\mathbf{Q}}[1,1]$
- projective arithmetic with two additional elements
  - the unique 0-dimensional subspace $\bot = \{ (0,0) \}$
  - The unique 2-dimensional subspace $\top = \{ (x,y) \mid x,y \in \mathbf{Q} \}$



+	0	r/s	∞	$\top$	$\bot$
0	0	r/s	∞	$\top$	$\bot$
p/q	–	(sp+qr)/qs	∞	$\top$	$\bot$
∞	–	–	∞	∞	∞
$\top$	–	–	–	$\top$	∞
$\bot$	–	–	–	–	$\bot$

×	0	r/s	∞	$\top$	$\bot$
0	0	0	$\bot$	0	$\bot$
p/q	0	pr/qs	∞	$\top$	$\bot$
∞	$\top$	∞	∞	$\top$	∞
$\top$	$\top$	$\top$	∞	$\top$	∞
$\bot$	0	$\bot$	$\bot$	0	$\bot$

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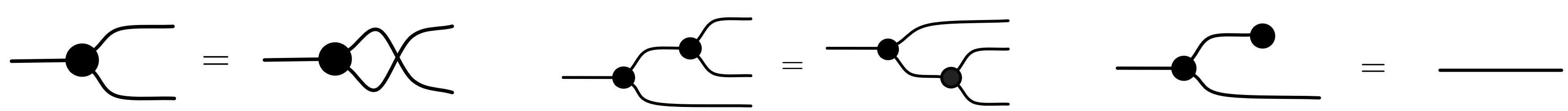
Cartesian categories

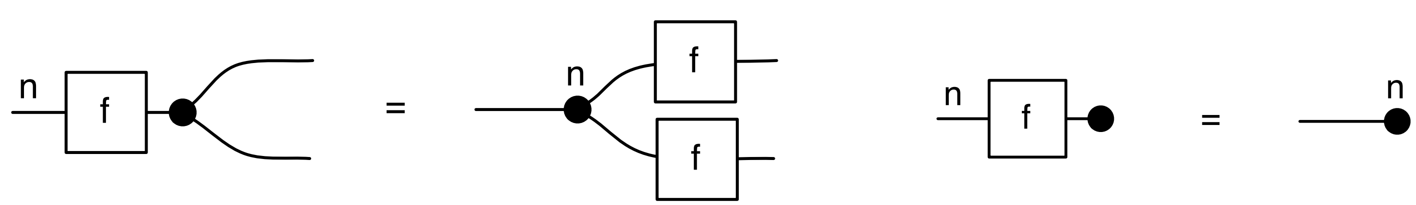
(Fox 1976)

cartesian categories are those sym. mon. cats. where every object has

commutative comonoid structure



and everything commutes with the structure

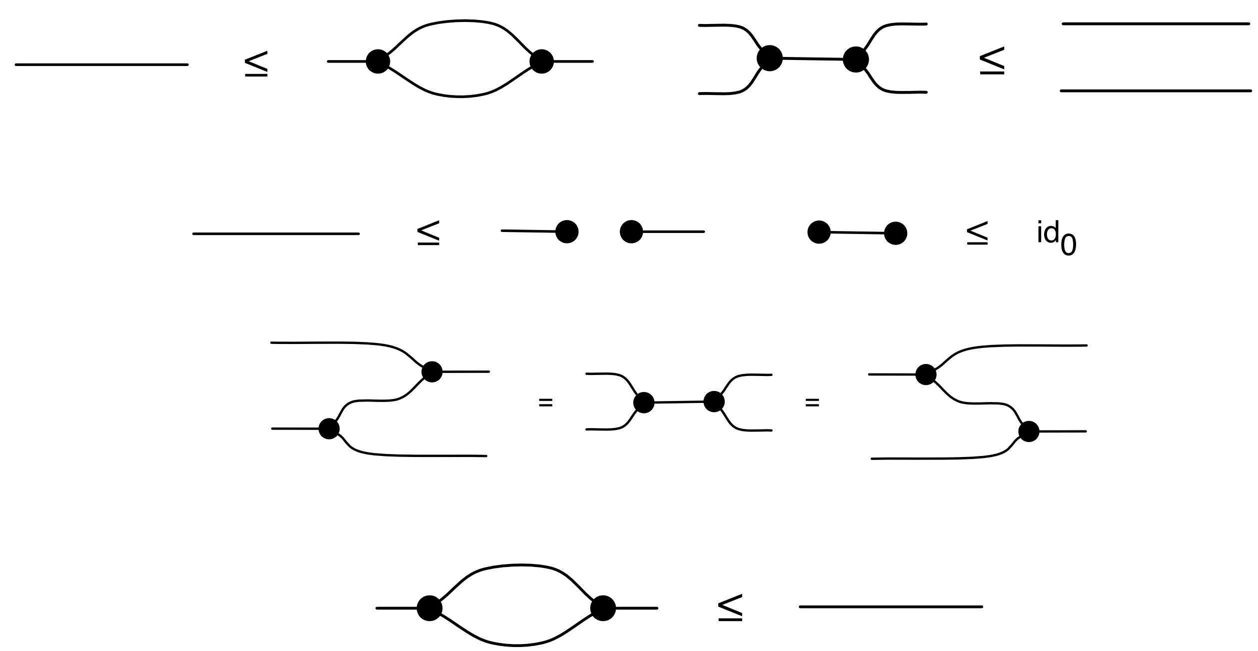


Example: $\mathbf{Set}_\times$

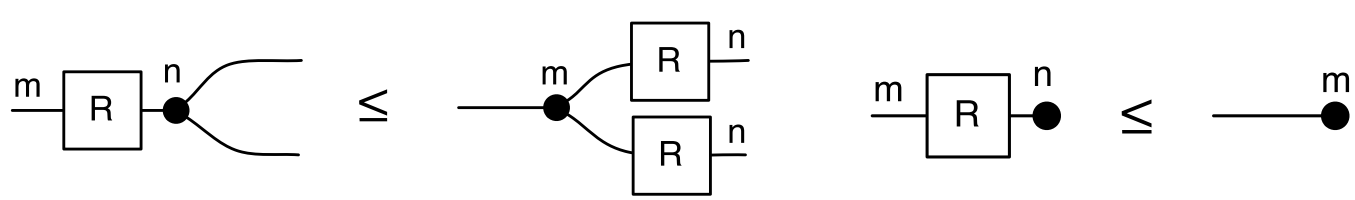
Cartesian bicategories

(Carboni, Walters 1987)

special Frobenius structure where monoid is right adjoint to comonoid



and everything laxly commutes with the structure



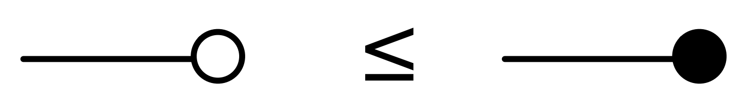
Example: **Rel**$_\times$

LinRel is a cartesian bicategory

- LinRel is a cartesian bicategory
  - In fact, it is an abelian bicategory

- To obtain a presentation we add just one inequality

(Bonchi, Holland, Pavlovic, S. 2017)



- This breaks the symmetry between white and black!

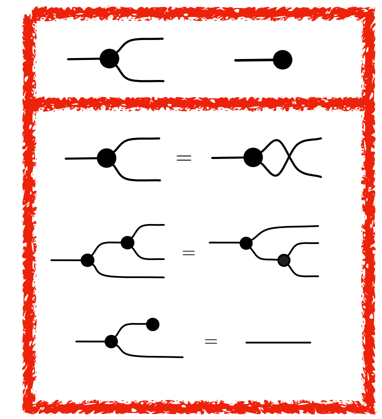
Lawvere theories

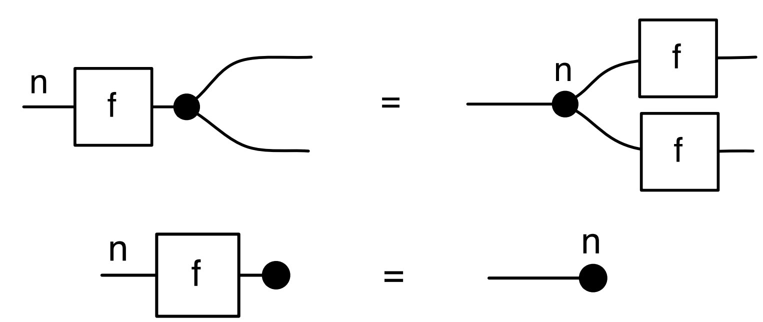
- recipe for Lawvere-theories-as-props

  1. add a cocommutative comonoid structure

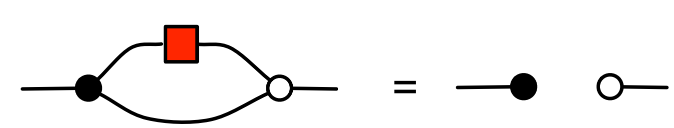
  2. make all generators commute with it

  3. add your other equations (which may make use of the comonoid structure)





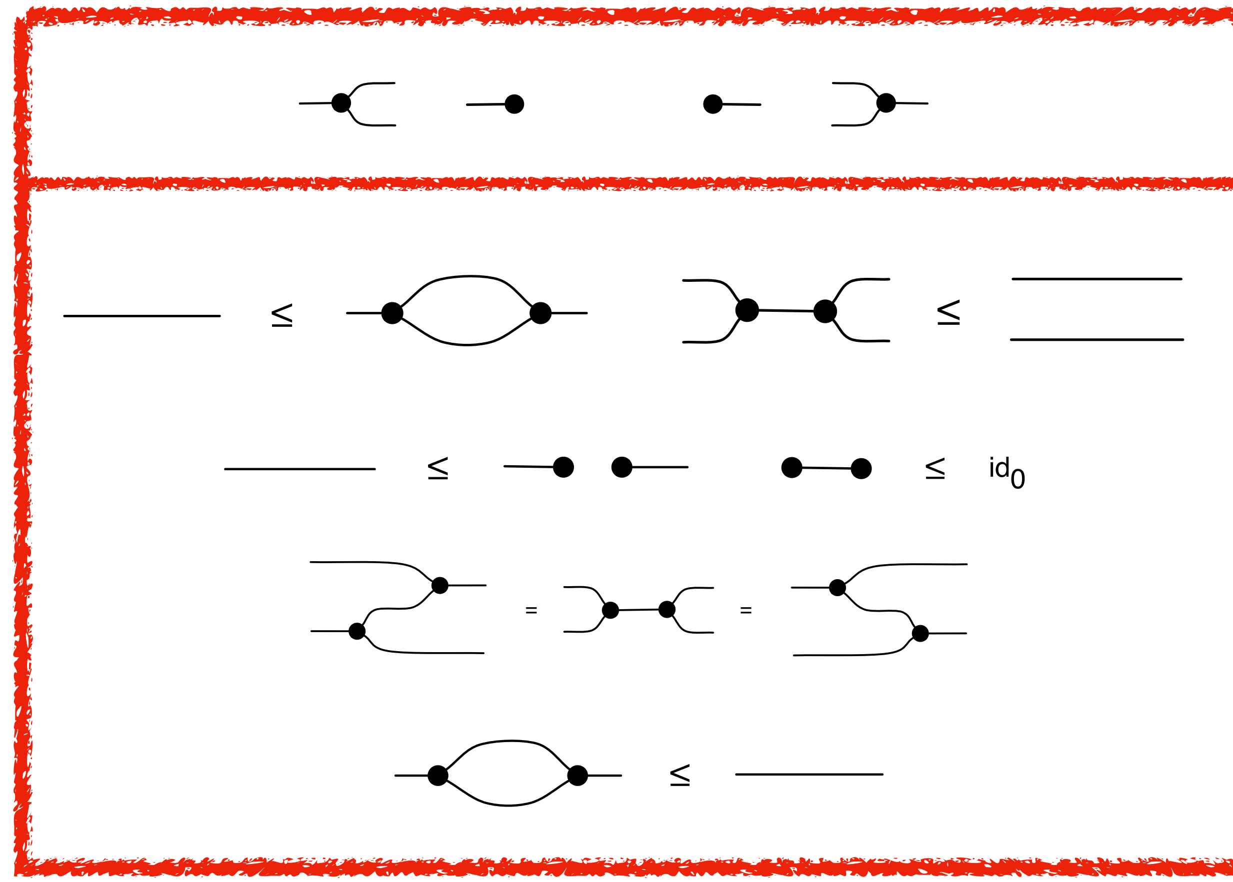
e.g. $x \cdot x^{-1} = e$

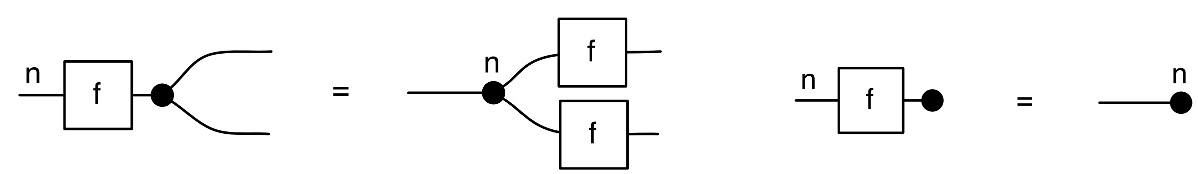


Frobenius theories

(Bonchi, Pavlovic, S. 2017)

- recipe for Frobenius-theories-as-locally-ordered-props
  - add a Frobenius bimonoid structure where monoid is right adjoint to comonoid
  - make all your generators laxly commute with it
  - add your other equations (which may make use of the Frobenius structure)





e.g. $id_0 \leq$ 

Functorial semantics

- For Lawvere theories
  - models = cartesian functors
  - homomorphisms = natural transformations
- For Frobenius theories
  - models = morphisms of cartesian bicategories
  - homomorphisms = lax natural transformations
- **Rel** models of GLA = $\mathbf{Vect}_{\mathbf{Q}}$

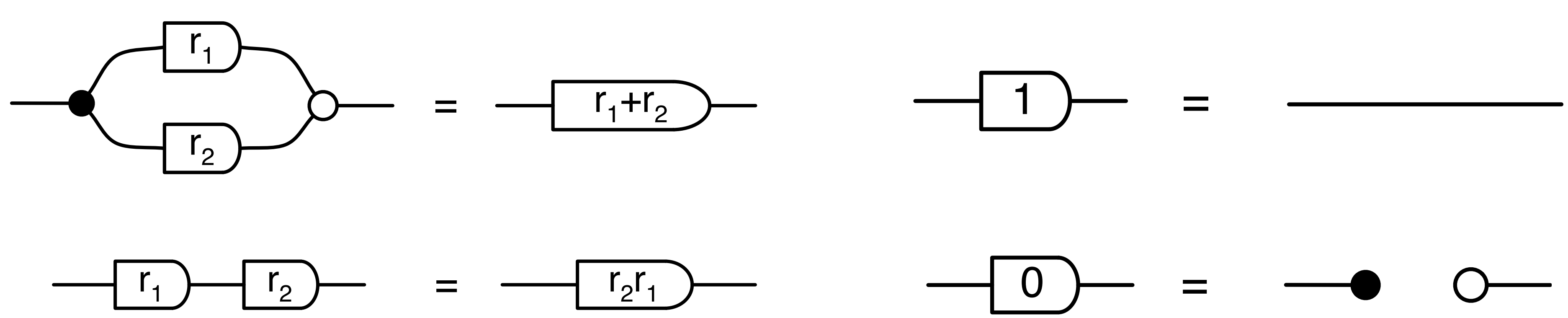
Plan

- composing props
- interacting Hopf algebras
- graphical linear algebra in action
- cartesian bicategories and Frobenius theories
- ***generating functions and signal flow graphs***
- *graphical affine algebra and non-passive electrical circuits*

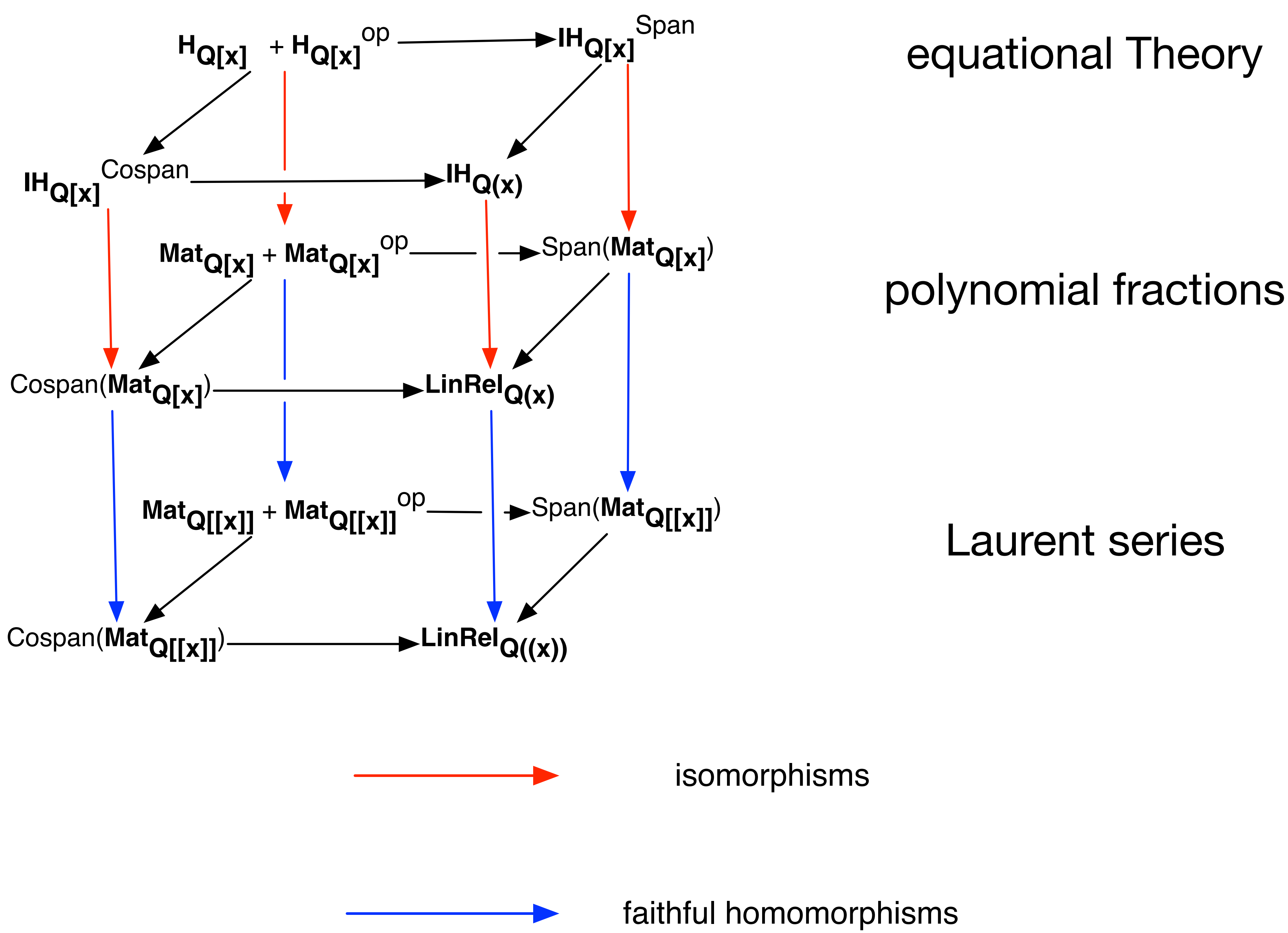
Generalising GLA

$$
\begin{tikzcd}
& \mathbf{H} + \mathbf{H}^{op} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathbf{IH}^{\mathrm{Span}} \arrow[dl] \arrow[dd] \\
\mathbf{IH}^{\mathrm{Cospan}} \arrow[rr] \arrow[dd] & & \mathbf{IH} \arrow[dd] & \\
& \mathbf{Mat}_{\mathbf{Z}} + \mathbf{Mat}_{\mathbf{Z}}{}^{op} \arrow[rr] \arrow[dl] & & \mathrm{Span}(\mathbf{Mat}_{\mathbf{Z}}) \arrow[dl] \\
\mathrm{Cospan}(\mathbf{Mat}_{\mathbf{Z}}) \arrow[rr] & & \mathbf{LinRel} &
\end{tikzcd}
$$

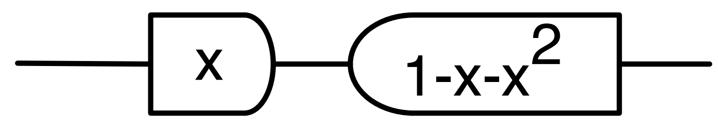
- The cube construction works for $\mathbf{Mat}_{R}$ whenever R is a PID



Generating Functions and Laurent series



Example



As linear relation over **Q**(x) is the space generated by

$$\boxed{\left(1 \quad , \quad x/(1-x-x^2)\right)}$$

As linear relation over **Q**((x)) is the space generated by

$$\boxed{\left(\underline{1},0,0,\ldots \quad , \quad \underline{0},1,1,2,3,5,8,\ldots\right)}$$

Signal flow graphs

(Shannon 1942)

- directed circuits with
  - addition gates
  - junctions
  - “registers”
    - act as integrators in the continuous semantics
    - act as one place buffers in the discrete semantics
  - guarded feedback

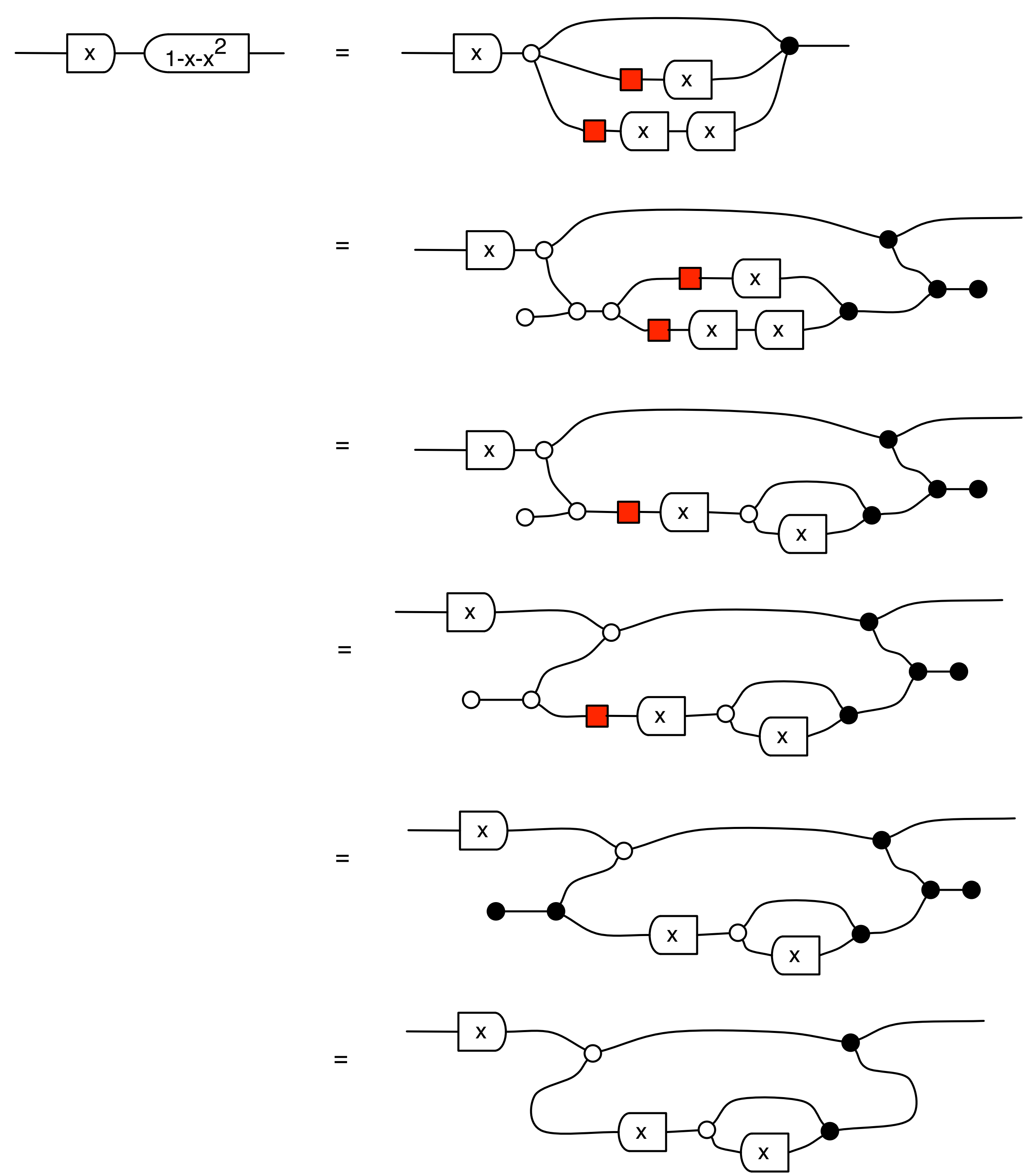
The Theory and Design of
Linear Differential Equation Machines*

Claude E. Shannon

* Report to National Defense Research Council, January, 1942.

Example - Fibonacci

(Bonchi, S., Zanasi 2015)



Plan

- composing props
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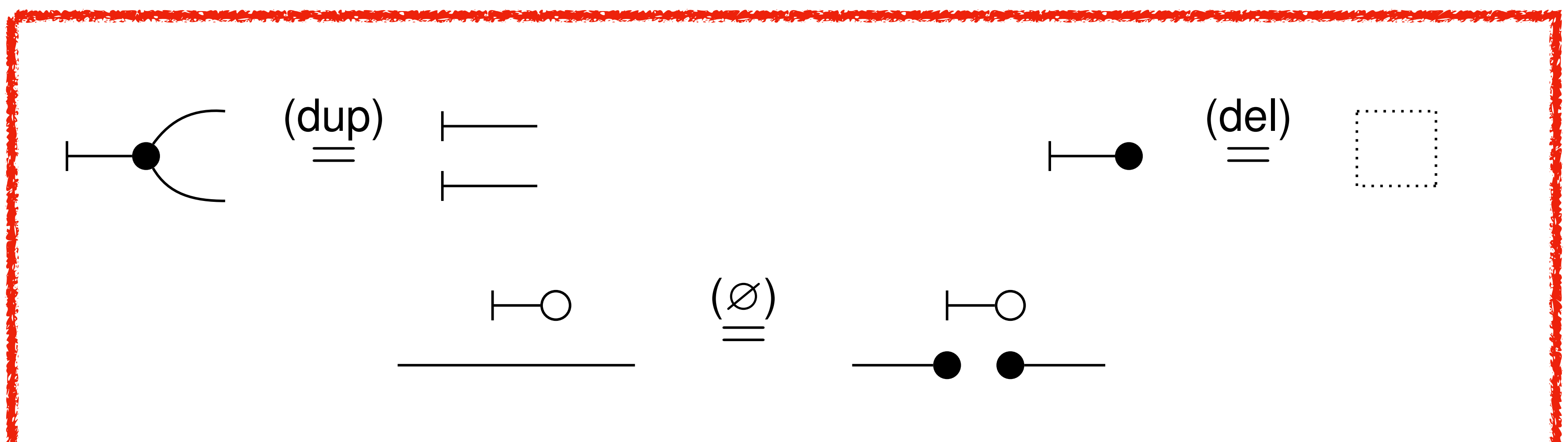
Graphical Affine Algebra

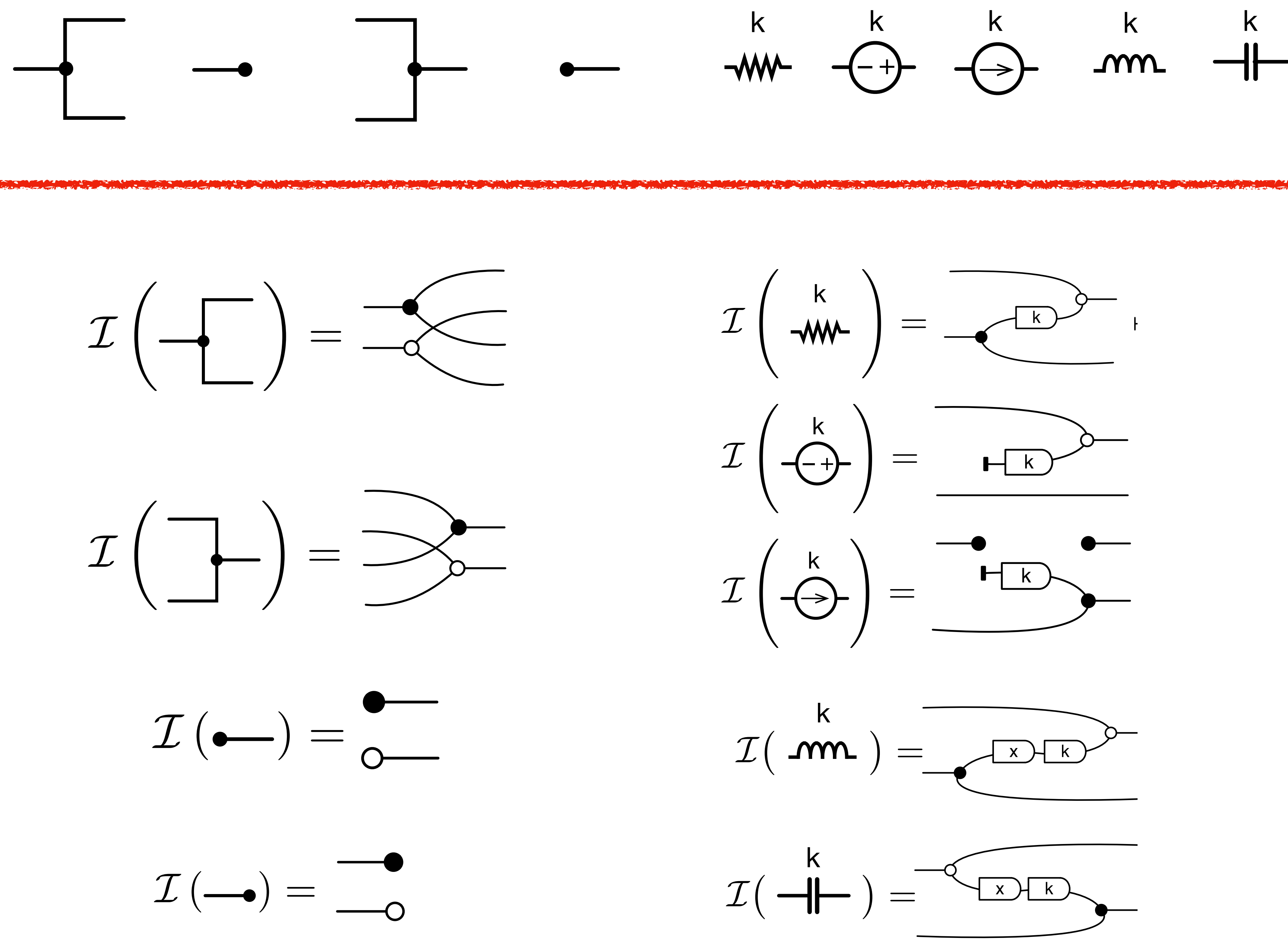
(Bonchi, Piedeleu, S., Zanasi 2019)

Definition. Given a field **k**, a **k**-affine relation k->l is a set $R \subseteq \mathbf{k}^k \times \mathbf{k}^l$ which is either empty, or s.t. there is a k-linear relation C and a vector $(\mathbf{a},\mathbf{b})$ s.t. $R = (\mathbf{a},\mathbf{b}) + C$

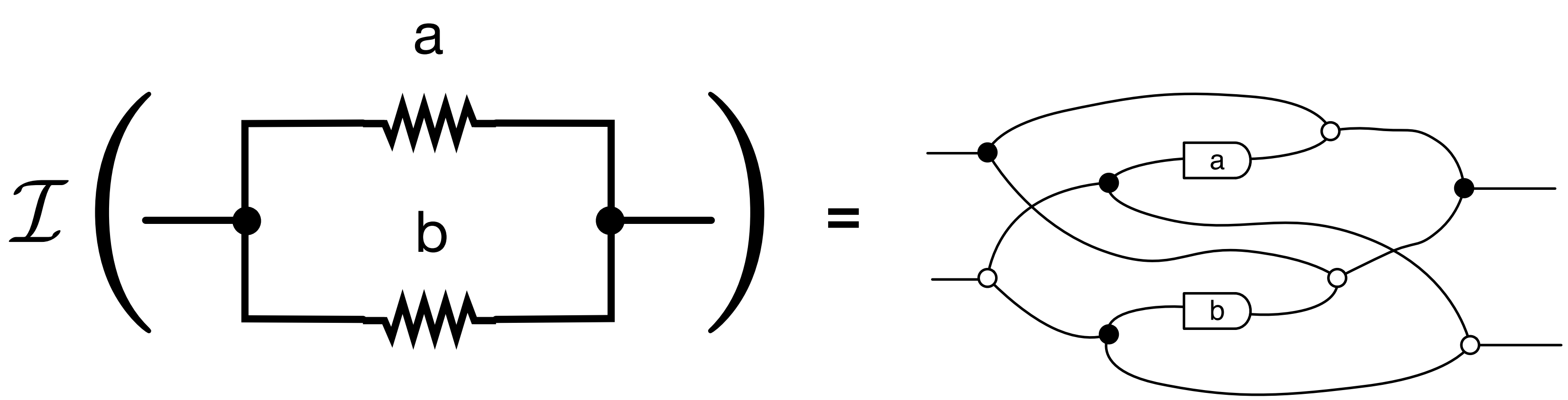


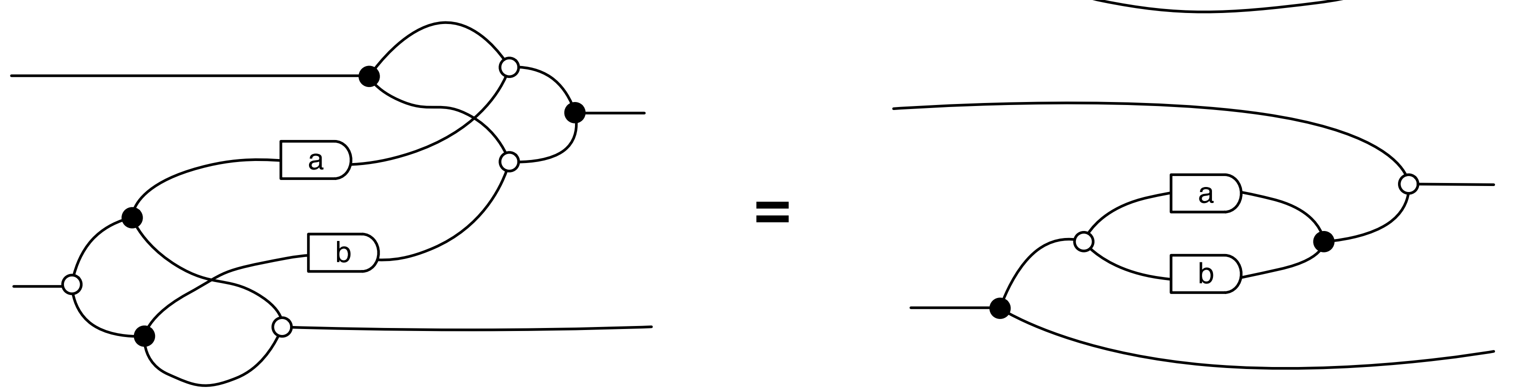
GLA + above ≅ $\mathbf{AffRel}_k$

Example: Non-passive electrical circuits

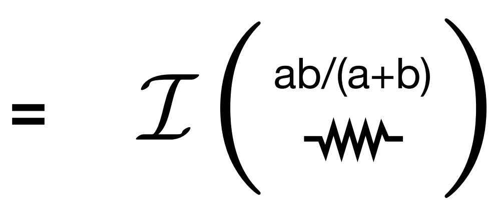


Resistors in parallel

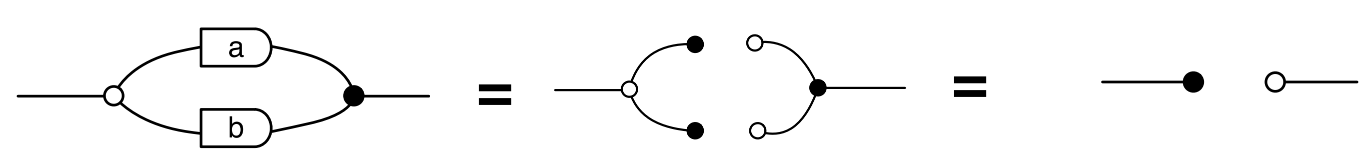




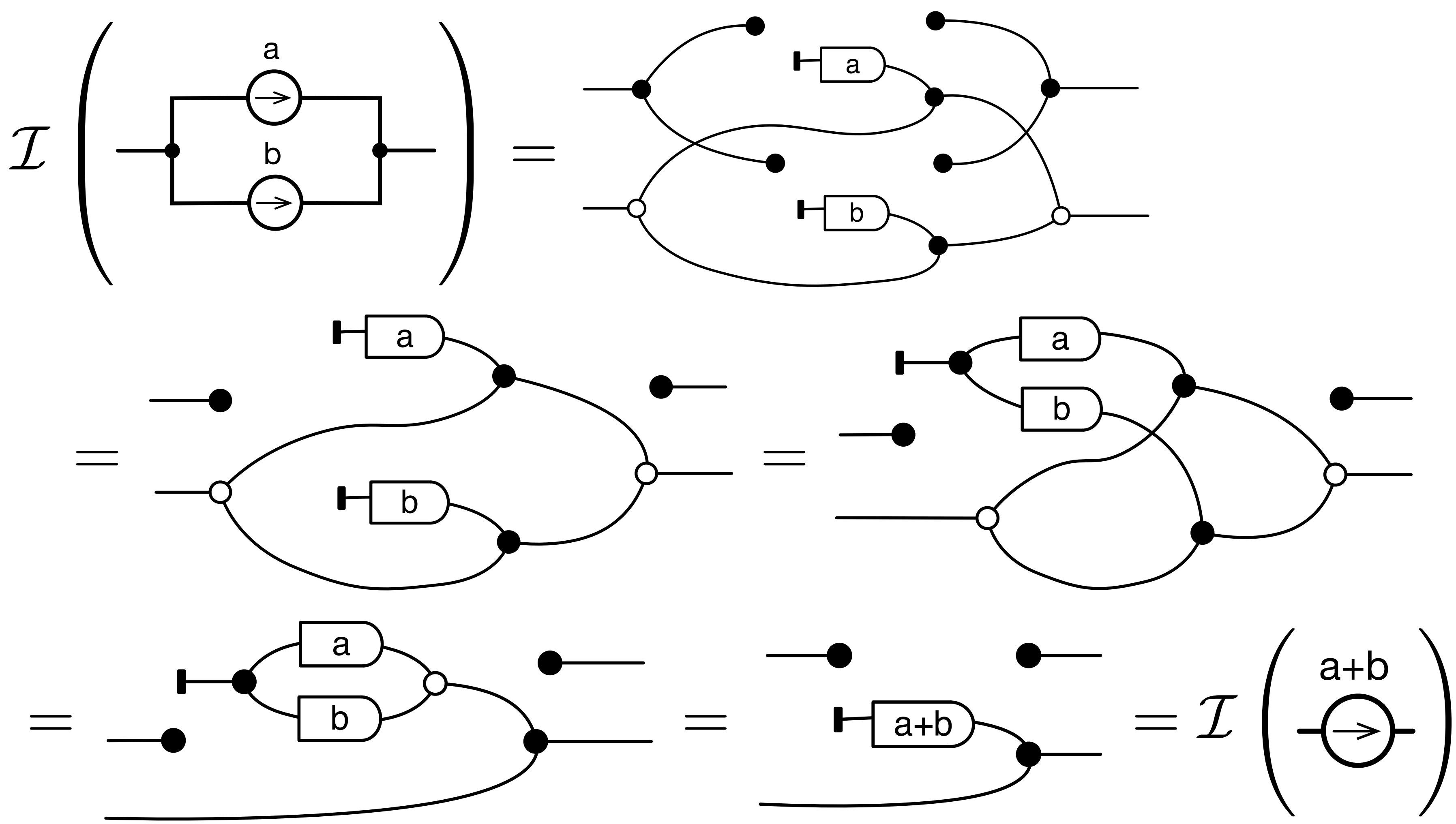
$$\left(\frac{1}{\frac{1}{a}+\frac{1}{b}}=\frac{ab}{a+b}\right)$$



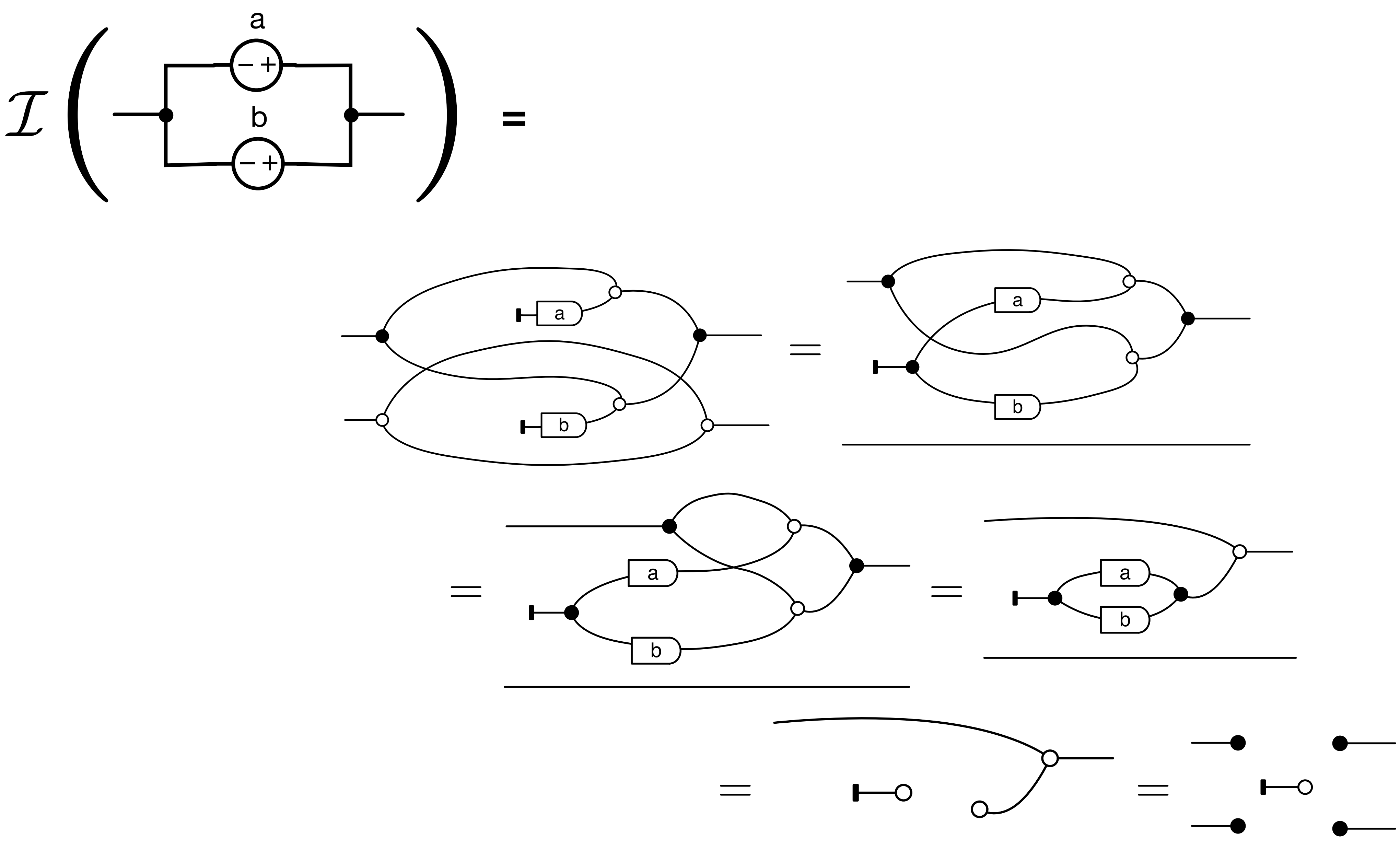
What if a=b=0?



Current sources in parallel are additive



Voltage sources in parallel are “illegal"



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See you in Tallinn!

PhD projects in open game theory, CT in programming, Frobenius theories, string diagrams in database theory and logic, …

Visitors welcome!