Non-semi-abelian split extensions in categorical algebra

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Split extensions vs actions

In the category of groups, there is a well-known equivalence

$$\mathrm{SplExt}(Grp) \sim \mathrm{Act}(Grp),$$

between the category of split extensions, that is diagrams

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift right, "p"'] & B \arrow[l, shift right, "s"'] \end{tikzcd},$$

with $k = ker(p)$, $p = coker(k)$ and $ps = 1_B$, and the category of group actions, i.e. group homomorphisms $\varphi\colon B \to Aut(X)$.

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Based on Bourn's theory of protomodular categories (1991) and on the theory of monads, this equivalence for groups was extended by D. Bourn and G.Janelidze (1998) to the context of semi-abelian categories in the sense of G. Janelidze, L. Márki and W. Tholen (2002).

Beyond the semi-abelian context

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But these actions are not equivalent to all split extensions of monoids.

The question naturally arises of characterizing the split extensions of monoids that correspond to monoid actions.

With Martins-Ferreira and Montoli we identified these split extensions.

Schreier split epimorphisms

Definition

A Schreier split epimorphism in the category of monoids is a split epimorphism (A, B, p, s) (also called a point) equipped with a unique set-theoretical map $q\colon A \dashrightarrow Ker[f]$, called the Schreier retraction,

$$X = Ker[p] \underset{k}{\overset{q}{\rightleftarrows}} A \underset{p}{\overset{s}{\leftrightarrows}} B,$$

such that, for every $a \in A$, $a = kq(a) + sp(a)$.

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Equivalently, the following conditions should be satisfied

(i) $a = kq(a) + sp(a)$

(ii) $q(k(x) + s(b)) = x$,

for all $a \in A, b \in B$ and $x \in X$, since (ii) gives de uniqueness of q.

The name was inspired by the Schreier internal categories in the category of monoids introduced by A. Patchkoria (1998).

Schreier split epis vs monoid actions

A Schreier split epimorphism

$$
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induces an action, $\varphi \colon B \to End(X)$, defined by

$$\varphi(b)(x) = q(s(b) + k(x)).$$

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Monoid actions determine Schreier split epimorphisms, via the semidirect product

$$\begin{tikzcd} X \arrow[r, tail, shift right, "\langle 1,0 \rangle"'] & X \rtimes_\varphi B \arrow[l, dotted, shift right, "\pi_X"'] \arrow[r, shift right, "\pi_B"'] & B, \arrow[l, tail, shift right, "\langle 0,1 \rangle"'] \end{tikzcd}$$

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$$X \underset{\langle 1,0 \rangle}{\overset{\pi_X}{\rightleftarrows}} X \rtimes_{\varphi} B \underset{\pi_B}{\overset{\langle 0,1 \rangle}{\leftrightarrows}} B,$$

This defines an equivalence between the category of Schreier split epimorphisms and the one of monoid actions.

Examples

Direct products $(X \times B, \pi_B, \langle 0, 1 \rangle)$ are Schreier split epimorphisms.

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For a monoid X, defining $Hol(X) = X \rtimes End(X)$, we obtain a Schreier split epimorphism:

$$\begin{tikzcd} X \arrow[r, shift right, "<1,0>"'] & Hol(X) \arrow[l, dotted, shift right, "\pi_X"'] \arrow[r, shift right, "\pi_{End(X)}"'] & End(X) \arrow[l, shift right, "<0,1>"'] \end{tikzcd}$$

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The split epimorphism

$$\begin{tikzcd} \mathbb{N} \arrow[r, "0"] & \mathbb{N} \times \mathbb{N} \arrow[r, shift right, "+"'] & \mathbb{N}, \arrow[l, shift right, "<0,1>"'] \end{tikzcd}$$

is not a Schreier split epimorphism.

First properties

Given a Schreier split epimorphism in the category of monoids

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{p}{\overset{s}{\leftrightarrows}} B$$

we have that, for $a, a' \in A$, $x \in X$ and $b \in B$,

(a) $qk = 1_X$;

(b) $qs = 0$;

(c) $q(0) = 0$;

(d) $kq(s(b) + k(x)) + s(b) = s(b) + k(x)$;

(d) $q(a + a') = q(a) + q(sp(a) + kq(a'))$.

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(d) $q(a + a') = q(a) + q(sp(a) + kq(a'))$.

A Schreier split epimorphism is a *strong split epimorphism* (also a *strong point*): the pair (k, s) is jointly strongly epimorphic.

Schreier split sequences are exact, that is $p = Coker(k)$ and so we recover the equivalence between SplExt$(Mon) \sim$ Act(Mon), with split extensions = Schreier split extensions.

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If (gf, st) is a Schreier split epimorphism then (g, t) is also a Schreier split epimorphism.

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The full subcategory of Schreier points $SPt(Mon)$ is closed under limits in the category of all points $Pt(Mon)$.

The Schreier Split Short Five Lemma

Theorem

Consider the following commutative diagram, where the two rows are Schreier split extensions:

$$\begin{tikzcd} X \arrow[r, shift right, "k"'] \arrow[d, "w"'] & A \arrow[l, shift right, dotted, "q"'] \arrow[r, shift right, two heads, "p"'] \arrow[d, "u"'] & B \arrow[l, shift right, "s"'] \arrow[d, "v"] \\ X' \arrow[r, shift right, "k'"'] & A' \arrow[l, shift right, dotted, "q'"'] \arrow[r, shift right, two heads, "p'"'] & B' \arrow[l, shift right, "s'"'] \end{tikzcd}.$$

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X' \arrow[r, shift right, "k'"'] & A' \arrow[l, dotted, shift right, "q'"'] \arrow[r, two heads, shift right, "p'"'] & B' \arrow[l, shift right, "s'"'] .
\end{tikzcd}
$$

We have that u is an isomorphism if and only if both v and w are.

An internal reflexive graph in the category of monoids

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 \longrightarrow}{\longrightarrow}}} X_0 \ , \ d_0 s_0 = 1_{X_0} = d_1 s_0,$$

is a *Schreier reflexive graph* if the split epimorphism (d_0, s_0) is a Schreier split epimorphism.

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is a *Schreier reflexive graph* if the split epimorphism (d_0, s_0) is a Schreier split epimorphism.

An internal reflexive relation, category or groupoid in *Mon* is a *Schreier reflexive relation, category or groupoid* if the underlying reflexive graph is a Schreier reflexive graph.

Mal'tsev-type properties

Theorem

Any Schreier reflexive relation

$$\begin{tikzcd} R \arrow[r, shift left=3, "r_0"] \arrow[r, shift right=3, "r_1"'] & X \arrow[l, "s_0"' description] \end{tikzcd}$$

is transitive. It is a congruence if and only if $\mathrm{Ker}(r_0)$ *is a group.*

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Example

The usual order between natural numbers:

$$\begin{tikzcd} \mathcal{O}_{\mathbb{N}} \arrow[r, shift left=2, "r_0"] \arrow[r, shift right=2, "r_1"'] & \mathbb{N}, \arrow[l, "s_0" description] \end{tikzcd}$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\},$$

is a Schreier order relation, with Schreier retraction defined by $q(x, y) = y - x$.

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Mal'tsev-type properties

Recall that a pointed finitely complete category is unital if, for every pair of objects X, Y, the morphisms

$$X \xrightarrow{\langle 1,0\rangle} X \times Y \xleftarrow{\langle 0,1\rangle} Y$$

are jointly strongly epimorphic.

A category $\mathbb{C}$ is Mal'tsev if and only every fiber $Pt_B(\mathbb{C})$ with respect to the fibration of points $cod\colon Pt(\mathbb{C}) \to \mathbb{C}$ is unital (Bourn, 1996).

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In the category of monoids all fibers $SPt_B(Mon)$ w.r. to the subfibration of Schreier points, $S\text{-}cod\colon SPt(Mon) \to Mon$, are unital categories.

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In the category of monoids all fibers $SPt_B(Mon)$ w.r.t. to the subfibration of Schreier points, $S\text{-}cod\colon SPt(Mon) \to Mon$, are unital categories. That is, for all pullback diagram of two Schreier split epimorphisms (f, r) and (g, s)

$$\begin{tikzcd}
A \times_B C \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "\pi_1"'] & C \arrow[l, shift right, "e_2"'] \arrow[d, shift right, "g"'] \\
A \arrow[u, shift right, "e_1"'] \arrow[r, shift right, "f"'] & B \arrow[l, shift right, "r"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

the morphisms induced by the universal property of the pullback $e_1 = \langle 1_A, sf \rangle$, $e_2 = \langle rg, 1_C \rangle$ are jointly strongly epimorphic.

Special Schreier homomorphisms

Definition

A homomorphism $f\colon A \to B$ is special Schreier if its kernel congruence

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\begin{tikzcd}
Eq(f) \arrow[r, shift left=2, "f_0"] \arrow[r, shift right=2, "f_1"'] & A \arrow[l, "\langle 1,1\rangle" description]
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This is equivalent to the existence of a partial subtraction on A: if $f(a_1) = f(a_2)$, then there exists a unique $x \in \mathrm{Ker}(f)$ such that $a_2 = x + a_1$. In particular, $\mathrm{Ker}(f)$ is a group.

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If $f\colon A \to B$ is a surjective special Schreier homomorphism, then it is the cokernel of its kernel.

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If $f\colon A \to B$ is a surjective special Schreier homomorphism, then it is the cokernel of its kernel. Hence we get an extension of monoids

$$X \xrightarrow{k} A \xrightarrow{f} B.$$

Properties of special Schreier extensions

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Special Schreier morphisms are used to characterize Schreier groupoids among the Schreier internal categories: they are exactly those Schreier internal categories for which d_0 is special Schreier.

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Looking at a monoid as a category with one object our approach can be compared with the one of G. Hoff (1974) where the low-dimensional cohomology of small categories was described by means of suitable extensions that are the special Schreier extensions in the case of monoids.

Special Schreier extensions with abelian kernel

A special Schreier extension of monoids $f\colon A \to B$ with abelian kernel X determines an action of B on X, $\varphi\colon B \to End(X)$, defined by

$$\varphi(b)(x) = q(a + x, a),$$

where q is the Schreier retraction of $(Eq(f), A, f_1, \langle 1, 1 \rangle)$, and $a \in A$ is such that $f(a) = b$.

Special Schreier extensions with abelian kernel

Theorem (Bourn, Martins-Ferreira, Montoli, S. (2013))

When X is an abelian group, the set $SpSExt(B, X, \varphi)$, of isomorphic classes of special Schreier extensions of B by X inducing a fixed action φ, has an abelian group structure.

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The Nine Lemma was then proved for special Schreier extensions by Martins-Ferreira, Montoli and S. (2018) and it was used to describe a push forward construction for special Schreier extensions with abelian kernel in monoids, an alternative, functorial description of the Baer sum of such extensions.

The special Schreier Nine Lemma

Theorem (Martins-Ferreira, Montoli, S.(2018))

Consider the following commutative diagram, where the three columns are special Schreier extensions:

$$\begin{tikzcd}
N \arrow[r, "\eta"] \arrow[d, tail, "l"'] & H \arrow[r, "\lambda"] \arrow[d, tail, "r"'] & K \arrow[d, tail, "s"] \\
X \arrow[r, "\sigma"] \arrow[d, two heads, "f"'] & Y \arrow[r, "\varphi"] \arrow[d, two heads, "g"'] & Z \arrow[d, two heads, "p"] \\
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1. *If the first two rows are special Schreier extensions, then the lower also is;*
2. *if the last two rows are special Schreier extensions, then the upper also is;*
3. *if $\varphi\sigma = 0$ and the first and the last rows are special Schreier extensions, then the middle also is.*

The push forward construction

Theorem

Consider the following situation:

$$\begin{tikzcd} X \arrow[r, "k", tail] \arrow[d, "g"'] & A \arrow[r, "f", two heads] & B, \\ Y & & \end{tikzcd}$$

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where

- f is a special Schreier extension with abelian kernel;

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- φ is the corresponding action of B on X;

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- φ is the corresponding action of B on X;

- Y is an abelian group, equipped with an action ψ of B on it;

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Consider the following situation:

$$\begin{tikzcd} X \arrow[r, "k", tail] \arrow[d, "g"'] & A \arrow[r, "f", two heads] & B, \\ Y \end{tikzcd}$$

where

- f is a special Schreier extension with abelian kernel;
- φ is the corresponding action of B on X;
- Y is an abelian group, equipped with an action ψ of B on it;
- g is a morphism which is equivariant, that is, for all $b \in B$ and all $x \in X$, $g(b \cdot_{\varphi} x)) = (b \cdot_{\psi} g(x))$.

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Then there exists a special Schreier extension f' with kernel Y and codomain B, which induces the action ψ and is universal among all such extensions.

It means that, given any diagram of the form

$$
\begin{tikzcd}
X \arrow[r, tail, "k"] \arrow[d, "g"] \arrow[dd, bend right, "u"'] & A \arrow[r, two heads, "f"] \arrow[d, "g'"'] \arrow[dd, bend left, "v"] & B \arrow[d, equal] \\
Y \arrow[r, tail, "k'"] \arrow[d, "r"] & C \arrow[r, two heads, "f'"] \arrow[d, dotted, "\alpha"'] & B \arrow[d, equal] \\
Z \arrow[r, tail, "l"'] & E \arrow[r, two heads, "p"'] & B,
\end{tikzcd}
$$

where p is a special Schreier extension with abelian kernel Z, (u, v) is a morphism of extensions and $u = rg$, then there exists a unique homomorphism α such that $v = \alpha g'$ and (r, α) is a morphism of extensions.

Looking for a conceptual notion

Many properties of all split epimorphisms in a protomodular category are satisfied by the Schreier split epimorphisms in the category of monoids.

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Monoids with operations are monoids $(M, +, 0)$ that may be equipped with other binary and unary operations such that

- every binary operation $* \neq +$ is distributive with respect to the monoid operation and $x * 0 = 0$ for all $x \in M$,
- for every unary operation w, $w(x + y) = w(x) + w(y)$, and $w(x * y) = w(x) * y$.

This is the counterpart for monoids of Porter's "groups with operations"(1987).

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A conceptual notion to capture this algebraic context was introduced, in the pointed case, by Bourn, Martins-Ferreira, Montoli and S. (2013), under the name of $\mathcal{S}$-protomodular category.

Let $\mathbb{C}$ be a pointed finitely complete category and S be a class of points in $\mathbb{C}$ which is stable under pullbacks.

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Examples are the categories of monoids, semirings (indeed, all categories of monoids with operations), and also the Jónsson-Tarski varieties of algebras as proved by Martins-Ferreira and Montoli(2017). All of them are S-protomodular for the class S of Schreier split epimorphisms.

Protomodularity relative to a class S

When $\mathbb{C}$ is S-protomodular then any change-of-base functor with respect to the subfibration of S-points, $S\text{-}cod\colon SPt(\mathbb{C}) \to \mathbb{C}$, is conservative.

Internal S-structures are defined in an analogous way as the ones defined when S is the class of Schreier split epimorphisms and have similar properties.

For example, we say that a morphism $f\colon X \to Y$ is S-special if its kernel equivalence relation is an S-special equivalence relation.

An object X is S-special if the terminal morphism, $X \to 1$, is S-special.

If the category $\mathbb{C}$ is S-protomodular then

- Every S-reflexive relation (R, r_0, r_1, s_0) is transitive. It is an S-equivalence relation if and only if r_0 is S-special.
- The full subcategory of S-special objects is protomodular and was called the *protomodular core of* $\mathbb{C}$ *with respect to* S.

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- If $\mathbb{C}$ is the category of monoids (semirings), then its protomodular core with respect to the class S of Schreier split epimorphisms is the category of groups (rings, respectively).

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- If $\mathbb{C}$ is the category of monoids (semirings), then its protomodular core with respect to the class S of Schreier split epimorphisms is the category of groups (rings, respectively).
- Indeed, in any category of monoids with operations, the protomodular core with respect to the class S of Schreier split epimorphisms is the corresponding subcategory of groups with operations.

Adding some conditions to S-protomodularity

Several conditions have been added to semi-abelian categories in order to get a closer group-like behaviour, like

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- (locally, fiberwise) algebraic cartesian closedness, (Gray (2012); Bourn, Gray (2012));
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Martins-Ferreira, Montoli and S. (2018) studied "relative" versions of above conditions in the framework of S-protomodular categories in parallel with the "absolute" semi-abelian context.

Relative notions

Definition

An S-protomodular category $\mathcal{C}$ is

1. *locally S-algebraically cartesian closed (S-lacc) if, for every morphism f in $\mathcal{C}$, the change-of-base functor f^* for the subfibration of points in S has a right adjoint.*
2. *fiberwise S-algebraically cartesian closed (S-fwacc) if, for every split epimorphism f in $\mathcal{C}$, the change-of-base functor f^* for the subfibration of points in S has a right adjoint;*
3. *S-algebraically coherent if, for every morphism f in $\mathcal{C}$, the change-of-base functor f^* for the subfibration of points in S preserves jointly strongly epimorphic pairs.*

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$$\begin{tikzcd} S\text{-(lacc)} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & S\text{-alg. coherent} \\ & S\text{-(fwacc)} \end{tikzcd}$$

The relative versions of the conditions mentioned above enabled us to obtain a hierarchy among S-protomodular categories that, for S the class of Schreier split epimorphisms, is the following:

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Condition	**Examples**
S-protomodular	Jónsson-Tarski varieties
S-(SH) (Martins-Ferreira, Montoli)	monoids with operations
S-(fwacc)	*Mon*, *SRng*
S-alg. coherent	*Mon*, *SRng*
S-(lacc)	*Mon*

From monoids to unitary magmas

Our purpose now is to describe another generalization of the theory of split extensions, namely from monoids to unitary magmas, that is, to algebraic structures of the form $M = (M, 0, +)$, where the only axiom required is $x + 0 = x = 0 + x$.

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Definition

Let B and X be magmas. A map $h\colon B \times X \to X$, written as $(b, x) \mapsto bx$, is said to be an action of B on X if $0x = x,\ b0 = 0$, *for all $x \in X$ and $b \in B$.*

Actions and semidirect products of unitary magmas

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Definition

For magmas B and X and an action of B on X, the semidirect product diagram is the diagram

$$\begin{tikzcd} X \arrow[r, shift right, "\langle 1,0\rangle"'] & X \rtimes B \arrow[l, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] & B \arrow[l, shift right, "\langle 0,1\rangle"'] \end{tikzcd}$$

in which $X \rtimes B$ is a magma whose underlying set is $X \times B$ and whose addition is defined by $(x, b) + (x', b') = (x + bx', b + b')$.

Definition

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$$X \underset{\kappa}{\overset{\lambda}{\leftrightarrows}} A \underset{\alpha}{\overset{\beta}{\leftrightarrows}} B$$

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Defining split extensions of magmas

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in which:

(a) *X, A, and B are magmas, α, β, and κ are magma homomorphisms, and λ preserves zero;*

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(b) *the equalities (1)* $\lambda\kappa = 1,\ \alpha\beta = 1,$
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(4) $\lambda(\kappa(x) + \beta(b)) = x,$

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(2) $\lambda\beta = 0,\ \alpha\kappa = 0$,
(3) $\kappa\lambda + \beta\alpha = 1$,
(4) $\lambda(\kappa(x) + \beta(b)) = x$,
(5) $\kappa(x) + (\beta(b) + a) = (\kappa(x) + \beta(b)) + a$,
(6) $\kappa(x) + (a + \beta(b)) = (\kappa(x) + a) + \beta(b)$,
(7) $a + (\kappa(x) + \beta(b)) = (a + \kappa(x)) + \beta(b)$,
hold for all $x, x' \in X$, $a \in A$ and $b, b' \in B$.

Split extensions vs semidirect products

Consider the diagram
$$
\begin{tikzcd}
X \arrow[r, shift right, "\kappa"'] \arrow[d, equal] & A \arrow[l, shift right, "\lambda"'] \arrow[r, shift right, "\alpha"'] \arrow[d, shift right, "\varphi"'] & B \arrow[l, shift right, "\beta"'] \arrow[d, equal] \\
X \arrow[r, shift right, "\iota_1"'] & X \rtimes B \arrow[l, shift right, "\pi_1"'] \arrow[r, shift right, "\pi_2"'] \arrow[u, shift right, "\psi"'] & B \arrow[l, shift right, "\iota_2"']
\end{tikzcd}
$$
in which:

- the top row is a split extension of magmas;
- the bottom row is a semidirect product diagram in which B acts on X as $bx = \lambda(\beta(b) + k(x))$, the action induced by the split extension;
- φ is defined by $\varphi(a) = (\lambda(a), \alpha(a))$;
- ψ is defined by $\psi(x, b) = \kappa(x) + \beta(b)$.

Then φ, ψ are homomorphisms of unitary magmas, inverse to each other.

The following lemma collects purely categorical properties of a split extension

$$X \underset{\kappa}{\overset{\lambda}{\leftrightarrows}} A \underset{\alpha}{\overset{\beta}{\leftrightarrows}} B$$

Lemma

(a) *κ and β are jointly strongly epic in the category of magmas;*

(b) *λ and α form a product diagram in the category of sets;*

(c) *κ is a kernel of α and α is a cokernel of κ in the category of magmas.*

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It is constructed as follows: to each morphisms of extensions $(f, u, p) : E \to E'$,

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X' \arrow[r, shift right, "\kappa'"'] & A' \arrow[l, shift right, "\lambda'"'] \arrow[r, shift right, "\alpha'"'] & B' \arrow[l, shift right, "\beta'"']
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assigns the morphism $(f, u) : (B, X, h) \to (B', X', h')$ between the corresponding actions.

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assigns the morphism $(f, u) : (B, X, h) \to (B', X', h')$ between the corresponding actions.
Conversely, to each morphism of actions $(f, u) : (B, X, h) \to (B', X', h')$ corresponds a morphisms (f, u, p) between the semidirect product extensions, where p is defined by $p(x, b) = (u(x), f(b))$.

Composition of split extensions

Lemma

The composite $(\gamma\alpha, \delta\gamma)$ of two split extensions

$$E: \; X \underset{\kappa}{\overset{\lambda}{\leftrightarrows}} A \underset{\alpha}{\overset{\beta}{\leftrightarrows}} B$$

$$F: \; Y \underset{\mu}{\overset{\nu}{\leftrightarrows}} B \underset{\gamma}{\overset{\delta}{\leftrightarrows}} D$$

is a split extension if and only if the equality

$$\mu(y)(\delta(d)x) = (\mu(y) + \delta(d))x$$

holds for all $y \in Y, d \in D$ and $x \in X$.

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So, in particular, it holds when the action induced by the extension E satisfies the condition

$$b(b'x) = (b + b')x.$$

Other classes of split extensions of magmas

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the corresponding subclasses $\mathcal{E}'$ and $\mathcal{E}''$ of split extensions have a nicer behaviour.

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Indeed, they are not only stable under pullbacks but also closed under composition.

For each of these three classes of split extensions, the category of unitary magmas is S-protomodular and so it satisfies the Split Short Five Lemma.

Final remarks

Everything is well known when we replace magmas with monoids. In particular, in the definition of split extensions, the three last conditions are automatically satisfied and they become simply Schreier split extensions.

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The group-theoretic case of our last Theorem is nothing but a categorical formulation of a first step towards a cohomological description of group extensions.

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Different approaches to a cohomology of monoids were defined by several authors, considering suitable notions of monoid extensions.

A generalization of the classical Eilenberg-Mac Lane cohomology theory from groups to monoids was developed by Martins-Ferreira, Montoli, Patchkoria and S. (2019), yielding a new, additional interpretation of this classical theory via some kind of monoid extensions, that are the special Schreier extensions when the kernel is a group.

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