Latent Force Models

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Outline

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Motivation and Review

Motion Capture Example

Styles of Machine Learning

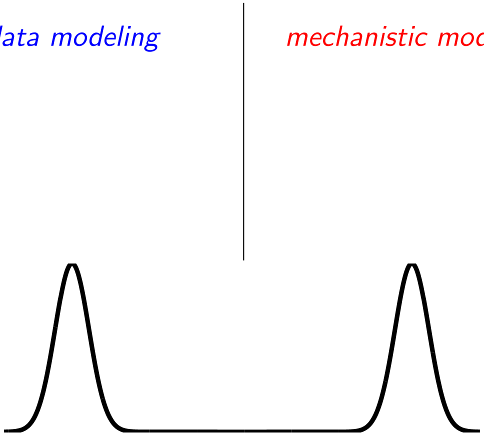
Background: interpolation is easy, extrapolation is hard

- Urs Hölzle keynote talk at NIPS 2005.
  - Emphasis on massive data sets.
  - Let the data do the work—more data, less extrapolation.
- Alternative paradigm:
  - Very scarce data: computational biology, human motion.
  - How to generalize from scarce data?
  - Need to include more assumptions about the data (e.g. invariances).

General Approach

Broadly Speaking: Two approaches to modeling

data modeling | *mechanistic modeling*



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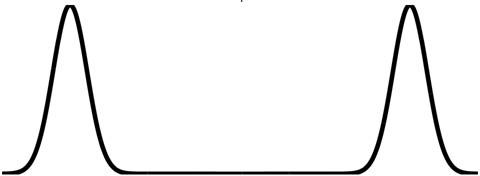
data modeling

let the data "speak"

mechanistic modeling



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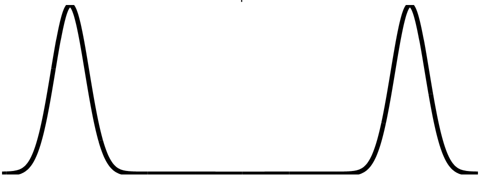

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General Approach

Broadly Speaking: Two approaches to modeling

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let the data "speak"

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adaptive models

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impose physical laws

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General Approach

Broadly Speaking: Two approaches to modeling

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impose physical laws

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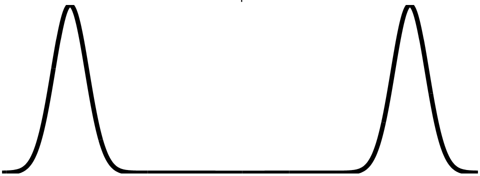
differential equations



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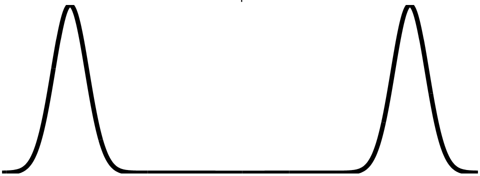
digit recognition	



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Weakly Mechanistic

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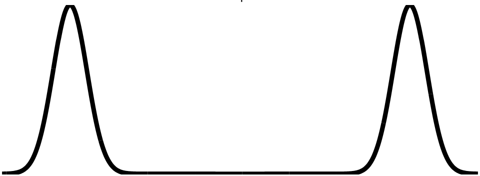


Weakly Mechanistic vs Strongly Mechanistic

- Underlying data modeling techniques there are *weakly mechanistic* principles (e.g. smoothness).
- In physics the models are typically *strongly mechanistic*.
- In principle we expect a range of models which vary in the strength of their mechanistic assumptions.
- This work is one part of that spectrum: add further mechanistic ideas to weakly mechanistic models.

Dimensionality Reduction

- Linear relationship between the data, $\mathbf{X} \in \Re^{n \times p}$, and a reduced dimensional representation, $\mathbf{F} \in \Re^{n \times q}$, where $q \ll p$.

$$\mathbf{X} = \mathbf{F}\mathbf{W} + \boldsymbol{\epsilon},$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$$

- Integrate out $\mathbf{F}$, optimize with respect to $\mathbf{W}$.
- For Gaussian prior, $\mathbf{F} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$
  - and $\Sigma = \sigma^2\mathbf{I}$ we have probabilistic PCA (Tipping and Bishop, 1999; Roweis, 1998).
  - and Σ constrained to be diagonal, we have factor analysis.

Dimensionality Reduction: Temporal Data

- Deal with temporal data with a temporal latent prior.
- Independent Gauss-Markov priors over each $f_i(t)$ leads to : Rauch-Tung-Striebel (RTS) smoother (Kalman filter).
- More generally consider a Gaussian process (GP) prior,

$$p(\mathbf{F}|\mathbf{t}) = \prod_{i=1}^{q} \mathcal{N}\left(\mathbf{f}_{:,i}|\mathbf{0}, \mathbf{K}_{f_{:,i},f_{:,i}}\right).$$

Joint Gaussian Process

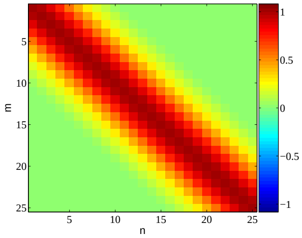
- Given the covariance functions for $\{f_i(t)\}$ we have an implied covariance function across all $\{x_i(t)\}$—(ML: semi-parametric latent factor model (Teh et al., 2005), Geostatistics: linear model of coregionalization).
- Rauch-Tung-Striebel smoother has been preferred
  - linear computational complexity in n.
  - Advances in sparse approximations have made the general GP framework practical. (Titsias, 2009; Snelson and Ghahramani, 2006; Quiñonero Candela and Rasmussen, 2005).

Gaussian Process: Exponentiated Quadratic Covariance

- Take, for example, exponentiated quadratic form for covariance.

$$k\left(t, t'\right) = \alpha \exp\left(-\frac{\left\|t - t'\right\|^2}{2\ell^2}\right)$$

- Gaussian process over latent functions.



Mechanical Analogy

Back to Mechanistic Models!

- These models rely on the latent variables to provide the dynamic information.
- We now introduce a further dynamical system with a *mechanistic* inspiration.
- Physical Interpretation:
  - the latent functions, $f_i(t)$ are q forces.
  - We observe the displacement of p springs to the forces.,
  - Interpret system as the force balance equation, $\mathbf{XD} = \mathbf{FS} + \epsilon$.
  - Forces act, e.g. through levers — a matrix of sensitivities, $\mathbf{S} \in \Re^{q \times p}$.
  - Diagonal matrix of spring constants, $\mathbf{D} \in \Re^{p \times p}$.
  - Original System: $\mathbf{W} = \mathbf{SD}^{-1}$.

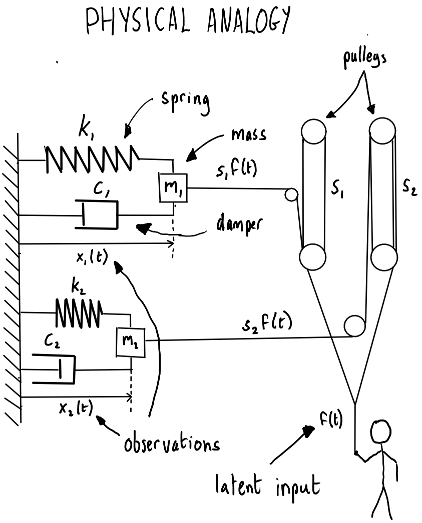
Extend Model

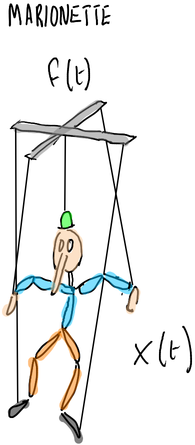
- Add a damper and give the system mass.

$$\mathbf{F}\mathbf{S} = \ddot{\mathbf{X}}\mathbf{M} + \dot{\mathbf{X}}\mathbf{C} + \mathbf{X}\mathbf{D} + \epsilon.$$

- Now have a second order mechanical system.
- It will exhibit inertia and resonance.
- There are many systems that can also be represented by differential equations.
  - When being forced by latent function(s), $\{f_i(t)\}_{i=1}^q$, we call this a *latent force model*.

Physical Analogy





- For Gaussian process we can compute the covariance matrices for the output displacements.
- For one displacement the model is

$$m_k \ddot{x}_k(t) + c_k \dot{x}_k(t) + d_k x_k(t) = b_k + \sum_{i=0}^{q} s_{ik} f_i(t), \qquad (1)$$

  where, m_k is the kth diagonal element from $\mathbf{M}$ and similarly for c_k and d_k. s_{ik} is the i, kth element of $\mathbf{S}$.
- Model the latent forces as q independent, GPs with exponentiated quadratic covariances

$$k_{f_i f_l}(t, t') = \exp\left(-\frac{(t-t')^2}{2\ell_i^2}\right)\delta_{il}.$$

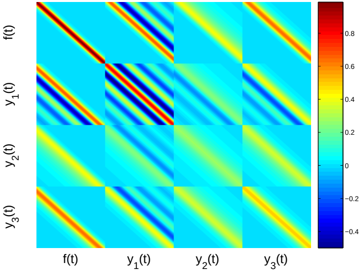
Covariance for ODE Model

- Exponentiated Quadratic Covariance function for $f(t)$

$$x_j(t) = \frac{1}{m_j \omega_j} \sum_{i=1}^{q} s_{ji} \exp(-\alpha_j t) \int_0^t f_i(\tau) \exp(\alpha_j \tau) \sin(\omega_j (t-\tau)) \mathrm{d}\tau$$

- Joint distribution for $x_1(t)$, $x_2(t)$, $x_3(t)$ and $f(t)$. Damping ratios:

ζ_1	ζ_2	ζ_3
0.125	2	1



Covariance for ODE Model

- Analogy

$$x = \sum_i \mathbf{e}_i^\top \mathbf{f}_i \quad \mathbf{f}_i \sim \mathcal{N}\left(\mathbf{0}, \Sigma_i\right) \to x \sim \mathcal{N}\left(0, \sum_i \mathbf{e}_i^\top \Sigma_i \mathbf{e}_i\right)$$

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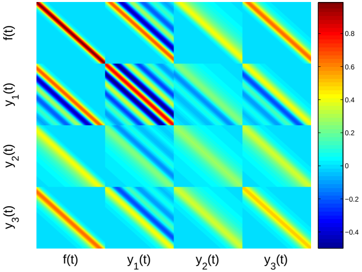

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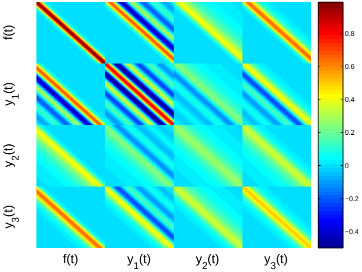

Joint Sampling of $x(t)$ and $f(t)$

- `lfmSample`

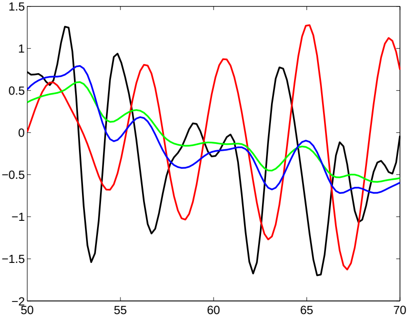

Figure: Joint samples from the ODE covariance, *black*: $f(t)$, *red*: $x_1(t)$ (underdamped), *green*: $x_2(t)$ (overdamped), and *blue*: $x_3(t)$ (critically damped).

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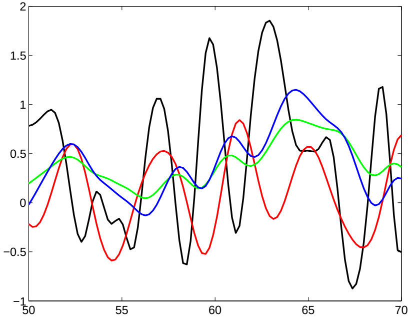

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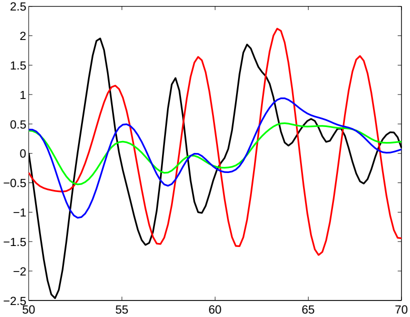

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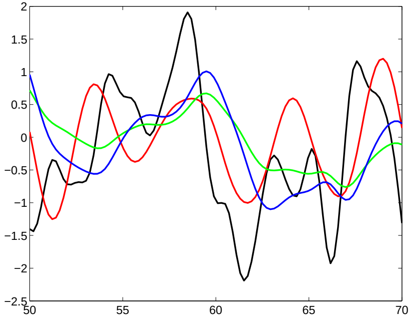

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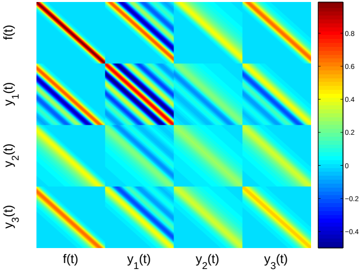

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Example: Motion Capture

Mauricio Alvarez and David Luengo (Álvarez et al., 2009)

- Motion capture data: used for animating human motion.
- Multivariate time series of angles representing joint positions.
- Objective: generalize from training data to realistic motions.
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Prediction of Test Motion

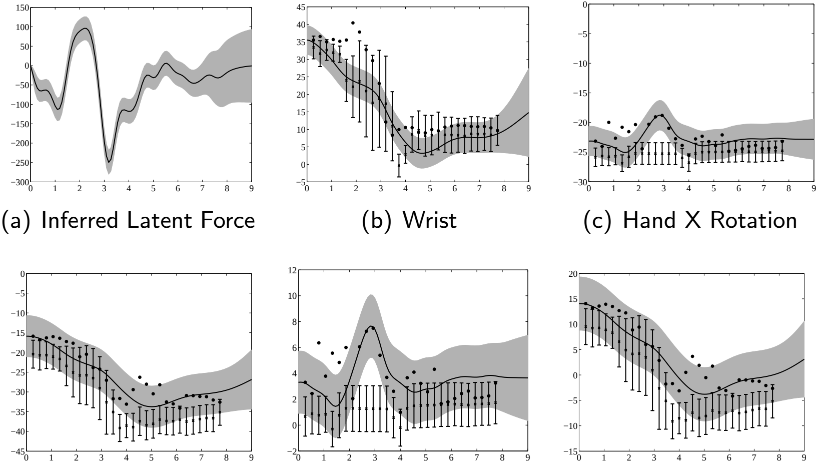
- Model left arm only.
- 3 balancing motions (18, 19, 20) from subject 49.
- 18 and 19 are similar, 20 contains more dramatic movements.
- Train on 18 and 19 and testing on 20
- Data was down-sampled by 32 (from 120 fps).
- Reconstruct motion of left arm for 20 given other movements.
- Compare with GP that predicts left arm angles given other body angles.

Mocap Results

Table: Root mean squared (RMS) angle error for prediction of the left arm's configuration in the motion capture data. Prediction with the latent force model outperforms the prediction with regression for all apart from the radius's angle.

Angle	Latent Force Error	Regression Error
Radius	4.11	4.02
Wrist	6.55	6.65
Hand X rotation	1.82	3.21
Hand Z rotation	2.76	6.14
Thumb X rotation	1.77	3.10
Thumb Z rotation	2.73	6.09

Mocap Results II



(a) Inferred Latent Force (b) Wrist (c) Hand X Rotation

(d) Hand Z Rotation (e) Thumb X Rotation (f) Thumb Z Rotation

Figure: Predictions from LFM (solid line, grey error bars) and direct regression (crosses with stick error bars).

Discussion and Future Work

- Integration of probabilistic inference with mechanistic models.
- Ongoing/other work:
  - Non linear response and non linear differential equations.
  - Scaling up to larger systems Álvarez et al. (2010); Álvarez and Lawrence (2009).
  - Discontinuities through Switched Gaussian Processes Álvarez et al. (2011b)
  - Robotics applications.
  - Applications to other types of system, *e.g.* spatial systems Álvarez et al. (2011a).
  - Stochastic differential equations Álvarez et al. (2010).

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