A category theoretic framework for noncommutative and nonassociative geometry

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Overview

Why nonassociativity?

- Flux compactifications of closed string theory
- Coordinate algebra probed by closed strings winding and propagating in R-flux compactification is noncommutative and nonassociative [Blumenhagen, Lüst ..., 2010, 2011, 2012]

$$[x^i, x^j] = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k$$
$$[x^i, x^j, x^k] = \ell_s^4 R^{ijk}$$

Actually associative

[Mylonas, Schupp, Szabo, 2012, 2014] found $\mathfrak{g}$ with $F \in U\mathfrak{g} \otimes U\mathfrak{g}$ (invertible, normalised) such that

- $\star = \mu \circ F^{-1}$ gives

$$[x^i, x^j]_\star = x^i \star x^j - x^j \star x^i = \frac{i\ell_s^4}{3\hbar} R^{ijk} \partial_k \ ,$$
$$[x^i, x^j, x^k]_\star = [[x^i, x^j]_\star, x^k]_\star + \text{cycl.} = \ell_s^4 R^{ijk}$$

- $R_F = F_{21} F^{-1} = \sum R_F^{(1)} \otimes R_F^{(2)}$ gives

$$\boxed{[x^i, x^j]_{R_F} = x^i \star x^j - \sum R_F^{(2)}(x^i) \star R_F^{(1)}(x^j) = 0}$$

- $\phi_F = (1 \otimes F) \cdot (\mathrm{id}_H \otimes \Delta)(F) \cdot (\Delta \otimes \mathrm{id}_H)(F^{-1}) \cdot (F^{-1} \otimes 1)$ gives

$$\boxed{[x^i, x^j, x^k]_{R_F, \phi_F} = 0}$$

Thm $(U\mathfrak{g}, R_F, \phi_F)$ is a triangular quasi-Hopf algebra

Toolbox

Abstract this: consider arbitrary algebra A, arbitrary triangular quasi-Hopf algebra (H, R, ϕ) such that, for $a, a', a'' \in A$

$$[a, a']_R = 0$$
$$[a, a', a'']_{R,\phi} = 0$$

Find that

- A is a commutative and associative algebra object in ${}^H\mathscr{M}$, the representation category of H

Strategy

- Consider the representation category of an **arbitrary** triangular quasi-Hopf algebra H
- Develop notions of **geometry** on **one** algebra object and **its** bimodule objects **internal** to such a representation category

Representation category of triangular quasi-Hopf algebra, ${}^H\mathscr{M}$

- Objects: All bounded $\mathbb{Z}$-graded left H-modules with $\triangleright : H \otimes V \to V$
- Morphisms: All H-equivariant degree preserving graded k-module maps

$$R = \sum R^{(1)} \otimes R^{(2)} \ , \phi = \sum \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)}$$

${}^H\mathscr{M}$ is Closed Symmetric Monoidal category:

- Monoidal functor $\otimes$
  - associator: $\Phi : (v \otimes w) \otimes x \mapsto \sum (\phi^{(1)} \triangleright v) \otimes ((\phi^{(2)} \triangleright w) \otimes (\phi^{(3)} \triangleright x))$
  - symmetric braiding: $\tau : v \otimes w \mapsto \sum (-1)^{|v|\,|w|} \left(R^{(2)} \triangleright w\right) \otimes \left(R^{(1)} \triangleright v\right)$
- Internal hom functor hom

$$\cdot \otimes V \dashv \mathrm{hom}(V, \cdot) : {}^H\mathscr{M} \to {}^H\mathscr{M} \quad , \forall \quad V \in \mathrm{Obj}({}^H\mathscr{M})$$

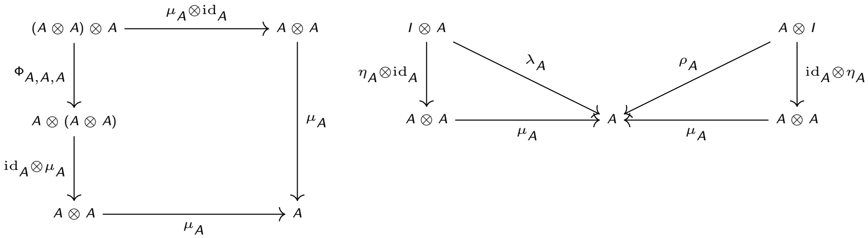
with currying natural bijection ζ

$$\zeta_{V,W,X} : \mathrm{Hom}_{{}^H\mathscr{M}}(V \otimes W, X) \longrightarrow \mathrm{Hom}_{{}^H\mathscr{M}}(V, \mathrm{hom}(W, X))$$

NB Internal homomorphisms are objects in ${}^H\mathscr{M}$

Commutative algebra object and its bimodule objects in ${}^H\mathscr{M}$

NCA space: **A commutative algebra object** in ${}^H\mathscr{M}$ is a triple (A, μ_A, η_A) consisting of an ${}^H\mathscr{M}$-object A together with two ${}^H\mathscr{M}$-morphisms $\mu_A : A \otimes A \to A$ (product) and $\eta_A : I \to A$ (unit) such that



and $\mu_A = \mu_A \circ \tau$.

NCA vector bundles: **A symmetric A-bimodule object** in ${}^H\mathscr{M}$ is is a triple (V, l_V, r_V) consisting of an ${}^H\mathscr{M}$-object V with two ${}^H\mathscr{M}$-morphisms $l_V : A \otimes V \to V$ (left A-action) and $r_V : V \otimes A \to V$ (right A-action) satisfying the usual bimodule relations and also $l_V = r_V \circ \tau$ and $r_V = l_V \circ \tau$.

Internal hom-objects in ${}^H\mathscr{M}$

Regard all geometric quantities as internal hom-objects.

- Indispensable when geometric quantities are **dynamical** (e.g. metric field in gravity or curvature field of connection in Yang-Mills theory)
- Richer framework for nonassociative geometry than [Beggs, Majid, 2010], **configuration space** of nca space as large as corresponding classical space

For any braided closed monoidal category there are **canonical**

- $\mathrm{ev} : \mathrm{hom}(V, W) \otimes V \to W$, $L(v)$ replaced by $\mathrm{ev}(L \otimes v)$
- $\bullet : \mathrm{hom}(V, W) \otimes \mathrm{hom}(X, V) \to \mathrm{hom}(X, W)$, $L \circ L'$ replaced by $L \bullet L'$
- $\circledast : \mathrm{hom}(V, W) \otimes \mathrm{hom}(X, Y) \to \mathrm{hom}(V \otimes X, W \otimes Y)$, $L \otimes L'$ replaced by $L \circledast L'$
- $\eta : k \to \mathrm{end}(V)$

morphisms for internal hom-objects.

Results:

1. $(\mathrm{end}(V), \bullet, \eta)$ is an algebra object in ${}^H\mathscr{M}$
2. Defining $[\cdot, \cdot] := \bullet - \bullet \circ \tau$, $(\mathrm{end}(V), [\cdot, \cdot])$ is a Lie algebra object in ${}^H\mathscr{M}$

Derivations of the commutative algebra object in ${}^H\mathscr{M}$

Guiding principle: Define geometry via universal constructions, e.g. equalisers (sensible since geometrical concepts are 'universal' in the sense that we can speak of the Leibniz rule for e.g.)

Leibniz rule for derivation X: $X(ab) = (-1)^{|a||X|} a\, X(b) + X(a)\, b$ for all $a, b \in A$

Observation: $X(a-) - (-1)^{|a||X|} a\, X(-) = X(a)\, -$. Content of Leibniz rule can be captured by two parallel morphisms

$$\mathrm{end}(A) \otimes A \underset{\zeta(\mu)\circ \mathrm{ev}}{\overset{[\cdot,\cdot]\circ(\mathrm{id}\otimes\zeta(\mu))}{\rightrightarrows}} \mathrm{end}(A)$$

$(a \to \zeta(\mu)(a) \in \mathrm{end}(A))$. Then we can use the currying natural bijection to obtain the derivations as the equaliser

$$\mathrm{der}(A) \longrightarrow \mathrm{end}(A) \underset{\zeta(\widehat{\mu}\circ \mathrm{ev})}{\overset{\zeta([\,\cdot\,,\,\cdot\,])}{\rightrightarrows}} \mathrm{hom}(A, \mathrm{end}(A))$$

Result: $(\mathrm{der}(A), [\cdot,\cdot])$ is a Lie algebra object in ${}^H\mathscr{M}$

Building up to the notion of connection

Need

- exterior derivative (differential calculus)
- configuration space for affine space of connections

Differential calculus in ${}^H\mathscr{M}$

Need an H-invariant, nilpotent derivation

- $k[1] = \bigoplus_n k[1]_n \in \mathrm{Obj}({}^H\mathscr{M})$, $k[1]_1 = k$, $k[1]_n = 0$ for all $n \neq 1$
- $d : k[1] \to \mathrm{der}(A)$ (H-invariant)
- $k[1] \otimes k[1] \xrightarrow{\mathrm{d} \otimes \mathrm{d}} \mathrm{der}(A) \otimes \mathrm{der}(A) \xrightarrow{[\cdot,\cdot]} \mathrm{der}(A)$ is 0 (nilpotent)

In summary $\mathrm{d}\,(c) \in \mathrm{der}(A)$ for any $c \in k$, and $\mathrm{d}\,(c) \bullet \mathrm{d}\,(c) = 0$. Note $\mathrm{d}\,(c) = c\,\mathrm{d}\,(1)$.

Differential calculus

$(A, \mathrm{d}\,(1))$

$\hom_A$

- for the configuration space of the affine space of connections

Idea: Ought to be the 'internal version of A-bimodule morphisms'

$$L(a\,v) = (-1)^{|a||L|} a\, L(v)$$

becomes

$$L \bullet \widehat{l}_W(a) - (-1)^{|a||L|} R^{(2)} \triangleright \widehat{l}_V(a) \bullet R^{(1)} \triangleright L = 0$$

Define

$$[\,\cdot\,,\,\cdot\,] := \bullet \circ \left(\mathrm{id} \otimes \widehat{l}\,\right) - \bullet \circ \left(\widehat{l} \otimes \mathrm{id}\right) \circ \tau : \hom(V,W) \otimes A \longrightarrow \hom(V,W)$$

and formulate property in terms of an equaliser:

$$\hom_A(V,W) \longrightarrow \hom(V,W) \underset{0}{\overset{\zeta([\,\cdot\,,\,\cdot\,])}{\rightrightarrows}} \hom(A, \hom(V,W))$$

Result: $U \subset \hom_A(V,W)$ if and only if $[L,a] = 0\,\forall L \in U$

Connections on A-bimodule objects in ${}^H\mathscr{M}$

Connections on noncommutative and nonassociative vector bundles...

$$\nabla(a\,v) = (-1)^{|a|} a\,\nabla(v) + \mathrm{d}\,a \otimes v (\cong (\mathrm{d}\,a)\,v)$$

Connections form an affine space, but our category is one of linear spaces. Obtain linear space of which connections is a subcollection in much same way as for derivations

$$(\mathrm{end}(V) \times k[1]) \otimes A \underset{\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\,\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id})}{\overset{[\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id})}{\rightrightarrows}} \mathrm{end}(V) \;,$$

Define the *k*ontinuous connections to be the equaliser

$$\mathrm{con}(V) \longrightarrow \mathrm{end}(V) \times k[1] \underset{\zeta(\widehat{l}\circ\mathrm{ev}\circ(\mathrm{d}\,\otimes\mathrm{id})\circ(\mathrm{pr}_2\otimes\mathrm{id}))}{\overset{\zeta([\,\cdot\,,\,\cdot\,]\circ(\mathrm{pr}_1\otimes\mathrm{id}))}{\rightrightarrows}} \mathrm{hom}(A, \mathrm{end}(V))$$

Result: $\{(L,1)\} \subset \mathrm{con}(V)$ is an affine space over $\{(L,0)\} \subset \mathrm{con}(V) = \mathrm{end}_A(V)$, "the usual affine space of connections"

Products of k-connections in ${}^H\mathscr{M}$

$\otimes_A$ defined via coequaliser

$$(V \otimes A) \otimes W \underset{r_V \otimes \mathrm{id}_W}{\overset{(\mathrm{id}_V \otimes l_W)\circ \Phi_{V,A,W}}{\rightrightarrows}} V \otimes W \longrightarrow V \otimes_A W .$$

Can we define a connection on $V \otimes_A W$ from connections on V and W?

subtlety: require a fibered product:

$$\begin{array}{ccc} \mathrm{con}(V) \times_{k[1]} \mathrm{con}(W) & \longrightarrow & \mathrm{con}(W) \\ \downarrow & & \downarrow \mathrm{pr}_2 \\ \mathrm{con}(V) & \xrightarrow[\mathrm{pr}_2]{} & k[1] \end{array}$$

Theorem (GEB, Schenkel, Szabo) (sum of k-connections)

Given two A-bimodule objects V, W in ${}^H\mathscr{M}$ there is an ${}^H\mathscr{M}$-morphism

$$\boxplus : \mathrm{con}(V) \times_{k[1]} \mathrm{con}(W) \longrightarrow \mathrm{con}(V \otimes_A W) ,$$
$$\big((\nabla_V, c), (\nabla_W, c)\big) \longmapsto \big(\nabla_V \circledast 1 + 1 \circledast \nabla_W, c\big) .$$

k-connections on dual modules in ${}^H\mathscr{M}$

Can we define a connection on $\hom_A(V,W)$ from connections on V and W?

Theorem (GEB, Schenkel, Szabo)

Given two A-bimodule objects V, W in ${}^H\mathscr{M}$ there is an ${}^H\mathscr{M}$-morphism

$$\begin{aligned} \mathrm{ad}_\bullet : \mathrm{con}(W) \times_{k[1]} \mathrm{con}(V) &\longrightarrow \mathrm{con}(\hom_A(V,W)) \;, \\ ((\nabla_W, c), (\nabla_V, c)) &\longmapsto \big(\mathscr{L}(\nabla_W) - \mathscr{R}(\nabla_V), c\big) \;. \end{aligned}$$

Cor: For dual modules $V^\vee := \hom_A(V,A)$ we can define the dual connection

$$\nabla^\vee := \mathrm{ad}_\bullet\big((\nabla,1)\,,\,(\mathrm{d}\,(1),1)\big) \in \mathrm{con}(V^\vee)$$

Existence?

We can build such a toolbox but does this toolbox exist?

Result: There is an **equivalence** of closed braided monoidal categories

$$\mathcal{F} :^{H} \mathscr{M} \to^{H_F} \mathscr{M}$$

where H_F is obtained from H using a cochain twist, F (Kassel).

- Interested in the case H is a Hopf algebra $(R = 1 \otimes 1, \phi = 1 \otimes 1 \otimes 1)$ where tools exist.
- Note: F is a cochain twist in motivating example.

Reflections

- Framework for noncommutative and nonassociative geometry.
- Description fits naturally into the context of a certain closed braided monoidal category, the representation category of a quasi-Hopf algebra.
- Twist deformation quantisation explains the present noncommutativity and nonassociativity and is simply a functor between two such categories.
- Exploring the syntax of category theory enables one to express well known geometrical concepts in terms of universal constructions internal to the category.

Further results:

- Able to provide action functionals for Yang-Mills theory and Einstein-Cartan gravity on noncommutative and nonassociative spaces
- Provides a local description. For global description use functor of points which also offers interpretation for what noncommutative and nonassociative space is.

Summary

- Why nonassociativity (flux compactifications of string theory)
- Actually associative (in the correct category)
- Toolbox for an arbitrary triangular quasi-Hopf algebra (derivations, connections)
- Existence due to an equivalence of categories
- Reflections

Thank you



questions and comments