Infinite-dimensional Categorical Quantum Mechanics

A talk for CLAP Scotland

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Introduction - motivation for this work

We want to do (diagrammatic) CQM in ∞-dimensions, but...

[1] Although there is a characterisation of orthonormal bases in terms of $H^{\star}$-algebras.

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Can we recover all of this (using non-standard analysis)? **YES, WE CAN.**

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Introduction - limit constructions, algebraically

Non-standard analysis: an algebraic way to handle limit constructions[2].

[2] Regardless of topological convergence. The sceptics out there might prefer to think directly in terms of the ultraproduct construction: we work in spaces of sequences, quotiented by a notion of "asymptotic equality", or "equality almost everywhere", determined by some non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$.

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(c) Some genuinely new finite vectors arise in non-standard Hilbert spaces:

$$\text{e.g. } \frac{1}{\sqrt{\nu}} \sum_{n=1}^{\nu} |e_n\rangle, \text{ where } \begin{cases} |e_n\rangle & \text{form an orthonormal basis} \\ \nu & \text{is an infinite natural} \end{cases}$$

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Introduction - the Transfer Theorem

The heavy lifting in non-standard analysis is done by the following result.

Theorem (Transfer Theorem)

A sentence φ holds in the standard model M of some theory—with quantifiers ranging over standard elements, functions, relations and subsets—if and only if the sentence φ holds in any/all non-standard models $^\star M$ of the theory—with quantifiers ranging over internal non-standard elements, functions, relations and subsets.

Introduction - the Transfer Theorem

Example (Natural predecessors)

Consider the sentence defining predecessors in the natural numbers:

$$\forall n \in \mathbb{N}.\left[n \neq 0 \Rightarrow [\exists m \in \mathbb{N}.\, n = m + 1]\right]$$

By TT, the following sentence holds in the non-standard model ${}^{\star}\mathbb{N}$:

$$\forall n \in {}^{\star}\mathbb{N}.\left[n \neq 0 \Rightarrow [\exists m \in {}^{\star}\mathbb{N}.\, n = m + 1]\right]$$

Hence all non-zero non-standard naturals have predecessors.

Introduction - the Transfer Theorem

Example (Well-ordering of naturals)

Consider the sentence defining the well-order property for the natural numbers, i.e. saying that every non-empty subset of $\mathbb{N}$ has a minimum:

$$\forall A \subseteq \mathbb{N}. \left[A \neq \emptyset \Rightarrow \left[\exists m \in A. \forall a \in A. m \leq a \right] \right]$$

By TT, the following sentence holds in the non-standard model ${}^\star\mathbb{N}$:

$$\forall A \subseteq {}^\star\mathbb{N}. \left[{}^\star A \neq \emptyset \Rightarrow \left[\exists m \in {}^\star A. \forall a \in {}^\star A. m \leq a \right] \right]$$

Hence all non-empty internal subsets $A \subseteq {}^\star\mathbb{N}$ have a minimum. (The requirement that A be internal is key here: e.g. the subset of all infinite non-standard naturals has no minimum, but it is also not internal.)

Introduction - the Transfer Theorem

Example (Partial sums)

Consider the sentence defining the sequence $s : \mathbb{N} \to \mathbb{R}$ of partial sums for every sequence $f : \mathbb{N} \to \mathbb{R}$ in the standard model $\mathbb{R}$:

$$\forall f : \mathbb{N} \to \mathbb{R}. \exists s : \mathbb{N} \to \mathbb{R}. \\ \big[s(0) = f(0) \wedge [\forall n \in \mathbb{N}. s(n+1) = s(n) + f(n+1)] \big]$$

By TT, the following sentence holds in the non-standard model ${}^\star\mathbb{R}$:

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Hence every internal sequence $f : {}^\star\mathbb{N} \to {}^\star\mathbb{R}$ admits a corresponding internal sequence of partial sums $s : {}^\star\mathbb{N} \to {}^\star\mathbb{R}$, i.e. the notation $\sum_{n=0}^{m} f(n)$ is legitimate for all $m \in {}^\star\mathbb{N}$.

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- there are a non-standard natural $D \in {}^\star\mathbb{N}$ and a family $(|e_d\rangle)_{d=1}^{D}$ of non-standard vectors in $|\mathcal{H}|$ (an **orthonormal basis** for $\mathcal{H}$) such that

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By Transfer Theorem we have that D is unique, and we define the **dimension** of object $\mathcal{H}$ to be the non-standard natural $\dim \mathcal{H} := D$.

The category *Hilb - morphisms

Morphisms $F : \mathcal{H} \to \mathcal{K}$ in *Hilb are the those internal non-standard linear maps $F : |\mathcal{H}| \to |\mathcal{K}|$ such that:

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This makes *Hilb a full subcategory of the Karoubi envelope for the category of non-standard Hilbert spaces and $^\star\mathbb{C}$-linear maps.

The category *Hilb - $\dagger$-symmetric monoidal structure

Morphisms $F : \mathcal{H} \rightarrow \mathcal{K}$ in *Hilb can be expressed as matrices with non-standard dimensions, using orthonormal bases for $\mathcal{H}$ and $\mathcal{K}$:

$$F = \sum_{d'=1}^{\dim \mathcal{K}} \sum_{d=1}^{\dim \mathcal{H}} |e'_{d'}\rangle F_{d'd} \langle e_d|$$

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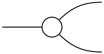
Equipped with Kronecker product, conjugate transpose, and the ${}^{\star}\mathbb{C}$-linear structure of matrices, *Hilb is an enriched †-symmetric monoidal category, with ${}^{\star}\mathbb{C}$ as its field of scalars.

The category *Hilb - some classical structures

If $|e_d\rangle_{d=1}^{\dim \mathcal{H}}$ is an orthonormal basis for $\mathcal{H}$, the following comultiplication and counit define a unital special commutative †-Frobenius algebra on $\mathcal{H}$:



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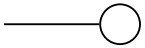

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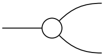

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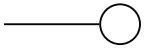

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When $|e_d\rangle_{d=1}^{\dim \mathcal{H}}$ is the non-standard extension of a standard complete orthonormal basis $|e_d\rangle_{d=1}^{\infty}$, the comultiplication is the non-standard extension of the standard isometry given by the H*-algebra associated with $|e_d\rangle_{d=1}^{\infty}$. In that case, the counit is the genuinely non-standard object.

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[3] By Transfer Theorem, this definition is independent of the choice of basis.

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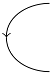
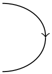
 $:= \sum_{n=1}^{\dim \mathcal{H}} |\xi_n\rangle \otimes |e_n\rangle$

 $:= \sum_{n=1}^{\dim \mathcal{H}} \langle e_n| \otimes \langle \xi_n|$

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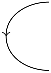 $:= \sum_{n=1}^{\dim \mathcal{H}} |\xi_n\rangle \otimes |e_n\rangle$

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(v) The category-theoretic dimension for $\mathcal{H}$ is $\operatorname{Tr} P_{\mathcal{H}} = \dim \mathcal{H}$.

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Case study - wavefunctions with periodic boundary

Wavefunctions in an *n*-dimensional box with periodic boundary conditions.

(i) Underlying Hilbert space ${}^{\star}\mathsf{L}^2[(\mathbb{R}/\mathbb{Z})^n]$.

(ii) Complete orthonormal basis of **momentum eigenstates**:

$$|\chi_{\underline{k}}\rangle := \underline{x} \to e^{-i2\pi\, \underline{k}\cdot\underline{x}}$$

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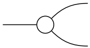
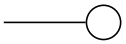
Classical structure corresponding to the **momentum observable**:


$$:= \sum_{k_1=-\omega}^{+\omega} \dots \sum_{k_n=-\omega}^{+\omega} |\chi_{\underline{k}}\rangle \otimes |\chi_{\underline{k}}\rangle \otimes \langle\chi_{\underline{k}}|$$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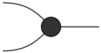

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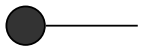

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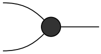

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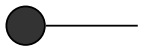

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The addition used here is that of the abelian group ${}^{\star}\mathbb{Z}^n_{2\omega+1}$:

- from the point of view of ${}^{\star}\mathbb{Z}^n$, it is cyclic on $\{-\omega, ..., +\omega\}^n$;
- from the point of view of $\mathbb{Z}^n$, it cycles "beyond infinity".

In particular, it contains $\mathbb{Z}^n$ as a proper subgroup.

Case study - wavefunctions with periodic boundary

The classical states for ● are those in the following form, where $\underline{x}$ takes the form $\underline{x} = \frac{1}{2\omega+1}\underline{q}$ for some $\underline{q} \in {}^{\star}\mathbb{Z}^{n}_{2\omega+1}$ (i.e. we have $\underline{x} \in \frac{1}{2\omega+1}\,{}^{\star}\mathbb{Z}^{n}_{2\omega+1}$):

$$|\delta_{\underline{x}}\rangle := \sum_{k_1=-\omega}^{+\omega} \dots \sum_{k_n=-\omega}^{+\omega} \chi_{\underline{k}}(\underline{x})^{*} |\chi_{\underline{k}}\rangle$$

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The classical states for $\bullet$ behave as Dirac deltas:

$\langle \delta_{\underline{x}_0} | f \rangle \simeq f(\underline{x}_0)$, for near-standard smooth f and near-standard $\underline{x}_0$

We call them the **position eigenstates**, and $\bullet$ the **position observable**.

Interlude - approximating tori by periodic lattices

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Case study - wavefunctions with periodic boundary

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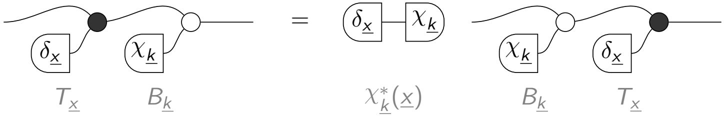

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Wavefunctions on an n-dimensional lattice $\mathbb{Z}^n$.

(i) Underlying Hilbert space ${}^\star\mathsf{L}^2[\mathbb{Z}^n]$.

(ii) Complete orthonormal basis of **position eigenstates**:

$$|\delta_{\underline{k}}\rangle := \underline{h} \mapsto \begin{cases} 1 & \text{if } \underline{k} = \underline{h} \\ 0 & \text{otherwise} \end{cases}$$

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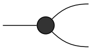

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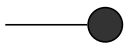

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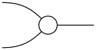
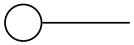
 $:= \sum_{k_1,h_1=-\omega}^{+\omega} \cdots \sum_{k_n,h_n=-\omega}^{+\omega} |\delta_{\underline{k}+\underline{h}}\rangle \otimes \langle\delta_{\underline{k}}| \otimes \langle\delta_{\underline{h}}|$

 $:= |\delta_{\underline{0}}\rangle$

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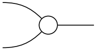
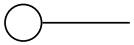
 $$:= \sum_{k_1,h_1=-\omega}^{+\omega} \dots \sum_{k_n,h_n=-\omega}^{+\omega} |\delta_{\underline{k}+\underline{h}}\rangle \otimes \langle\delta_{\underline{k}}| \otimes \langle\delta_{\underline{h}}|$$  $$:= |\delta_{\underline{0}}\rangle$$

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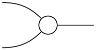
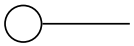
 $:= \sum_{k_1,h_1=-\omega}^{+\omega} \dots \sum_{k_n,h_n=-\omega}^{+\omega} |\delta_{\underline{k}+\underline{h}}\rangle \otimes \langle \delta_{\underline{k}}| \otimes \langle \delta_{\underline{h}}|$  $:= |\delta_{\underline{0}}\rangle$

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A common trick in non-standard analysis sees standard real space approximated by non-standard lattices of infinitesimal mesh.

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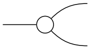

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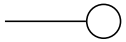

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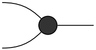

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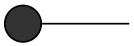

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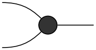
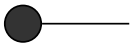
 $:= \sum_{p_1,q_1=-\omega_{ir}}^{+\omega_{ir}} \dots \sum_{p_n,q_n=-\omega_{ir}}^{+\omega_{ir}} |\chi_{\underline{p+q}}\rangle \otimes \langle\chi_{\underline{p}}| \otimes \langle\chi_{\underline{q}}|$  $:= |\chi_{\underline{0}}\rangle$

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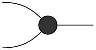

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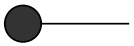

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More stuff out there, and a lot more to come

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And even more material is currently being worked out:

- position/momentum duality, quantum symmetries and dynamics;
- applications to other quantum protocols (e.g. RFI quantum teleport'n);
- wavefunctions/fields over general locally compact abelian Lie groups;
- wavefunctions/fields over Minkowski space;
- connections with Feynman diagrams.

Thanks for Your Attention!

Any Questions?

S Gogioso, F Genovese. *Infinite-dimensional CQM*. arXiv:1605.04305
S Gogioso, F Genovese. *Towards Quantum Field Theory in CQM*[5]. arXiv:1703.09594v2
S Abramsky, C Heunen. *H*-algebras and nonunital FAs*. arXiv:1011.6123
A Robinson. *Non-standard analysis*. Princeton University Press, 1974

CQM := "Categorical Quantum Mechanics"
FA := "Frobenius algebra"

[5] This is a revised and extended version, and will be out by the end of the week.