A categorical semantics for causal structure

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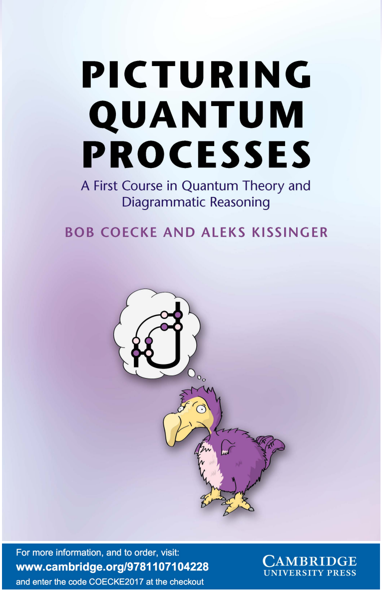
Process theory

$$:=$$

Symmetric monoidal category

$+$

interepretation of morphisms as *processes*



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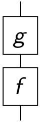
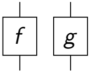
Symmetric monoidal categories

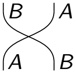
$f : A \to B \quad :=$ 

$g \circ f :=$ 

$f \otimes g :=$ 

$1_A :=$ 

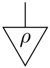
$1_I :=$

$\sigma_{A,B} :=$ 

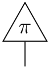
States, effects, numbers

Morphisms in/out of the monoidal unit get special names:

state :=



effect :=

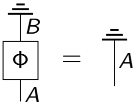


number := λ

Interpretation: discarding + causality

Consider a special family of *discarding* effects:

This enables us to say when a process is *causal*:

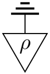


“If the output of a process is discarded,
it doesn’t matter which process happened.”

The classical case

Mat$(\mathbb{R}_+)$ is the category whose objects are natural numbers and morphisms are *matrices of positive numbers*. Then:

 $= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$

 $= \sum_i \rho^i = 1$

Causal states = probability distributions
Causal processes = stochastic maps

The quantum case

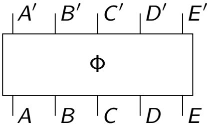
CPM is the category whose objects are Hilbert spaces and morphisms are *completely postive maps*. Then:

 $= \mathrm{Tr}(-)$ 

$= \mathrm{Tr}(\rho) = 1$

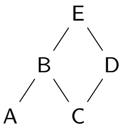
Causal states = density operators
Causal processes = CPTPs

Causal structure of a process



A **causal structure** on Φ associates input/output pairs with a set of ordered *events*:

$$\mathcal{G} := \left\{ \begin{array}{ccc} (A, A') & \leftrightarrow & \mathrm{A} \\ (B, B') & \leftrightarrow & \mathrm{B} \\ (C, C') & \leftrightarrow & \mathrm{C} \\ (D, D') & \leftrightarrow & \mathrm{D} \\ (E, E') & \leftrightarrow & \mathrm{E} \end{array} \right.$$

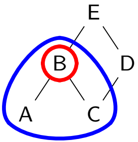


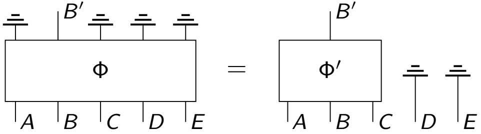
$$\left. \right\}$$

Causal structure of a process

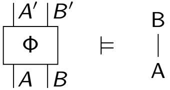
Definition

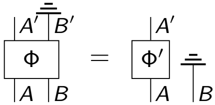
Φ *admits causal structure* $\mathcal{G}$, written $\Phi \vDash \mathcal{G}$ if the output of each event only depends on the inputs of itself and its causal ancestors.





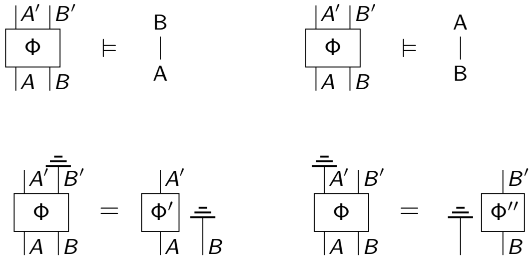
Example: one-way signalling





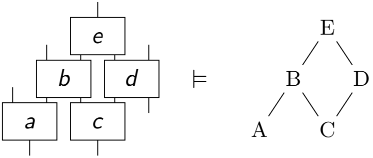
$$P(A'|AB) = P(A'|A)$$

Example: non-signalling



$P(A'|AB) = P(A'|A)$ $\qquad$ $P(B'|AB) = P(B'|B)$

An acyclic diagram comes with a canonical choice of causal structure:

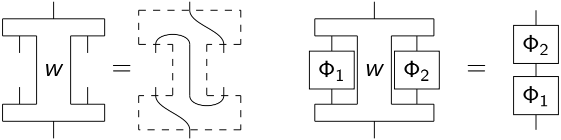


Theorem

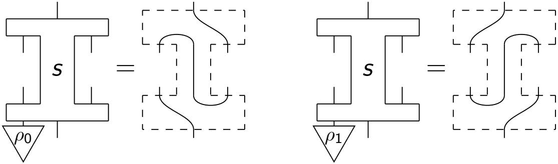
All acyclic diagrams of processes admit their associated causal structure if and only if all processes are causal.

Higher-order causal structure

We can also define (super-)processes with *higher-order causal structure*:



These can introduce definite, or **indefinite** causal structure:



e.g. Quantum Switch, OCB W-matrix, ...

The questions

Q1: Can we define a category whose *types* express causal structure?

Q2: Can we define a category whose *types* express **higher-order** causal structure?

It turns out answering **Q2** gives the answer to **Q1**.

Compact closed categories

An easy way to get higher-order processes is to use compact closed categories:

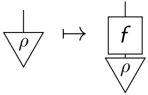
Definition

An SMC $\mathcal{C}$ is *compact closed* if every object A has a *dual* object A^*, i.e. there exists $\eta_A : I \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to I$, satisfying:

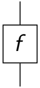
$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \qquad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

Higher-order processes

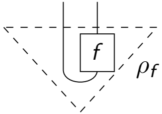
Processes send states to states:



In compact closed categories, everything is a state, thanks to *process-state duality*:

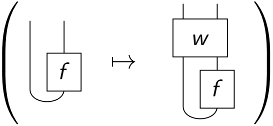


$: A \multimap B \quad \leftrightarrow$



$: \; A^* \otimes B$

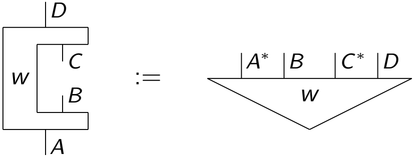
⇒ **higher order processes** are the same as **first-order processes**:



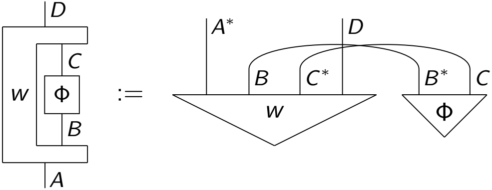
$: \; (A \multimap B) \multimap (C \multimap D)$

Some handy notation

We can treat *everything* as a state, and write states in any shape we like:

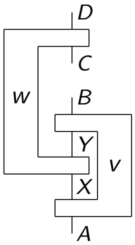


Then plugging shapes together means composing the appropriate caps:



Some handy notation

It looks like we can now freely work with higher-order causal processes:



$: \; A \multimap (B \multimap C) \multimap D$

...but theres a problem.

The compact collapse

In a compact closed category:

$$(A \otimes B)^* = A^* \otimes B^*$$

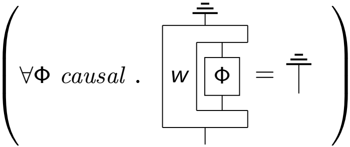
Which gives:

$$\begin{aligned}
(A \multimap B) \multimap C &\cong (A \multimap B)^* \otimes C \\
&\cong (A^* \otimes B)^* \otimes C \\
&\cong A \otimes B^* \otimes C \\
&\cong B^* \otimes A \otimes C \\
&\cong B \multimap A \otimes C
\end{aligned}$$

$\Rightarrow$ everything collapses to first order!

The compact collapse

But first-order causal $\neq$ second-order causal:



So, *causal types* are richer than compact-closed types. In particular:

$$A \multimap B := (A \otimes B^*)^* \not\cong A^* \otimes B$$

If we drop this iso from the definition of compact closed, we get a $*$-*autonomous category*.

Definition

A $*$-*autonomous category* is a symmetric monoidal category equipped with a full and faithful functor $(-)^* : \mathcal{C}^{\text{op}} \to \mathcal{C}$ such that, by letting:

$$A \multimap B := (A \otimes B^*)^* \tag{1}$$

there exists a natural isomorphism:

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B \multimap C) \tag{2}$$

The recipe

Precausal category $\mathcal{C}$	$\mapsto$	$\mathrm{Caus}[\mathcal{C}]$
compact closed category of 'raw materials'		$*$*-autonomous category capturing 'logic of causality'*
Mat$(\mathbb{R}_+)$	$\mapsto$	higher-order stochastic maps
CPM	$\mapsto$	higher-order quantum channels

Precausal categories

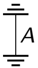
Precausal categories give 'good' raw materials, i.e. discarding behaves well w.r.t. the categorical structure. The standard examples are **Mat**$(\mathbb{R}_+)$ *and* **CPM**.

Definition

A *precausal category* is a compact closed category $\mathcal{C}$ such that:

(C1) $\mathcal{C}$ has discarding processes for every system

(C2) For every (non-zero) system A, the *dimension* of A:

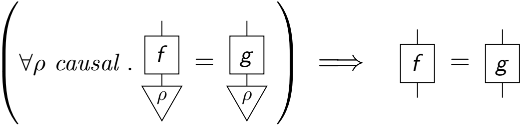
$d_A \ :=$ 

is an invertible scalar.

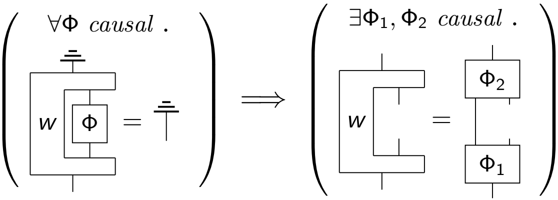
(C3) $\mathcal{C}$ has *enough causal states*

(C4) *Second-order causal* processes factorise

Enough causal states

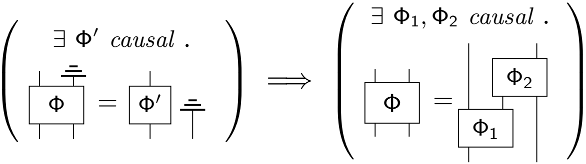


Second-order causal processes factorise

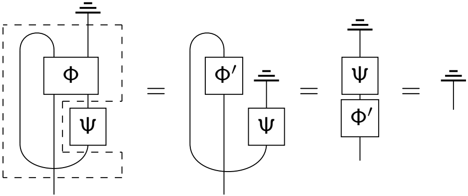


Theorem

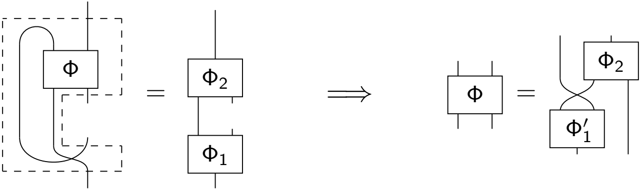
In a pre-causal category, one-way signalling processes factorise:



Proof. Treat Φ as a second-order process by bending wires. Then for any causal Ψ, we have:

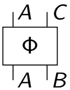


So Φ is second-order causal. By (C4):

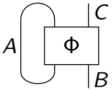


Theorem (No time-travel)

No non-trivial system A in a precausal category $\mathcal{C}$ admits time travel. That is, if there exist systems B and C such that:



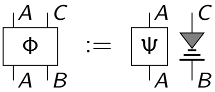
causal $\implies$



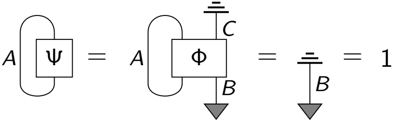
causal

then $A \cong I$.

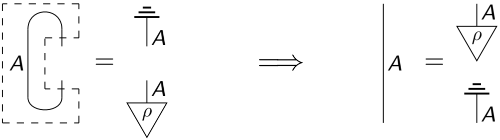
Proof. For any causal process Ψ and causal state  :



is causal.So:



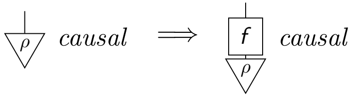
Applying (C4):



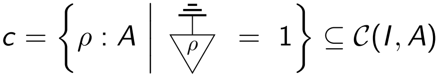
for some ρ causal.So $\rho \circ \overline{\top} = 1_A$and $\overline{\top} \circ \rho = 1_I$ is causality.

Causal states

A process is causal, a.k.a. *first order causal*, if and only if it preserves the set of causal states:



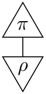
That is, it preserves:



We define $\mathrm{Caus}[\mathcal{C}]$ by equipping each object with *a generalisation of* the set c, and requiring processes to preserve it.

Duals and closure

Note *any* set of states $c \subseteq \mathcal{C}(I, A)$ admits a *dual*, which is a set of effects:

$$c^* \;:=\; \left\{ \pi : A^* \;\middle|\; \forall \rho \in c \;.\right.$$


$$\left. =\; 1 \right\}$$

The double-dual c^{**} is a set of states again.

Definition

A set of states $c \subseteq \mathcal{C}(I, A)$ is *closed* if $c = c^{**}$.

Flatness

If c is the set of causal states, discarding $\in c^*$, and up to some rescaling, discarding-transpose:

$\frac{1}{D}$ 

i.e. the maximally mixed state $\in c$.

We make this symmetric $c \leftrightarrow c^*$, and call this propery flatness:

Definition

A set of states $c \subseteq \mathcal{C}(I, A)$ is *flat* if there exist invertible numbers λ, μ such that:

λ  $\in c$ $\qquad$ μ  $\in c^*$

The main definition

Definition

For a precausal category $\mathcal{C}$, the category $\mathrm{Caus}[\mathcal{C}]$ has as objects pairs:

$$\boldsymbol{A} := (A, c_{\boldsymbol{A}} \subseteq \mathcal{C}(I, A))$$

where $c_{\boldsymbol{A}}$ is closed and flat. A morphism $f : \boldsymbol{A} \to \boldsymbol{B}$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that:

$$\rho \in c_{\boldsymbol{A}} \implies f \circ \rho \in c_{\boldsymbol{B}}$$

The main theorem

Theorem

Caus[$\mathcal{C}$] is a $*$-autonomous category, where:

$$\boldsymbol{A} \otimes \boldsymbol{B} := (A \otimes B, (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**}) \qquad \boldsymbol{I} := (I, \{1_I\})$$

$$\boldsymbol{A}^* := (A^*, c_{\boldsymbol{A}}^*)$$

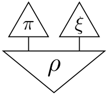
Connectives

One connective $\otimes$ becomes 3 interrelated ones:

$$\begin{aligned} &\boldsymbol{A} \otimes \boldsymbol{B} \\ &\boldsymbol{A} \parr \boldsymbol{B} := (\boldsymbol{A}^* \otimes \boldsymbol{B}^*)^* \\ &\boldsymbol{A} \multimap \boldsymbol{B} := \boldsymbol{A}^* \parr \boldsymbol{B} \cong (\boldsymbol{A} \otimes \boldsymbol{B}^*)^* \end{aligned}$$

- $\otimes$ is the smallest joint state space that contains all product states
- $\parr$ is the biggest joint state space normalised on all product effects:

$$c_{\boldsymbol{A} \parr \boldsymbol{B}} = \left\{ \rho : A \otimes B \;\middle|\; \forall \pi \in c_{\boldsymbol{A}}^*, \xi \in c_{\boldsymbol{B}}^* \,.\right.$$

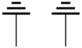


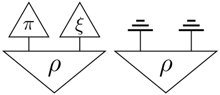
$$\left. = 1 \right\}$$

- $\multimap$ is the space of causal-state-preserving maps

Example: first-order systems

First order := systems of the form $\boldsymbol{A} = (A, \{$  $\}^*)$

$c_{\boldsymbol{A}\otimes\boldsymbol{B}} := (c_{\boldsymbol{A}} \otimes c_{\boldsymbol{B}})^{**} = ($  $)^* =$ all causal states

$c_{\boldsymbol{A}\parr\boldsymbol{B}} := \left\{ \rho : A \otimes B \;\middle|\; \forall \pi \in c_{\boldsymbol{A}}^*, \xi \in c_{\boldsymbol{B}}^* \,.\right.$ 

$\left. = 1 \right\} =$ all causal states

Theorem

For first order systems, $\boldsymbol{A} \otimes \boldsymbol{B} \cong \boldsymbol{A} \parr \boldsymbol{B}$.

When $\otimes \neq \parr$

For f.o. $\boldsymbol{A}, \boldsymbol{A}', \boldsymbol{B}, \boldsymbol{B}'$:

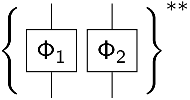
$$
\begin{aligned}
(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}') &\cong \boldsymbol{A}^* \parr \boldsymbol{A}' \parr \boldsymbol{B}^* \parr \boldsymbol{B}' \\
&\cong \boldsymbol{A}^* \parr \boldsymbol{B}^* \parr \boldsymbol{A}' \parr \boldsymbol{B}' \\
&\cong (\boldsymbol{A} \otimes \boldsymbol{B})^* \parr \boldsymbol{A}' \parr \boldsymbol{B}' \\
&\cong (\boldsymbol{A} \otimes \boldsymbol{B})^* \parr (\boldsymbol{A}' \otimes \boldsymbol{B}') \\
&\cong \boldsymbol{A} \otimes \boldsymbol{B} \multimap \boldsymbol{A}' \otimes \boldsymbol{B}'
\end{aligned}
$$

$(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}') =$ all causal processes

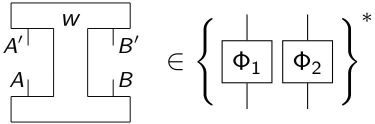
Theorem

$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') =$ *causal*, **non-signalling** *processes*

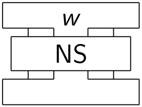
Proof. (idea) The causal states for $(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$ are:



We show:



is also normalised for all non-signalling processes:



This follows from a graphical proof using all 4 precausal axioms.

Refining causal structure

Since $\boldsymbol{I} \cong \boldsymbol{I}^* = (I, \{1\})$, a standard theorem of $*$-autonomous gives a canonical embedding:

$$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \hookrightarrow (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$$

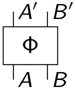
What about in between?

$$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}') \hookrightarrow \cdots \hookrightarrow (\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$$

One-way signalling

Theorem

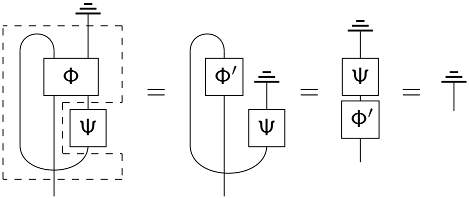
One-way signalling processes are processes of the form:



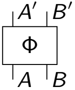
$$: \ \boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'$$

One-way signalling

Proof. Exploiting the relationship between one-way signalling and second-order causal:



we have:



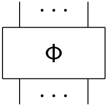
$$: \ (A' \multimap B) \multimap (A \multimap B')$$

Then $*$-autonomous structure gives a canonical iso:

$$(A' \multimap B) \multimap (A \multimap B') \ \cong \ A \multimap (A' \multimap B) \multimap B'$$

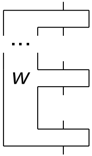
Further examples

- n-party non-signalling:



$$: \; (\boldsymbol{A}_1 \multimap \boldsymbol{A}'_1) \otimes \cdots \otimes (\boldsymbol{A}_n \multimap \boldsymbol{A}'_n)$$

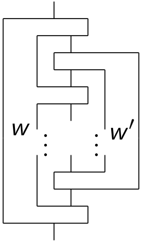
- Quantum n-combs:



$$: \; \boldsymbol{A}_1 \multimap (\boldsymbol{A}'_1 \multimap (\cdots) \multimap \boldsymbol{A}_n) \multimap \boldsymbol{A}'_n$$

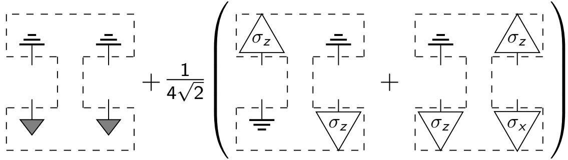
Further examples

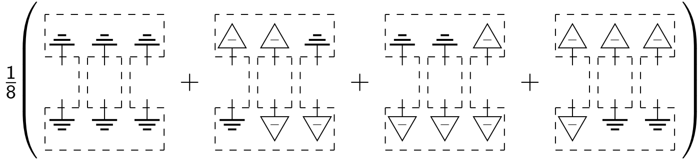
- Compositions of those things:



Further examples

- Indefinite causal structures (e.g. quantum switch, OCB W-process, Baumeler-Wolf):





$$\left[(\boldsymbol{A}_1 \multimap \boldsymbol{A}'_1) \otimes \ldots \otimes (\boldsymbol{A}_n \multimap \boldsymbol{A}'_n)\right]^*$$

Automation

The internal logic of $*$-autonomous categories is multiplicative linear logic (MLL):

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

$$\frac{}{\vdash 1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}$$

$\Rightarrow$ use off-the-shelf theorem provers to prove causality theorems.

Automation

For example, we can show using `llprover` that:

$$(\boldsymbol{A} \multimap \boldsymbol{A}') \otimes (\boldsymbol{B} \multimap \boldsymbol{B}')$$

$$\downarrow$$

$$\boldsymbol{A} \multimap (\boldsymbol{A}' \multimap \boldsymbol{B}) \multimap \boldsymbol{B}'$$

$$\downarrow$$

$$(\boldsymbol{A} \multimap \boldsymbol{A}') \parr (\boldsymbol{B} \multimap \boldsymbol{B}')$$

Thanks

...and some refs:

- **A categorical semantics for causal structure.** arXiv:1701.04732
- *Causal structures and the classification of higher order quantum computation.* Paulo Perinotti. arXiv:1612.05099