

# A categorical semantics for causal structure

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# Process theory

$:=$

Symmetric monoidal category

+

interepretation of morphisms as *processes*

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# Symmetric monoidal categories

$$f : A \rightarrow B := \begin{array}{c} | \\ \boxed{f} \\ | \end{array} \begin{array}{l} B \\ A \end{array}$$

$$g \circ f := \begin{array}{c} \boxed{g} \\ | \\ \boxed{f} \\ | \end{array} \quad f \otimes g := \begin{array}{c} | \\ \boxed{f} \\ | \end{array} \begin{array}{c} | \\ \boxed{g} \\ | \end{array}$$

$$1_A := \begin{array}{c} | \\ | \\ | \end{array} A \quad 1_I := \begin{array}{c} \phantom{|} \\ \phantom{|} \\ \phantom{|} \end{array} \quad \sigma_{A,B} := \begin{array}{c} \begin{array}{cc} (B & A \\ \backslash & / \\ & \\ / & \backslash \\ (A & B) \end{array} \end{array}$$

# States, effects, numbers

Morphisms in/out of the monoidal unit get special names:

$$\textit{state} := \begin{array}{c} | \\ \triangle \\ \rho \end{array}$$

$$\textit{effect} := \begin{array}{c} \triangle \\ \pi \\ | \end{array}$$

$$\textit{number} := \lambda$$

## Interpretation: discarding + causality

Consider a special family of *discarding* effects:

$$\overline{\top}_A \quad \overline{\top}_{A \otimes B} := \overline{\top}_A \overline{\top}_B \quad \overline{\top}_I := 1$$

This enables us to say when a process is *causal*:

$$\begin{array}{c} \overline{\top}_B \\ | \\ \boxed{\Phi} \\ | \\ A \end{array} = \overline{\top}_A$$

“If the output of a process is discarded,  
it doesn't matter which process happened.”

## The classical case

$\mathbf{Mat}(\mathbb{R}_+)$  is the category whose objects are natural numbers and morphisms are *matrices of positive numbers*. Then:

$$\begin{array}{c} \equiv \\ \vdash \end{array} = (1 \quad 1 \quad \cdots \quad 1) \qquad \begin{array}{c} \equiv \\ \triangle \\ \rho \end{array} = \sum_i \rho^i = 1$$

Causal states = probability distributions

Causal processes = stochastic maps

## The quantum case

**CPM** is the category whose objects are Hilbert spaces and morphisms are *completely positive maps*. Then:

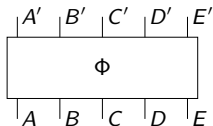
$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \text{Tr}(-) \qquad \begin{array}{c} \text{---} \\ \text{---} \\ \rho \\ \nabla \end{array} = \text{Tr}(\rho) = 1$$

Causal states = density operators

Causal processes = CPTPs



## Causal structure of a process



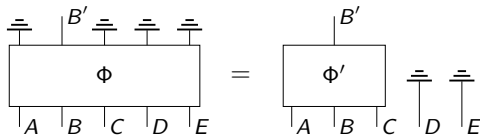
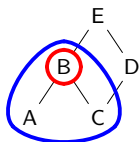
A **causal structure** on  $\Phi$  associates input/output pairs with a set of ordered *events*:

$$\mathcal{G} := \left\{ \begin{array}{l} (A, A') \leftrightarrow A \\ (B, B') \leftrightarrow B \\ (C, C') \leftrightarrow C \\ (D, D') \leftrightarrow D \\ (E, E') \leftrightarrow E \end{array} \right. \left. \begin{array}{c} \text{E} \\ \diagdown \quad \diagup \\ \text{B} \quad \text{D} \\ \diagup \quad \diagdown \\ \text{A} \quad \text{C} \end{array} \right\}$$

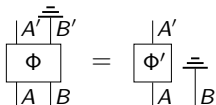
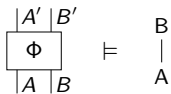
# Causal structure of a process

## Definition

$\Phi$  admits causal structure  $\mathcal{G}$ , written  $\Phi \models \mathcal{G}$  if the output of each event only depends on the inputs of itself and its causal ancestors.

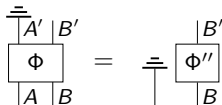
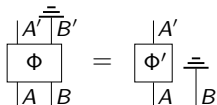


## Example: one-way signalling



$$P(A'|AB) = P(A'|A)$$

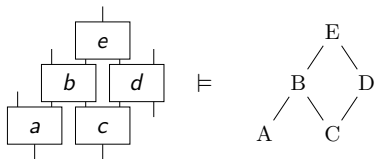
## Example: non-signalling



$$P(A'|AB) = P(A'|A)$$

$$P(B'|AB) = P(B'|B)$$

An acyclic diagram comes with a canonical choice of causal structure:

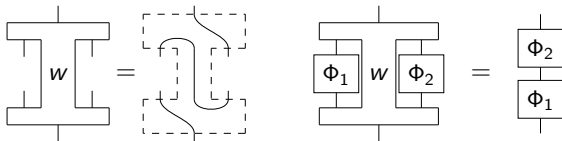


## Theorem

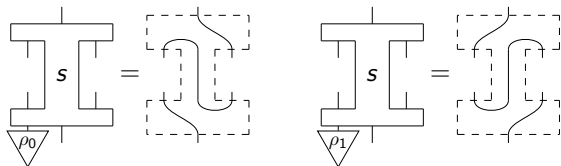
*All acyclic diagrams of processes admit their associated causal structure if and only if all processes are causal.*

## Higher-order causal structure

We can also define (super-)processes with *higher-order causal structure*:



These can introduce definite, or **indefinite** causal structure:



e.g. Quantum Switch, OCB  $W$ -matrix, ...

## The questions

**Q1:** Can we define a category whose *types* express causal structure?

**Q2:** Can we define a category whose *types* express **higher-order** causal structure?

It turns out answering **Q2** gives the answer to **Q1**.

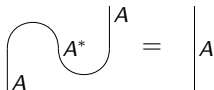
## Compact closed categories

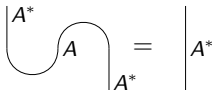
An easy way to get higher-order processes is to use compact closed categories:

### Definition

An SMC  $\mathcal{C}$  is *compact closed* if every object  $A$  has a *dual* object  $A^*$ , i.e. there exists  $\eta_A : I \rightarrow A^* \otimes A$  and  $\epsilon_A : A \otimes A^* \rightarrow I$ , satisfying:

$$(\epsilon_A \otimes 1_A) \circ (1_A \otimes \eta_A) = 1_A \quad (1_{A^*} \otimes \epsilon_A) \circ (\eta_A \otimes 1_{A^*}) = 1_{A^*}$$

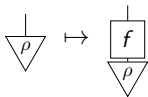




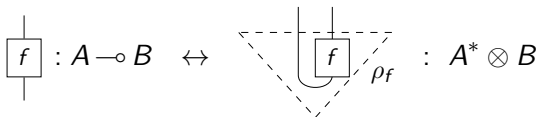


## Higher-order processes

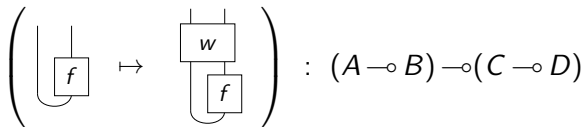
Processes send states to states:



In compact closed categories, everything is a state, thanks to *process-state duality*:

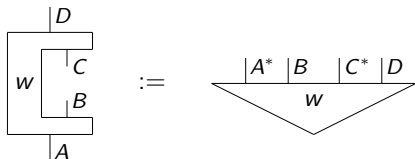


$\Rightarrow$  **higher order processes** are the same as **first-order processes**:

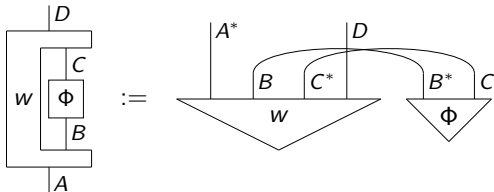


## Some handy notation

We can treat *everything* as a state, and write states in any shape we like:

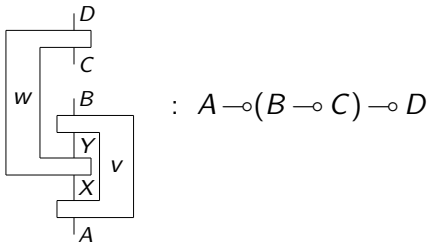


Then plugging shapes together means composing the appropriate caps:



## Some handy notation

It looks like we can now freely work with higher-order causal processes:



...but theres a problem.

## The compact collapse

In a compact closed category:

$$(A \otimes B)^* = A^* \otimes B^*$$

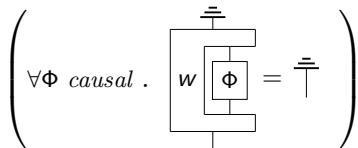
Which gives:

$$\begin{aligned}(A \multimap B) \multimap C &\cong (A \multimap B)^* \otimes C \\ &\cong (A^* \otimes B)^* \otimes C \\ &\cong A \otimes B^* \otimes C \\ &\cong B^* \otimes A \otimes C \\ &\cong B \multimap A \otimes C\end{aligned}$$

$\Rightarrow$  everything collapses to first order!

# The compact collapse

But first-order causal  $\neq$  second-order causal:



So, *causal types* are richer than compact-closed types. In particular:

$$A \multimap B := (A \otimes B^*)^* \not\cong A^* \otimes B$$

If we drop this iso from the definition of compact closed, we get a *\*-autonomous category*.

## Definition

A *\*-autonomous category* is a symmetric monoidal category equipped with a full and faithful functor  $(-)^* : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  such that, by letting:

$$A \multimap B := (A \otimes B^*)^* \quad (1)$$

there exists a natural isomorphism:

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, B \multimap C) \quad (2)$$

# The recipe

Precausal category  $\mathcal{C}$   $\mapsto$

$\text{Caus}[\mathcal{C}]$

*compact closed category  
of 'raw materials'*

*\*-autonomous category  
capturing 'logic of causality'*

$\text{Mat}(\mathbb{R}_+)$   
**CPM**

$\mapsto$   
 $\mapsto$

higher-order stochastic maps  
higher-order quantum channels

## Precausal categories

*Precausal categories give ‘good’ raw materials, i.e. discarding behaves well w.r.t. the categorical structure. The standard examples are  $\mathbf{Mat}(\mathbb{R}_+)$  and  $\mathbf{CPM}$ .*

### Definition

A *precausal category* is a compact closed category  $\mathcal{C}$  such that:

- (C1)  $\mathcal{C}$  has discarding processes for every system
- (C2) For every (non-zero) system  $A$ , the *dimension* of  $A$ :

$$d_A := \text{Diagram of system } A \text{ with discarding processes}$$

is an invertible scalar.

- (C3)  $\mathcal{C}$  has *enough causal states*
- (C4) *Second-order causal processes factorise*



## Enough causal states

$$\left( \forall \rho \text{ causal} . \begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \nabla \rho \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{g} \\ | \\ \nabla \rho \end{array} \right) \implies \begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{g} \\ | \\ \text{---} \end{array}$$

# Second-order causal processes factorise

$$\left( \forall \Phi \text{ causal . } \left( \text{Diagram 1} = \text{Diagram 2} \right) \right) \Rightarrow \left( \exists \Phi_1, \Phi_2 \text{ causal . } \left( \text{Diagram 3} = \text{Diagram 4} \right) \right)$$

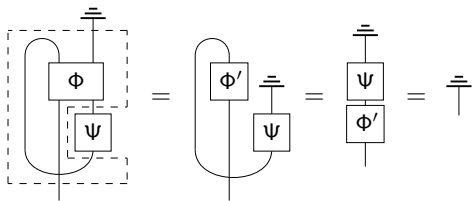
The diagram illustrates the factorization of a second-order causal process. On the left, a universal statement  $\forall \Phi \text{ causal .}$  is shown. Inside large parentheses, a complex diagram representing a process  $w$  with a box  $\Phi$  inside is equated to a simpler diagram consisting of a box with a top wire and a bottom wire. On the right, an implication  $\Rightarrow$  leads to an existential statement  $\exists \Phi_1, \Phi_2 \text{ causal .}$  Inside large parentheses, a similar process  $w$  is shown, but its internal structure is decomposed into two boxes,  $\Phi_2$  and  $\Phi_1$ , connected in series.

## Theorem

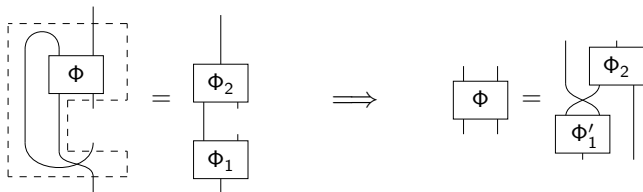
*In a pre-causal category, one-way signalling processes factorise:*

$$\left( \begin{array}{c} \exists \Phi' \text{ causal .} \\ \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{\Phi} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \boxed{\Phi'} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \end{array} \right) \Rightarrow \left( \begin{array}{c} \exists \Phi_1, \Phi_2 \text{ causal .} \\ \begin{array}{c} \boxed{\Phi} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \boxed{\Phi_2} \\ \text{---} \\ \boxed{\Phi_1} \\ \text{---} \\ \text{---} \end{array} \end{array} \right)$$

**Proof.** Treat  $\Phi$  as a second-order process by bending wires. Then for any causal  $\Psi$ , we have:

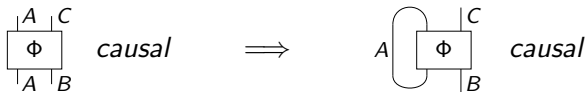


So  $\Phi$  is second-order causal. By (C4):



## Theorem (No time-travel)

No non-trivial system  $A$  in a precausal category  $\mathcal{C}$  admits time travel. That is, if there exist systems  $B$  and  $C$  such that:



then  $A \cong I$ .

**Proof.** For any causal process  $\Psi$  and causal state  $\Downarrow$ :

$$\begin{array}{c} A \quad C \\ | \quad | \\ \boxed{\Phi} \\ | \quad | \\ A \quad B \end{array} := \begin{array}{c} A \quad C \\ | \quad | \\ \boxed{\Psi} \quad \Downarrow \\ | \quad | \\ A \quad B \end{array}$$

is causal. So:

$$A \begin{array}{c} | \\ \boxed{\Psi} \\ | \\ A \end{array} = A \begin{array}{c} | \\ \boxed{\Phi} \\ | \\ A \end{array} = \begin{array}{c} \Downarrow \\ | \\ B \end{array} = 1$$

Applying (C4):

$$\begin{array}{c} \boxed{\begin{array}{c} | \\ \downarrow \\ | \end{array}} \\ A \end{array} = \begin{array}{c} \Downarrow \\ | \\ A \end{array} \Rightarrow \left| A \right. = \begin{array}{c} | \\ \downarrow \\ \rho \end{array} \begin{array}{c} \Downarrow \\ | \\ A \end{array}$$

for some  $\rho$  causal. So  $\rho \circ \ddagger = 1_A$  and  $\ddagger \circ \rho = 1_I$  is causality.

## Causal states

A process is causal, a.k.a. *first order causal*, if and only if it preserves the set of causal states:

$$\begin{array}{c} \downarrow \\ \rho \end{array} \text{ causal} \implies \begin{array}{c} \boxed{f} \\ \downarrow \\ \rho \end{array} \text{ causal}$$

That is, it preserves:

$$c = \left\{ \rho : A \mid \begin{array}{c} \overline{\overline{\overline{\downarrow}}} \\ \rho \end{array} = 1 \right\} \subseteq \mathcal{C}(I, A)$$

We define  $\text{Caus}[C]$  by equipping each object with a *generalisation* of the set  $c$ , and requiring processes to preserve it.

## Duals and closure

Note *any* set of states  $c \subseteq \mathcal{C}(I, A)$  admits a *dual*, which is a set of effects:

$$c^* := \left\{ \pi : A^* \mid \forall \rho \in c . \begin{array}{c} \triangle \pi \\ | \\ \rho \\ \triangle \end{array} = 1 \right\}$$

The double-dual  $c^{**}$  is a set of states again.

### Definition

A set of states  $c \subseteq \mathcal{C}(I, A)$  is *closed* if  $c = c^{**}$ .



# Flatness

If  $c$  is the set of causal states, discarding  $\in c^*$ , and up to some rescaling, discarding-transpose:

$$\frac{1}{D} \perp\!\!\!\perp$$

i.e. the maximally mixed state  $\in c$ .

We make this symmetric  $c \leftrightarrow c^*$ , and call this property flatness:

## Definition

A set of states  $c \subseteq \mathcal{C}(I, A)$  is *flat* if there exist invertible numbers  $\lambda, \mu$  such that:

$$\lambda \perp\!\!\!\perp \in c \qquad \mu \overline{\perp\!\!\!\perp} \in c^*$$

# The main definition

## Definition

For a precausal category  $\mathcal{C}$ , the category  $\text{Caus}[\mathcal{C}]$  has as objects pairs:

$$\mathbf{A} := (A, c_{\mathbf{A}} \subseteq \mathcal{C}(I, A))$$

where  $c_{\mathbf{A}}$  is closed and flat. A morphism  $f : \mathbf{A} \rightarrow \mathbf{B}$  is a morphism  $f : A \rightarrow B$  in  $\mathcal{C}$  such that:

$$\rho \in c_{\mathbf{A}} \implies f \circ \rho \in c_{\mathbf{B}}$$

# The main theorem

## Theorem

$\text{Caus}[\mathcal{C}]$  is a  $*$ -autonomous category, where:

$$\mathbf{A} \otimes \mathbf{B} := (A \otimes B, (c_A \otimes c_B)^{**}) \qquad \mathbf{I} := (I, \{1_I\})$$

$$\mathbf{A}^* := (A^*, c_A^*)$$

# Connectives

One connective  $\otimes$  becomes 3 interrelated ones:

$$\mathbf{A} \otimes \mathbf{B}$$

$$\mathbf{A} \wp \mathbf{B} := (\mathbf{A}^* \otimes \mathbf{B}^*)^*$$

$$\mathbf{A} \multimap \mathbf{B} := \mathbf{A}^* \wp \mathbf{B} \cong (\mathbf{A} \otimes \mathbf{B}^*)^*$$

- $\otimes$  is the smallest joint state space that contains all product states
- $\wp$  is the biggest joint state space normalised on all product effects:

$$c_{\mathbf{A}\wp\mathbf{B}} = \left\{ \rho : \mathbf{A} \otimes \mathbf{B} \mid \forall \pi \in c_{\mathbf{A}^*}, \xi \in c_{\mathbf{B}^*} . \begin{array}{c} \triangle \pi \quad \triangle \xi \\ | \quad | \\ \rho \\ \hline \end{array} = 1 \right\}$$

- $\multimap$  is the space of causal-state-preserving maps

## Example: first-order systems

First order := systems of the form  $\mathbf{A} = (A, \{\overset{\equiv}{\top}\}^*)$

$c_{\mathbf{A} \otimes \mathbf{B}} := (c_{\mathbf{A}} \otimes c_{\mathbf{B}})^{**} = (\overset{\equiv}{\top} \quad \overset{\equiv}{\top})^* = \text{all causal states}$

$c_{\mathbf{A} \wp \mathbf{B}} := \left\{ \rho : A \otimes B \mid \forall \pi \in c_{\mathbf{A}}^*, \xi \in c_{\mathbf{B}}^* . \begin{array}{c} \triangle \pi \quad \triangle \xi \\ | \quad | \\ \rho \quad \rho \end{array} = 1 \right\} = \text{all causal states}$

### Theorem

For first order systems,  $\mathbf{A} \otimes \mathbf{B} \cong \mathbf{A} \wp \mathbf{B}$ .

## When $\otimes \neq \wp$

For f.o.  $\mathbf{A}, \mathbf{A}', \mathbf{B}, \mathbf{B}'$ :

$$\begin{aligned}(\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}') &\cong \mathbf{A}^* \wp \mathbf{A}' \wp \mathbf{B}^* \wp \mathbf{B}' \\ &\cong \mathbf{A}^* \wp \mathbf{B}^* \wp \mathbf{A}' \wp \mathbf{B}' \\ &\cong (\mathbf{A} \otimes \mathbf{B})^* \wp \mathbf{A}' \wp \mathbf{B}' \\ &\cong (\mathbf{A} \otimes \mathbf{B})^* \wp (\mathbf{A}' \otimes \mathbf{B}') \\ &\cong \mathbf{A} \otimes \mathbf{B} \multimap \mathbf{A}' \otimes \mathbf{B}'\end{aligned}$$

$(\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$  = all causal processes

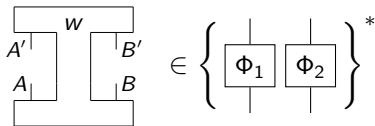
## Theorem

$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') = \text{causal, non-signalling processes}$

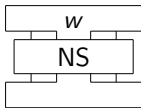
**Proof.** (idea) The causal states for  $(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$  are:

$$\left\{ \begin{array}{|c|} \hline \Phi_1 \\ \hline \end{array} \begin{array}{|c|} \hline \Phi_2 \\ \hline \end{array} \right\}^{**}$$

We show:

$$\begin{array}{|c|} \hline W \\ \hline \end{array} \begin{array}{|c|} \hline A' \\ \hline \end{array} \begin{array}{|c|} \hline B' \\ \hline \end{array} \in \left\{ \begin{array}{|c|} \hline \Phi_1 \\ \hline \end{array} \begin{array}{|c|} \hline \Phi_2 \\ \hline \end{array} \right\}^*$$


is also normalised for all non-signalling processes:



This follows from a graphical proof using all 4 precausal axioms.

## Refining causal structure

Since  $I \cong I^* = (I, \{1\})$ , a standard theorem of  $*$ -autonomous gives a canonical embedding:

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \hookrightarrow (\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$$

What about in between?

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}') \hookrightarrow \dots \hookrightarrow (\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$$



# One-way signalling

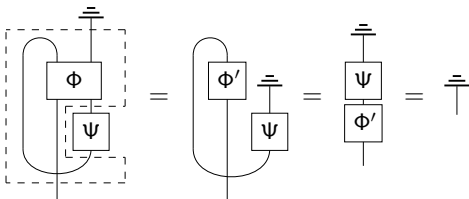
## Theorem

*One-way signalling processes are processes of the form:*

$$\begin{array}{|c|c|} \hline A' & B' \\ \hline \Phi & \\ \hline A & B \\ \hline \end{array} : \mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}'$$

## One-way signalling

**Proof.** Exploiting the relationship between one-way signalling and second-order causal:



we have:

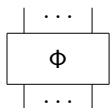
$$\begin{array}{c} A' \quad B' \\ \boxed{\Phi} \\ A \quad B \end{array} : (A' \multimap B) \multimap (A \multimap B')$$

Then  $*$ -autonomous structure gives a canonical iso:

$$(A' \multimap B) \multimap (A \multimap B') \cong A \multimap (A' \multimap B) \multimap B'$$

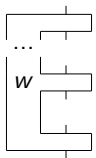
## Further examples

- $n$ -party non-signalling:



$$: (\mathbf{A}_1 \multimap \mathbf{A}'_1) \otimes \cdots \otimes (\mathbf{A}_n \multimap \mathbf{A}'_n)$$

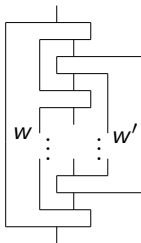
- Quantum  $n$ -combs:



$$: \mathbf{A}_1 \multimap (\mathbf{A}'_1 \multimap (\cdots) \multimap \mathbf{A}_n) \multimap \mathbf{A}'_n$$

## Further examples

- Compositions of those things:



## Further examples

- Indefinite causal structures (e.g. quantum switch, OCB  $W$ -process, Baumeler-Wolf):

$$\begin{aligned}
 & \left( \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) + \frac{1}{4\sqrt{2}} \left( \begin{array}{c} \text{Diagram 3} + \text{Diagram 4} \\ \text{Diagram 5} + \text{Diagram 6} \end{array} \right) \\
 & \otimes \left( \begin{array}{c} \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} \end{array} \right) \\
 & \left[ (\mathbf{A}_1 \multimap \mathbf{A}'_1) \otimes \dots \otimes (\mathbf{A}_n \multimap \mathbf{A}'_n) \right]^*
 \end{aligned}$$

The diagrams consist of nodes (circles with horizontal lines) and triangles (pointing up or down) connected by dashed lines, representing causal structures. The first diagram shows two parallel paths. The second diagram shows a more complex structure with a central node. The third and fourth diagrams show paths with  $\sigma_z$  operations. The fifth and sixth diagrams show paths with  $\sigma_z$  and  $\sigma_x$  operations. The seventh through tenth diagrams show various combinations of nodes and triangles.

## Automation

The internal logic of \*-autonomous categories is multiplicative linear logic (MLL):

$$\frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B}$$

$$\frac{}{\vdash 1}$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}$$

⇒ use off-the-shelf theorem provers to prove causality theorems.

# Automation

For example, we can show using `llprover` that:

$$(\mathbf{A} \multimap \mathbf{A}') \otimes (\mathbf{B} \multimap \mathbf{B}')$$

$$\Downarrow$$

$$\mathbf{A} \multimap (\mathbf{A}' \multimap \mathbf{B}) \multimap \mathbf{B}'$$

$$\Downarrow$$

$$(\mathbf{A} \multimap \mathbf{A}') \wp (\mathbf{B} \multimap \mathbf{B}')$$

# Thanks

...and some refs:

- **A categorical semantics for causal structure.** [arXiv:1701.04732](#)
- *Causal structures and the classification of higher order quantum computation.* Paulo Perinotti. [arXiv:1612.05099](#)