New Foundations for String Diagram Rewriting

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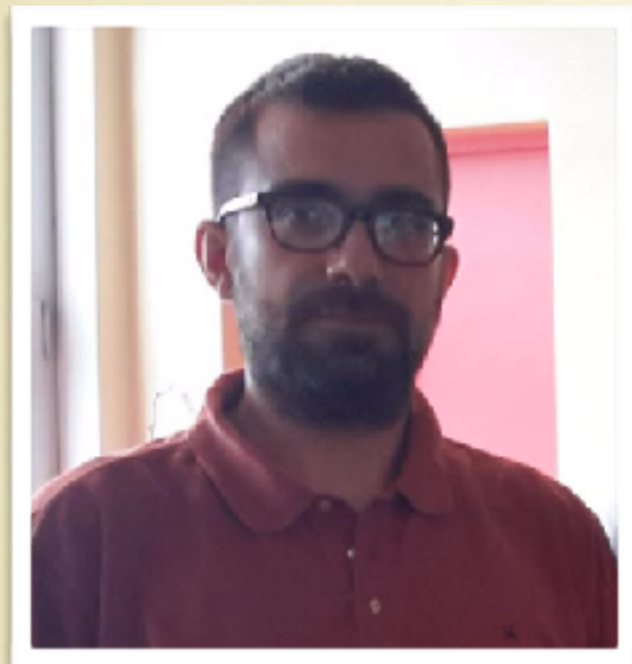
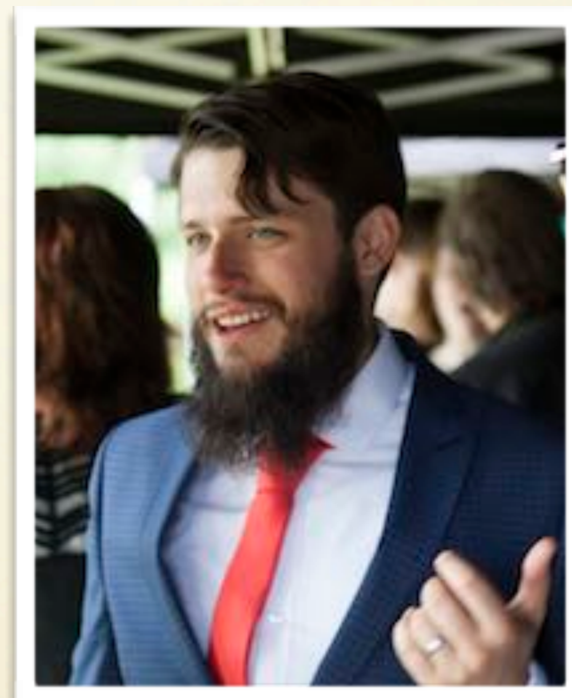
University College London

Collaborators

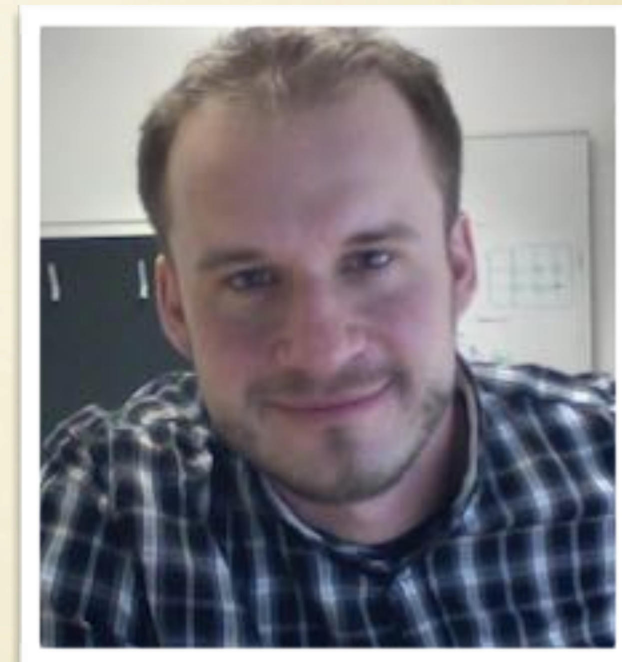


Filippo Bonchi

Fabio Gadducci

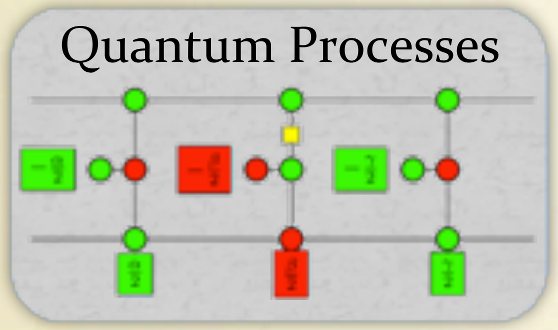


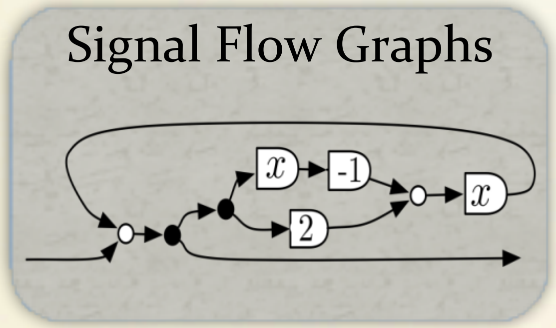
Aleks Kissinger

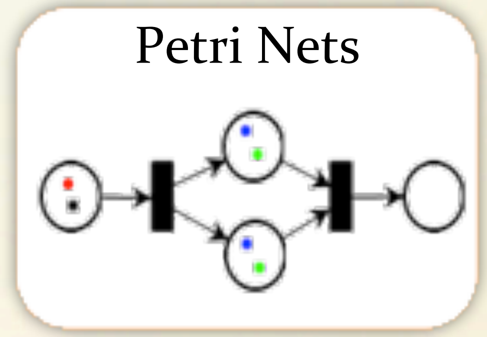


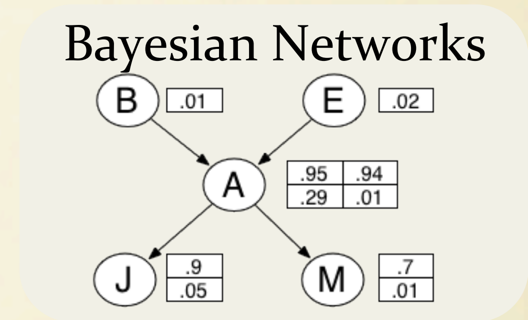
Pawel Sobocinski

Props: algebras of network diagrams





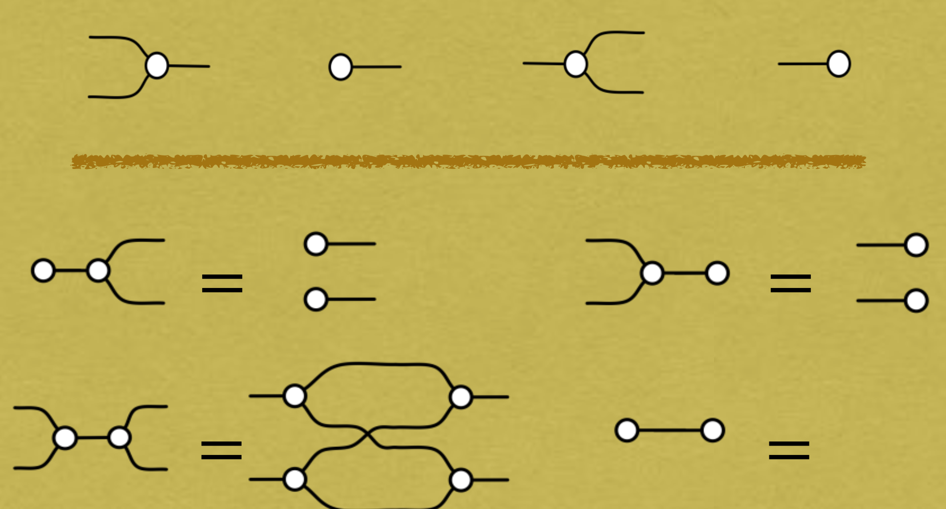




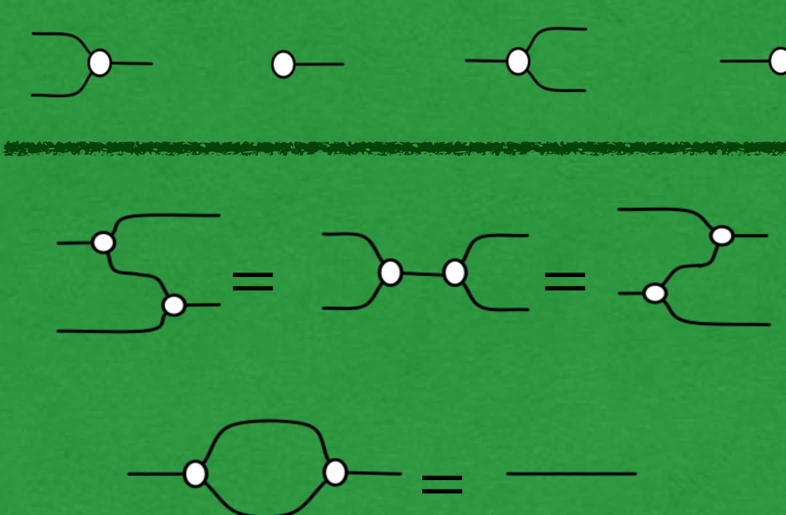
A prop is (just) a symmetric monoidal category with set of objects $\mathbb{N}$

Props can be freely constructed starting from a signature Σ and equations E

prop of bialgebras



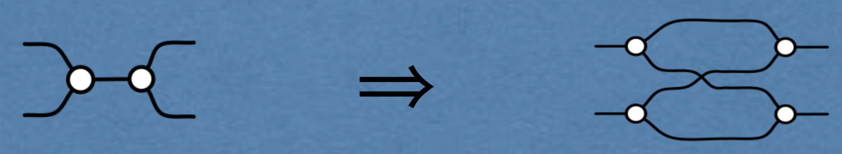
prop of special Frobenius algebras

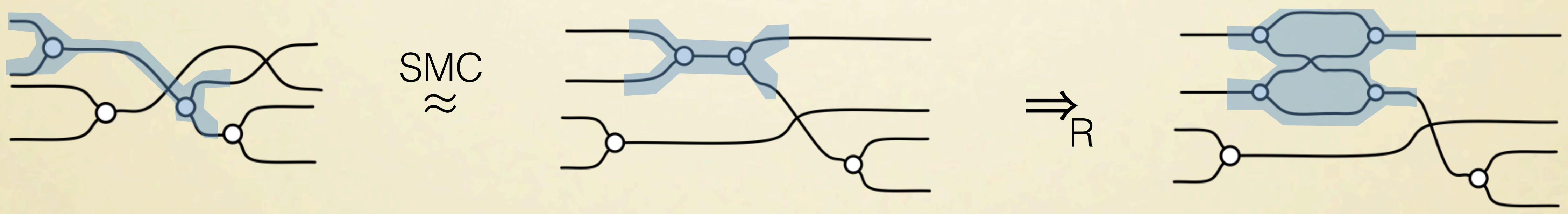


Rewriting in a prop

Perspective of this work:
see E as a **rewriting system** on diagrams

(R)

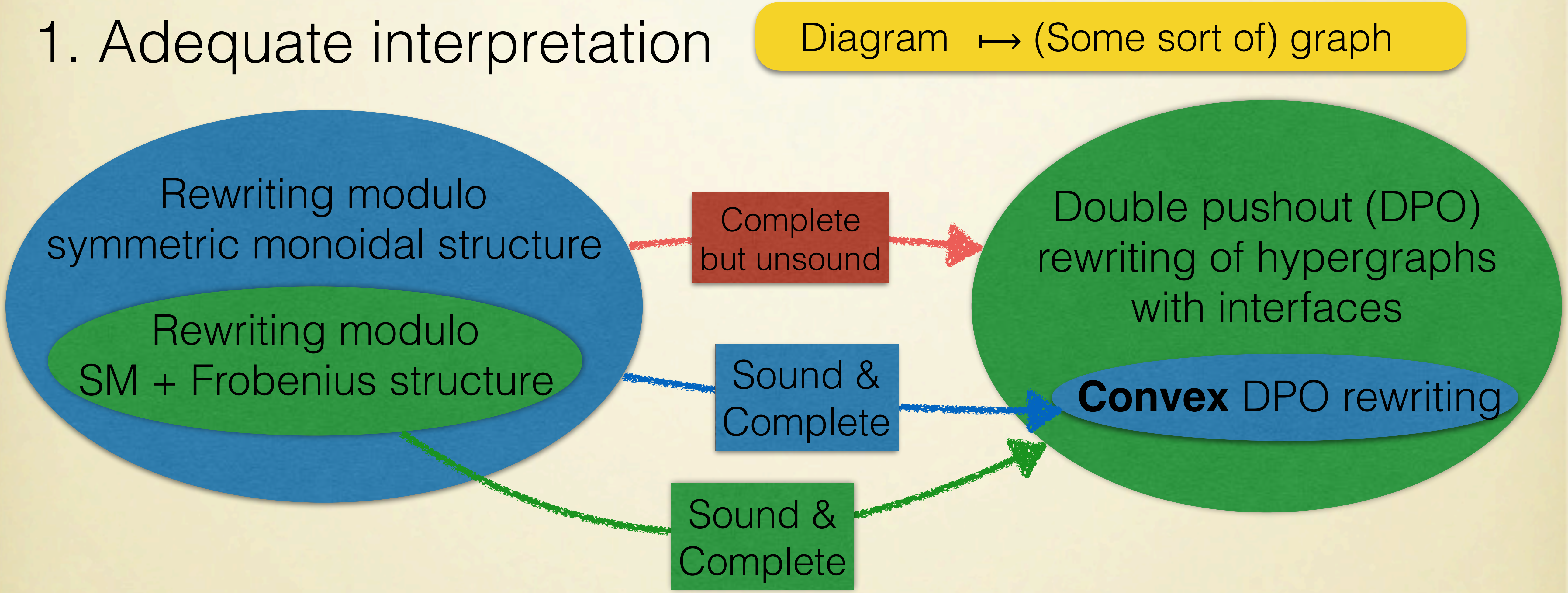




Our question
How to implement rewriting modulo symmetric monoidal structure in a simple, yet rigorous way?

Outline

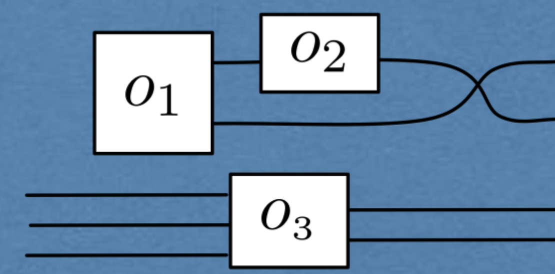
1. Adequate interpretation



2. Decidability of confluence

Hypergraph interpretation

prop ***Syn(Σ)*** of syntax
freely generated by
$\Sigma = \{ o_1, o_2, o_3 \}$

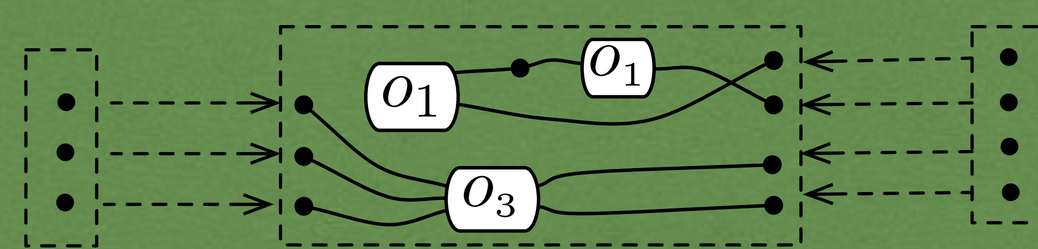


Operations in Σ ~ Hyperedges
L/R boundary ~ Cospan structure

$\langle\!\langle \cdot \rangle\!\rangle$

prop ***Csp(Hyp(Σ))*** of (discrete) cospans
of Σ-labelled hypergraphs

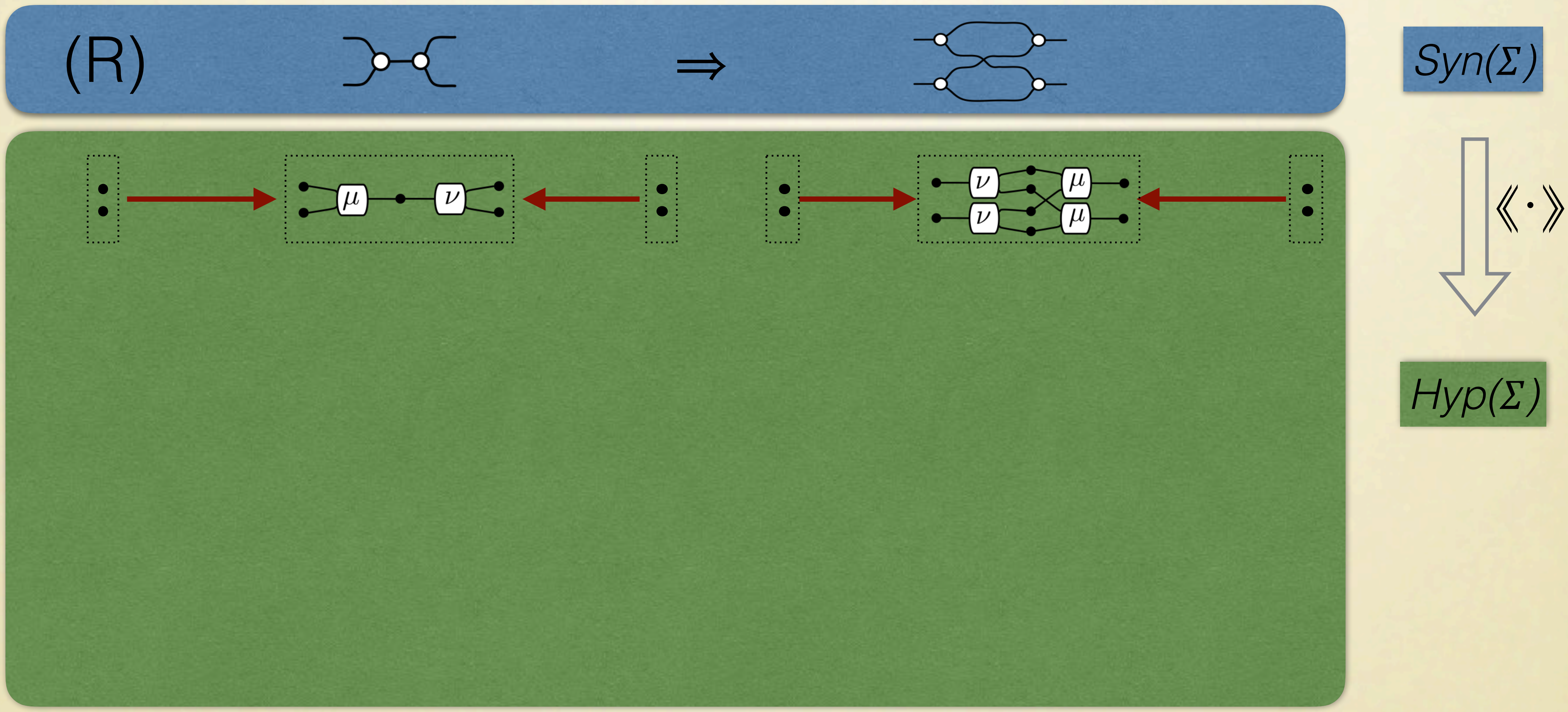
prop ***Hyp(Σ)*** of
Σ-labelled hypergraphs



Proposition $\langle\!\langle \cdot \rangle\!\rangle$: *Syn(Σ)* → *Csp(Hyp(Σ))* is faithful.

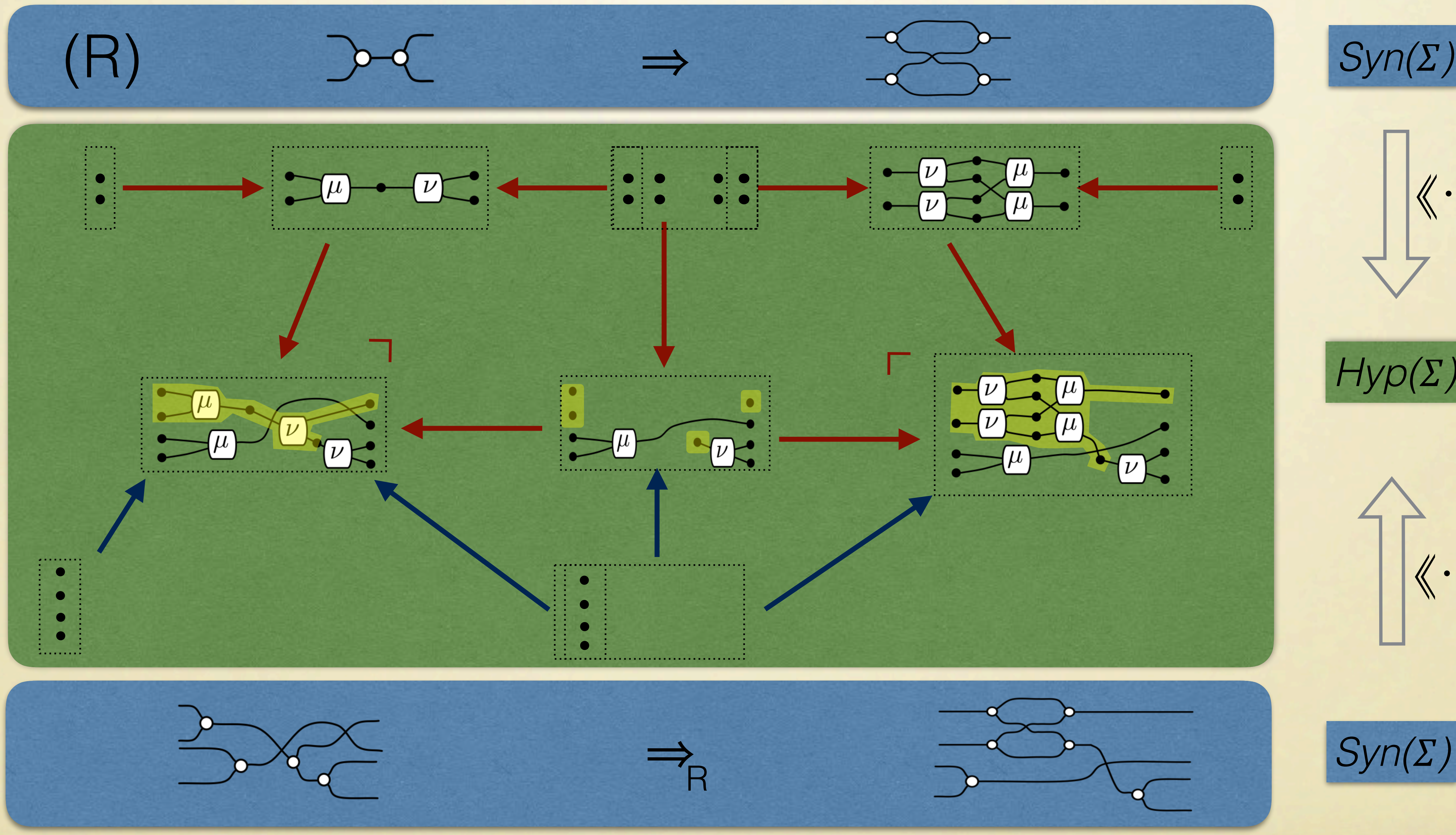
DPO rewriting with interfaces

Hyp(Σ) is an *ahdesive category* (Lack & Sobocinski)
and thus adapted to DPO rewriting.



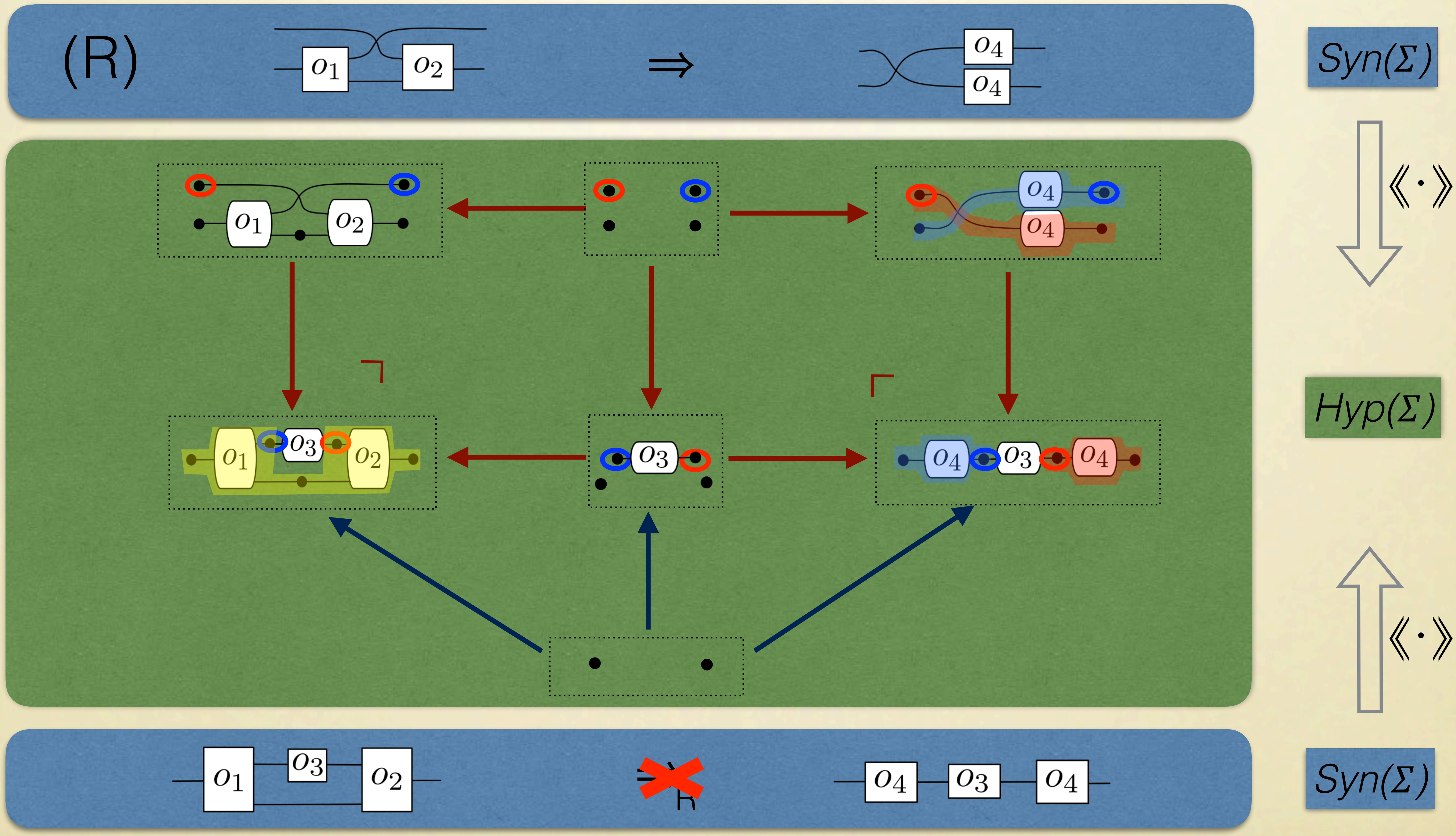
DPO rewriting with interfaces

Hyp(Σ) is an *ahdesive category* (Lack & Sobocinski)
and thus adapted to double-pushout rewriting.



DPO rewriting can be unsound

Rewriting in *Hyp(Σ)* is complete but generally not sound



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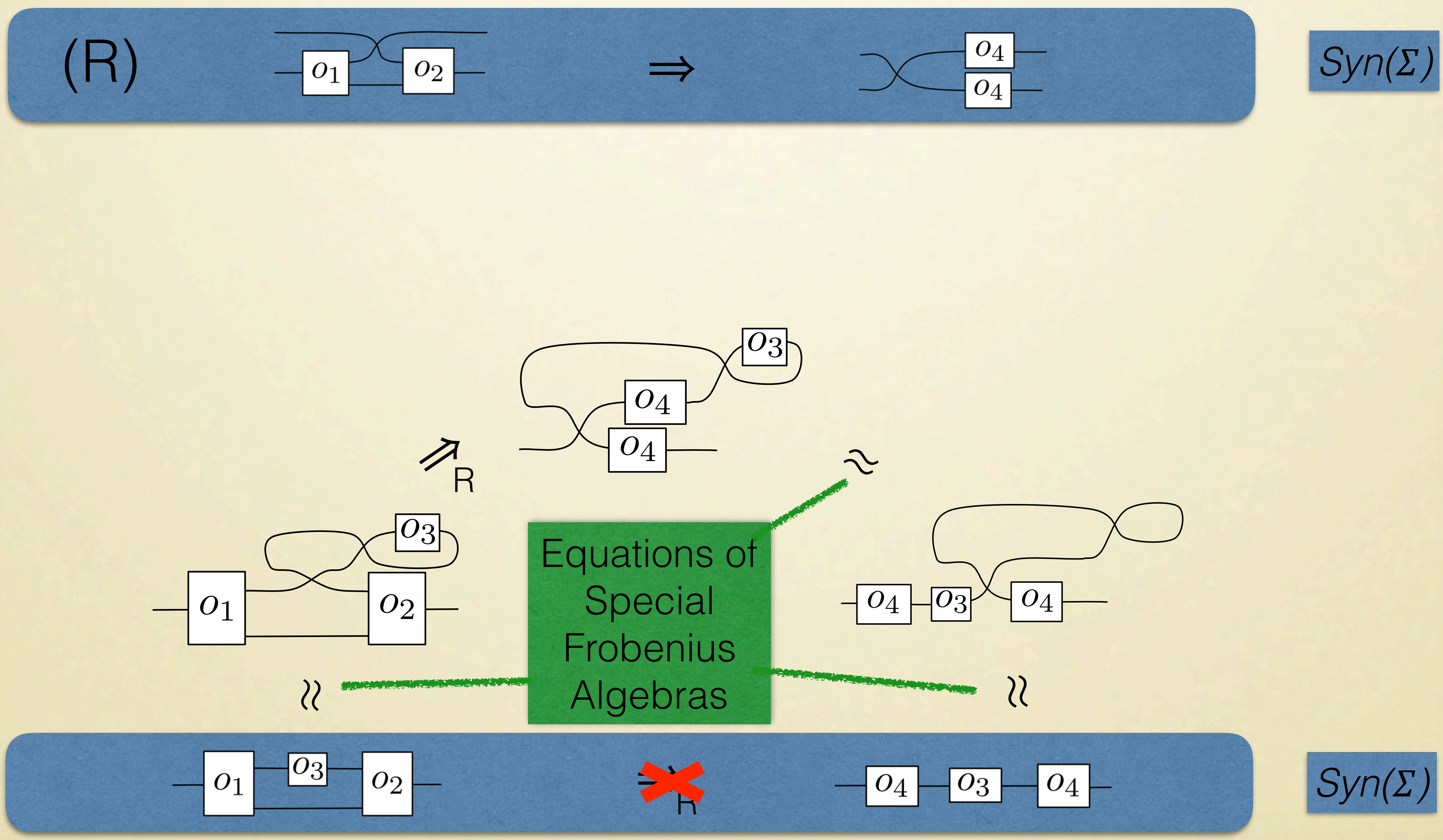
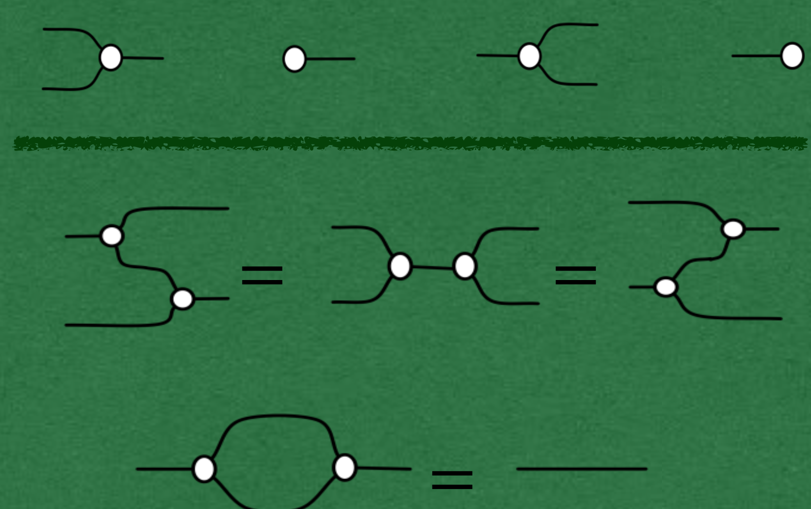


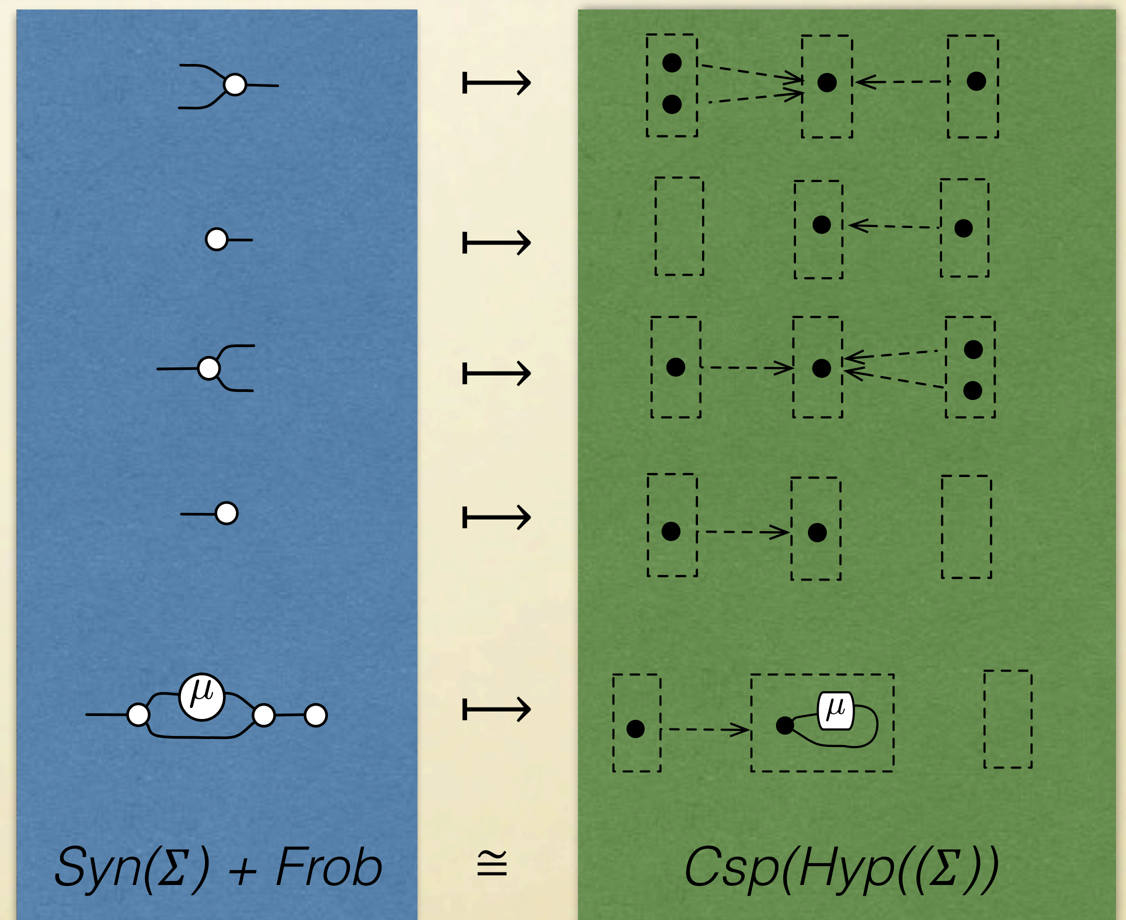
Frobenius makes DPO rewriting sound

Theorem I

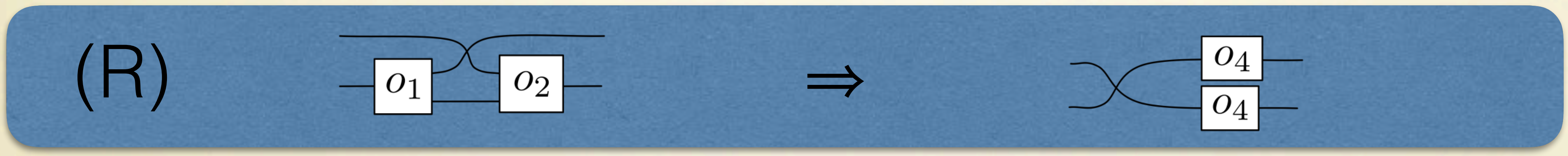
DPO rewriting with interfaces is *sound and complete* for symmetric monoidal categories with a chosen *special Frobenius structure*.

prop of special Frobenius algebras

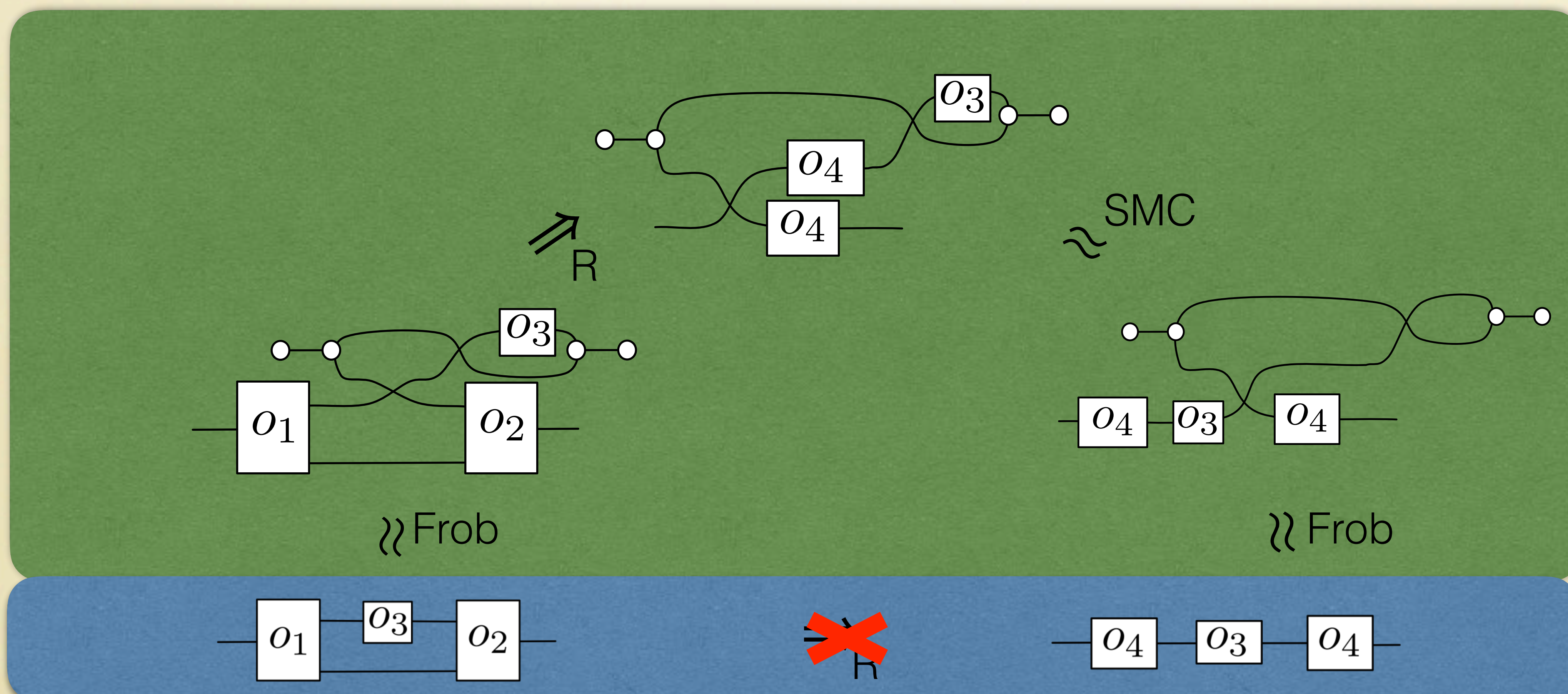




Frobenius makes DPO rewriting sound



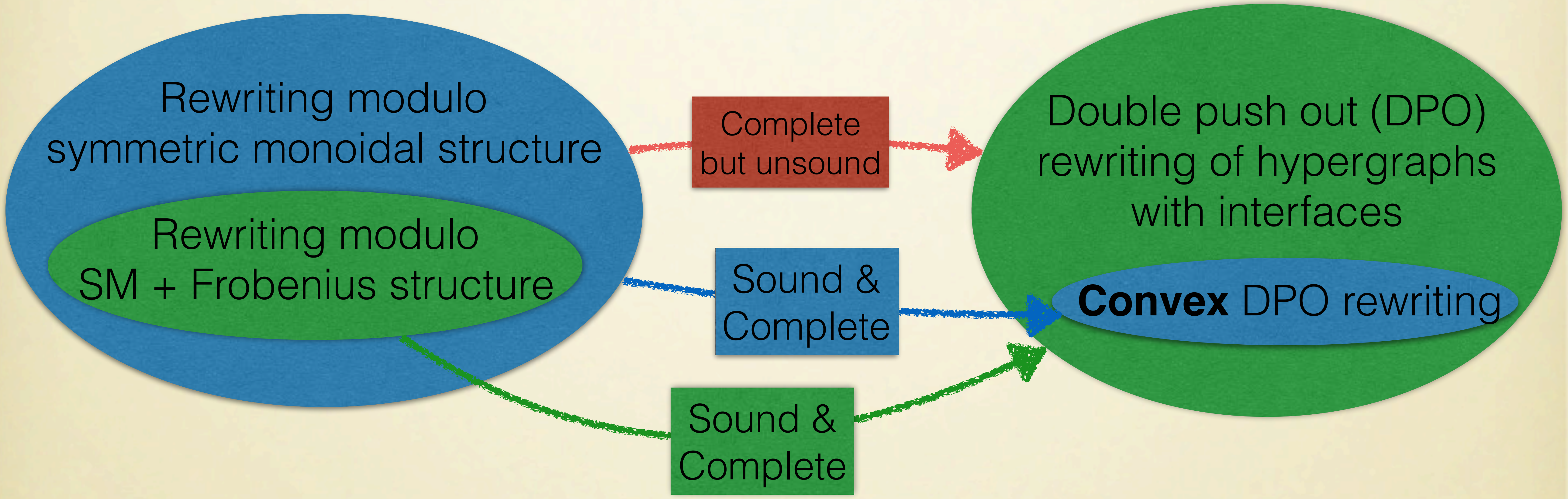
Syn(Σ)



Frob
+
Syn(Σ)

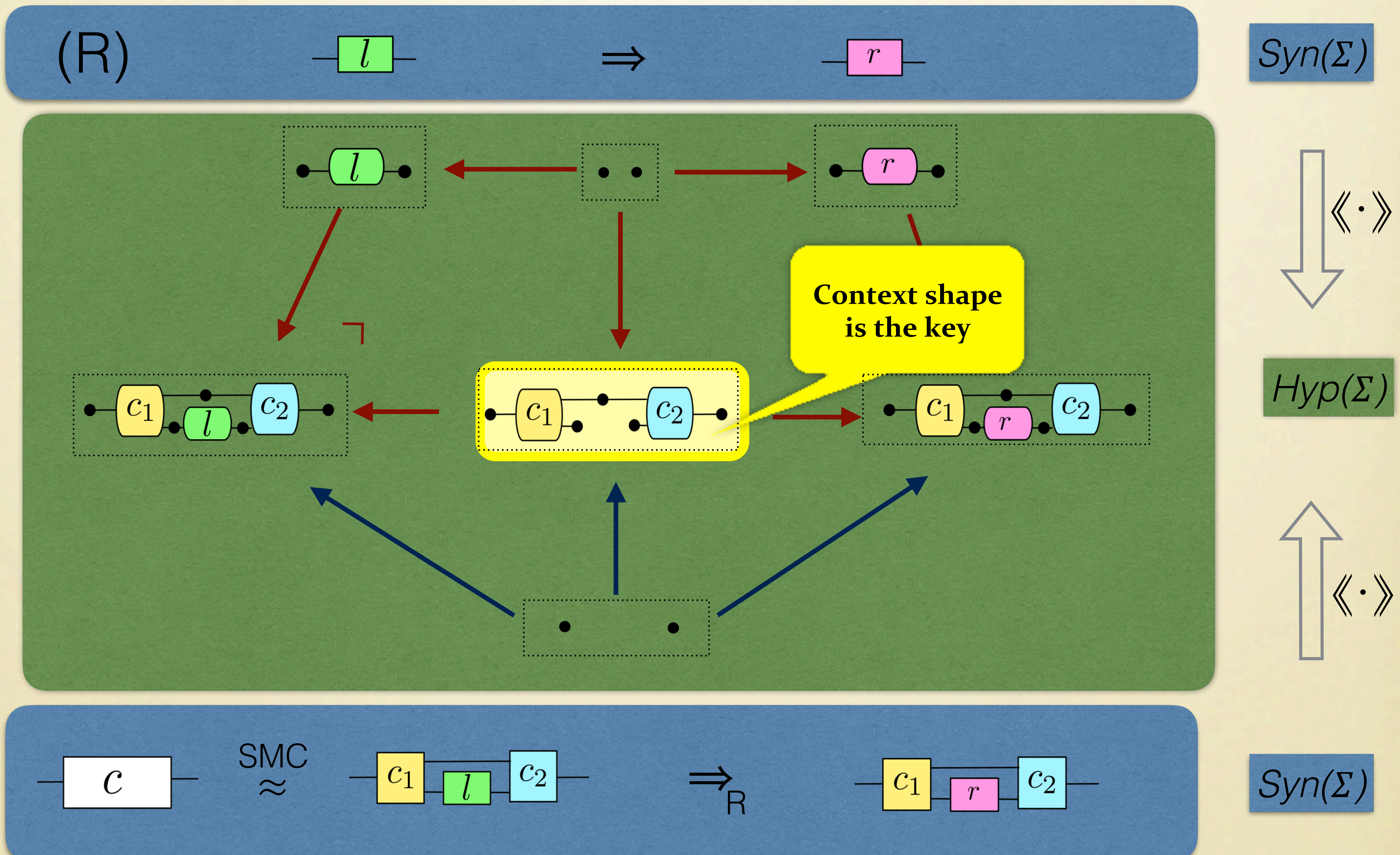
Where we are, so far

1. Adequate interpretation

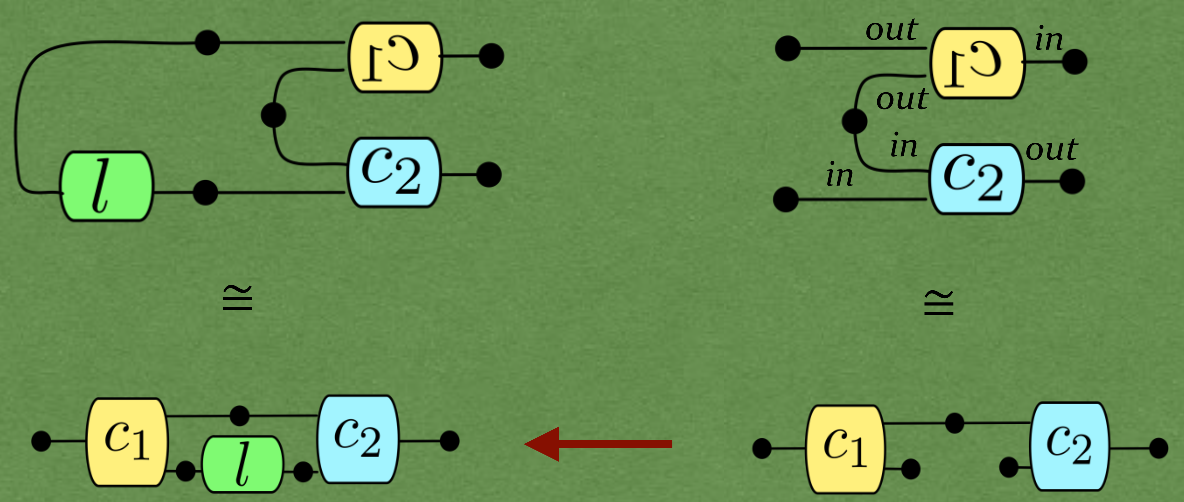


2. Decidability of confluence

How does sound DPO rewriting look like?



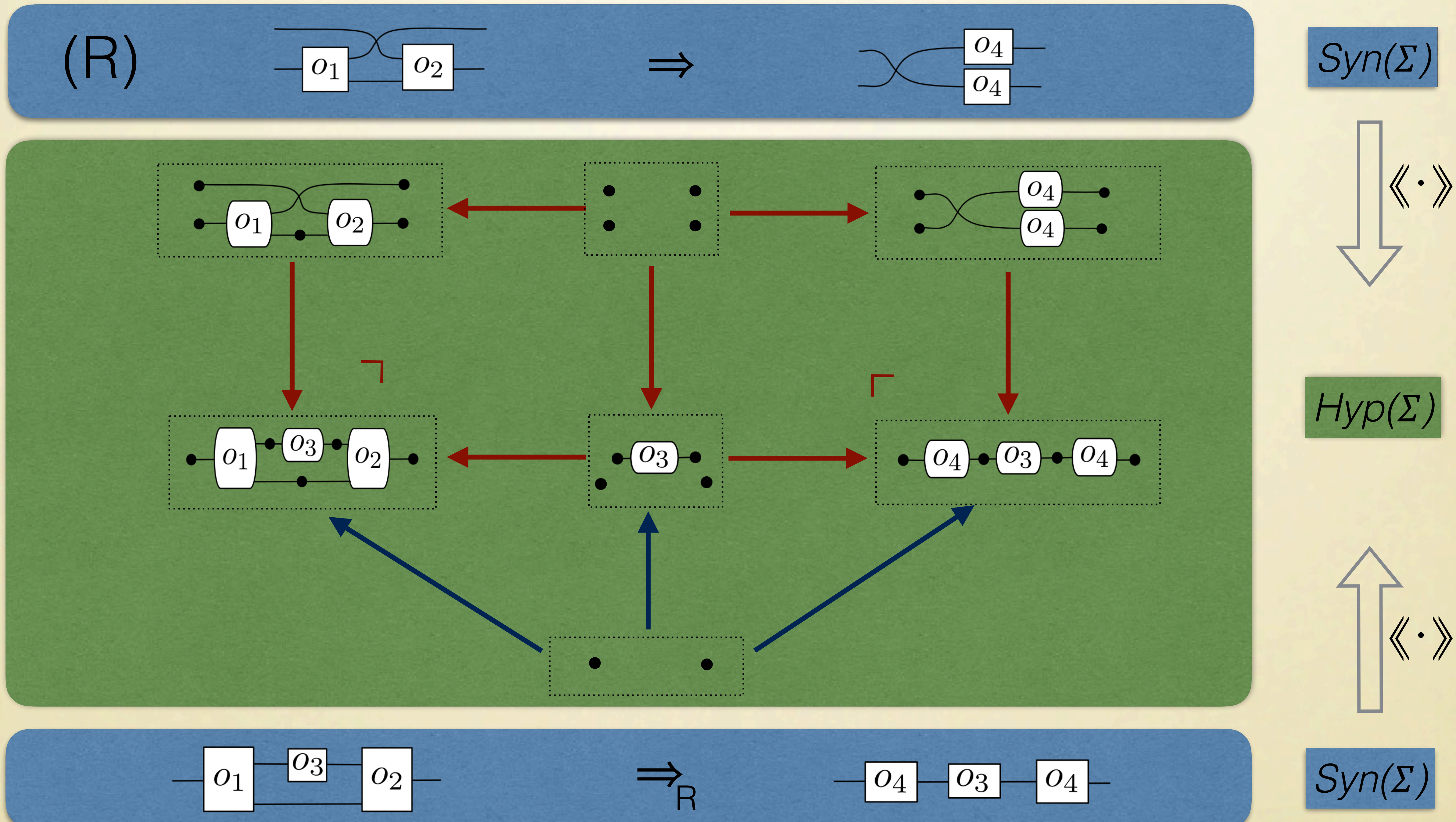
How does sound DPO rewriting look like?



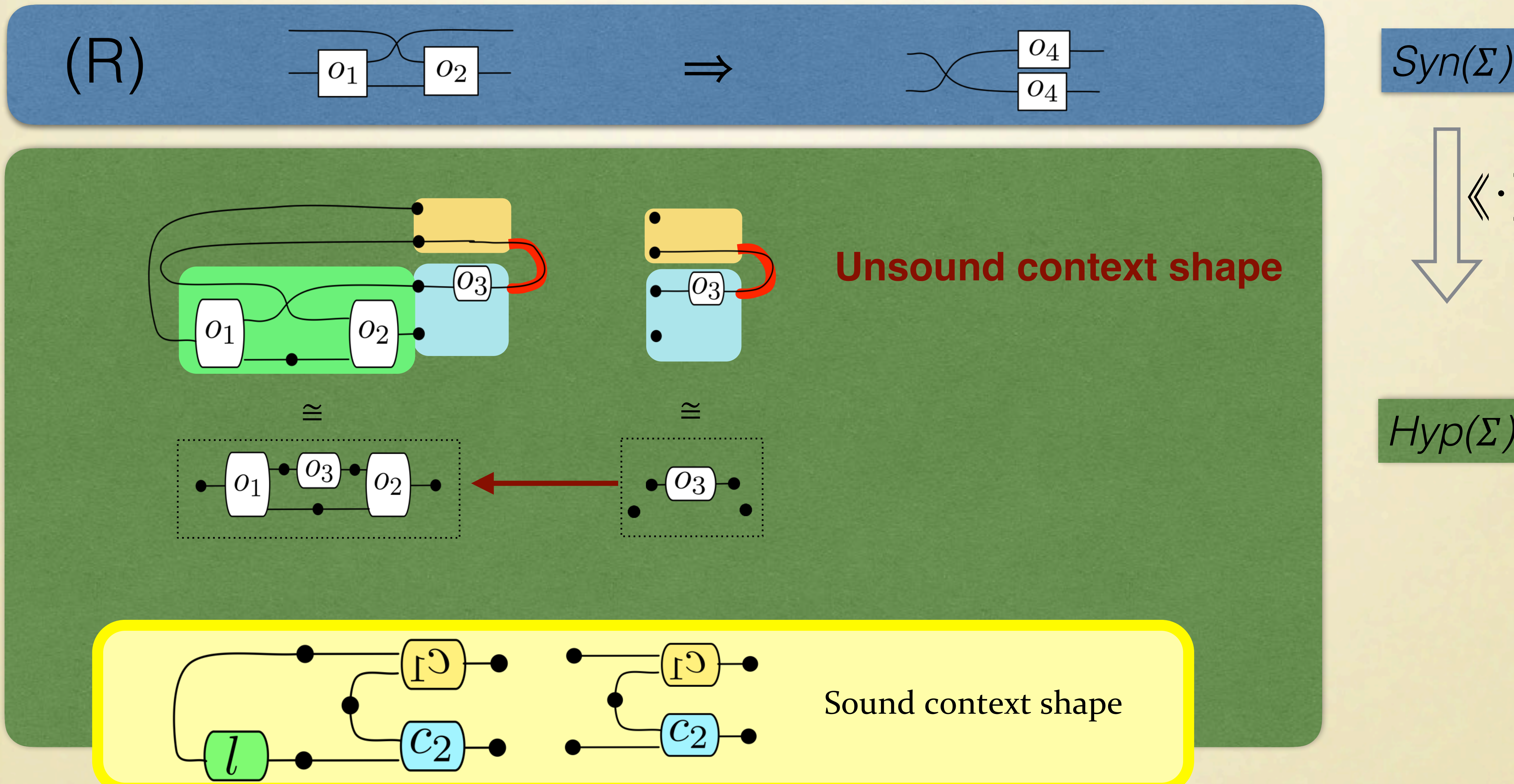
Leading Intuition
a rewriting steps
is sound
iff
the rewriting context
has this shape

Hyp(Σ)

Back to the soundness counterexample



Back to the soundness counterexample



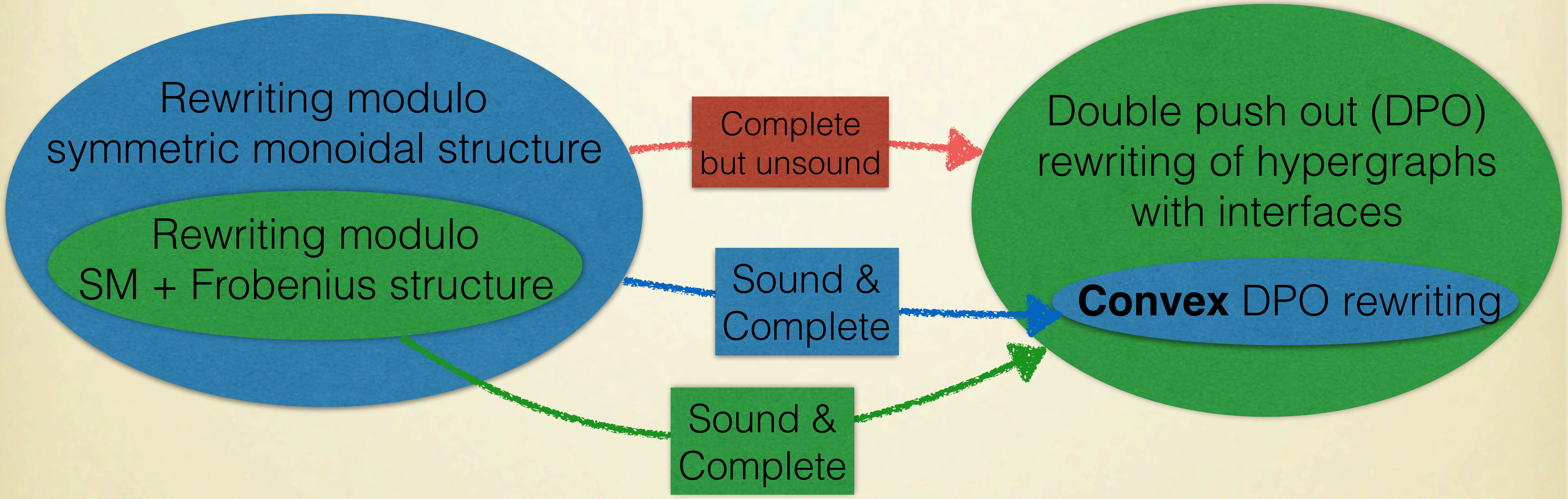
Convex DPO rewriting is sound

Theorem I
DPO rewriting with interfaces is *sound and complete*
for symmetric monoidal categories
with a chosen *special Frobenius structure*.

Theorem II
Convex DPO rewriting with interfaces is *sound and complete*
for symmetric monoidal categories.

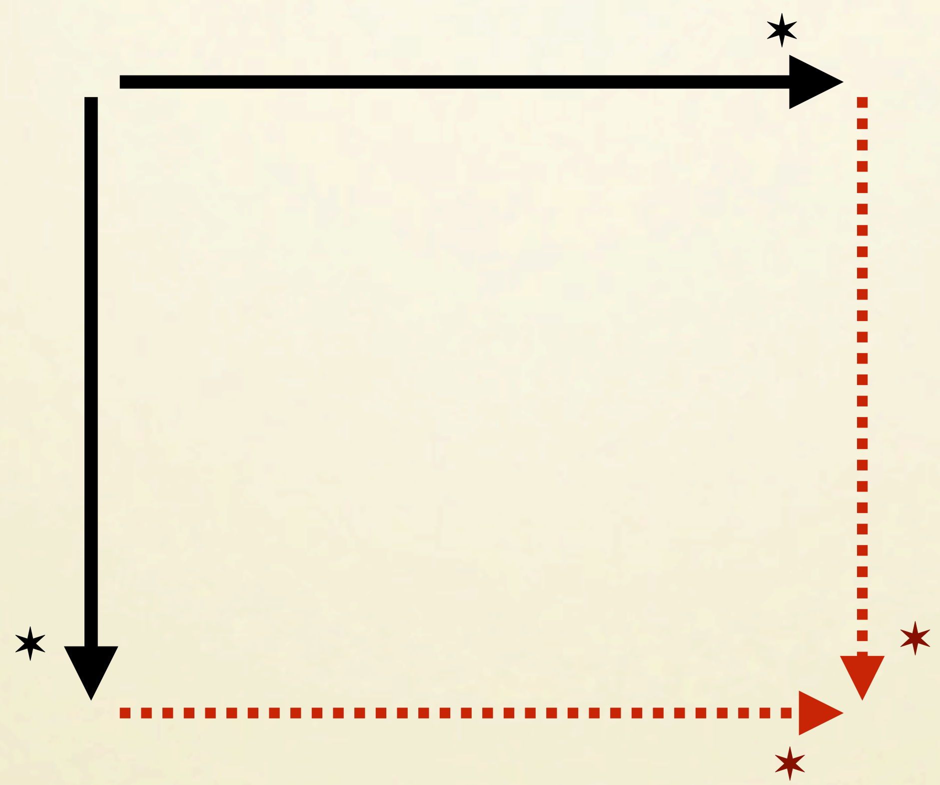
Where we are, so far

1. Adequate interpretation



2. Decidability of confluence

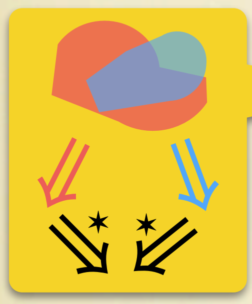
Confluence, abstractly



If *E* is confluent & terminating
then $x \stackrel{E}{=} y$ becomes decidable.

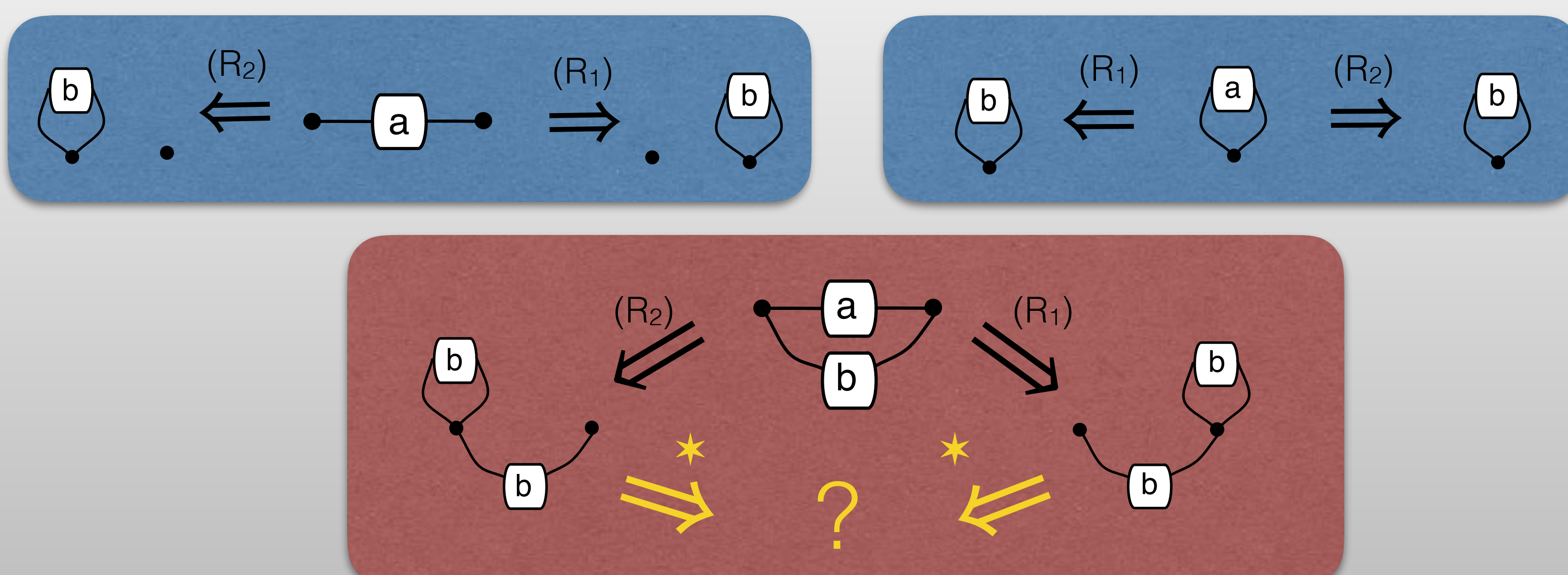
Decidability of Confluence

In term rewriting, confluence is **decidable** for terminating systems



All the critical pairs are joinable ⟺ The system is confluent (Knuth-Bendix)

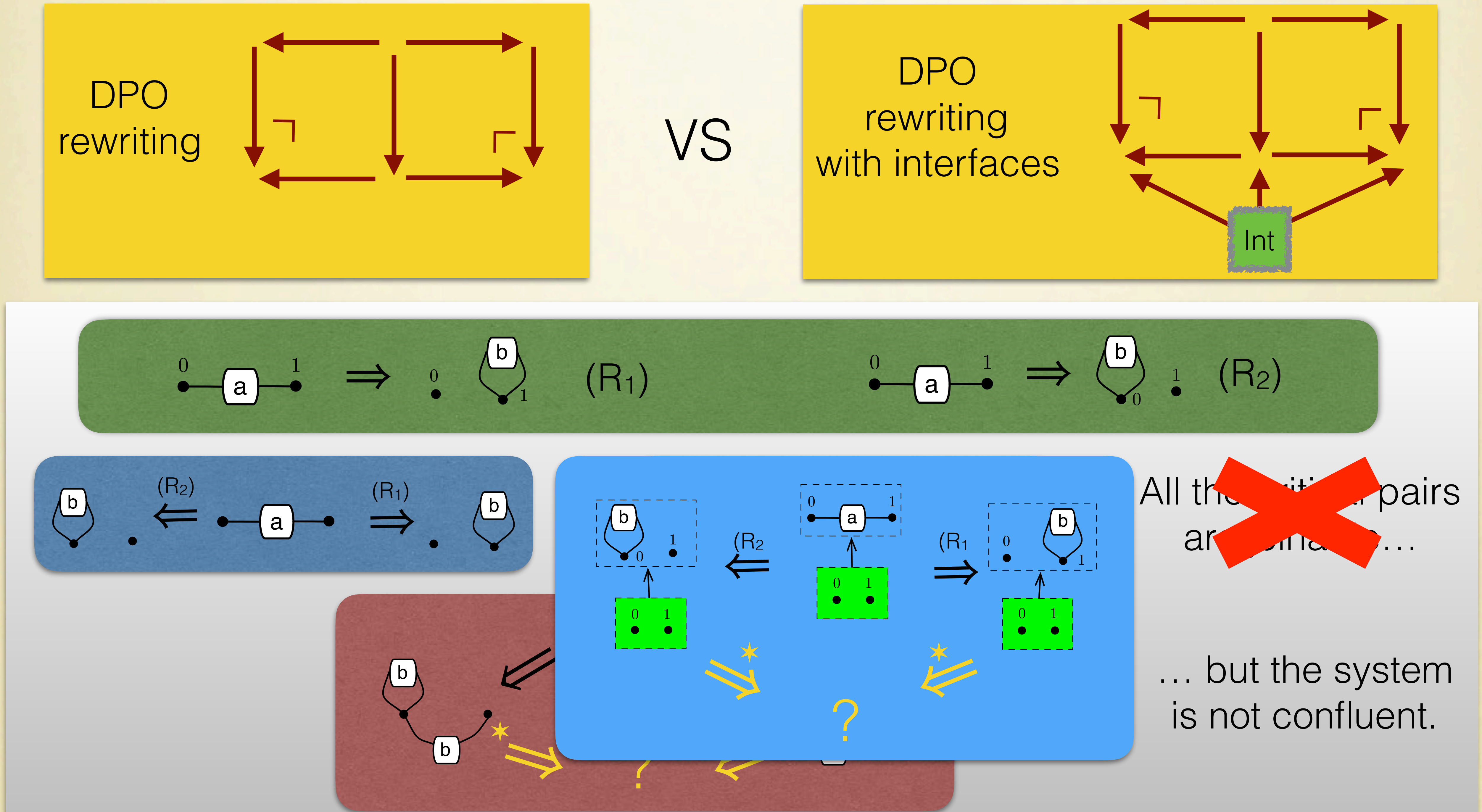
In DPO (hyper)graph rewriting, confluence is **undecidable** (Plump)



All the critical pairs are joinable…

… but the system is not confluent.

Interfaces to the Rescue



Theorem In DPO rewriting *with interfaces*, confluence is decidable.

Confluence is decidable

Theorem I
DPO rewriting with interfaces is *sound and complete*
for symmetric monoidal categories
with a chosen *special Frobenius structure*.

Theorem II
Convex DPO rewriting with interfaces is *sound and complete*
for symmetric monoidal categories.

Confluence is decidable

Theorem I bis
DPO rewriting with interfaces is *sound and complete* for symmetric monoidal categories with a chosen *special Frobenius structure*.

Confluence is decidable for terminating rewriting systems on such categories.

Theorem II
Convex DPO rewriting with interfaces is *sound and complete* for symmetric monoidal categories.

Confluence is decidable

Theorem I bis

DPO rewriting with interfaces is *sound and complete* for symmetric monoidal categories with a chosen *special Frobenius structure*.

Confluence is decidable for terminating rewriting systems on such categories.

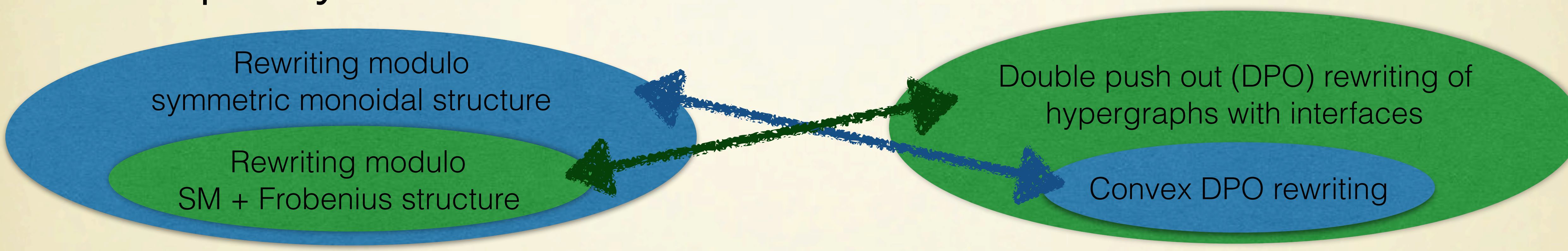
Theorem II bis

Convex DPO rewriting with interfaces is *sound and complete* for symmetric monoidal categories.

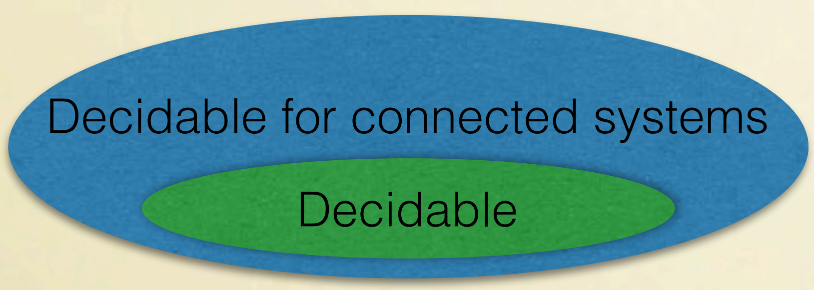
Confluence is decidable for *connected* terminating rewriting systems on such categories.

Conclusions

Adequacy



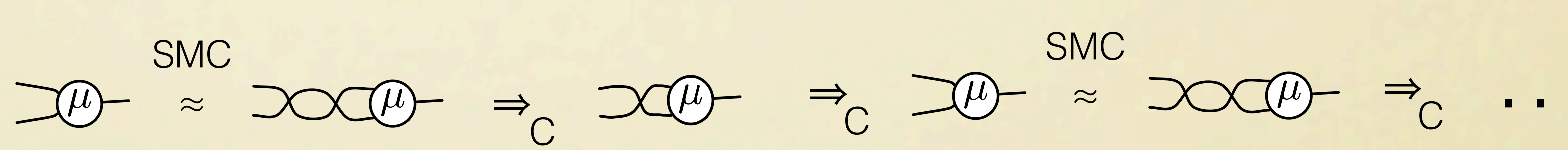
Confluence



	Terminating term rewriting systems	Terminating DPO-with-interface systems
Confluence for ground objects	*undecidable (Kapur et al.)*	*undecidable (Plump)*
Confluence	*decidable (Knuth-Bendix)*	***decidable***

Termination

Commutativity does not terminate



Proposal: interpret commutative operators as nodes of a new sort