Quantale-valued dissimilarity

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Abstract

Motivated by the theory of apartness relations of Scott [3], a positive theory of *dissimilarity* valued in an involutive quantale

$$\mathsf{Q} = (\mathsf{Q}, \&, k, {}^\circ)$$

is established without the aid of negation. The notion of Q-valued dissimilarity dualizes that of Q-*valued set* (i.e., a set equipped with a Q-valued similarity) in the sense of Höhle–Kubiak [2], whose prototype comes from the theory of Ω-*sets* of Fourman–Scott [1].

It is demonstrated that sets equipped with a Q-valued dissimilarity are symmetric categories enriched in the quantaloid

$$\mathbf{B}(\mathsf{Q})$$

of *back diagonals* of Q [4]. Moreover, it is shown that similarities and dissimilarities are interdefinable if Q is a *Girard quantale*, in which case there is an isomorphism

$$\mathbf{D}(\mathsf{Q}) \cong \mathbf{B}(\mathsf{Q})$$

of quantaloids, where $\mathbf{D}(\mathsf{Q})$ is the quantaloid of *diagonals* of Q [5]. In the case that Q is a commutative quantale, it is proved that the above isomorphism holds if, and only if, Q is a Girard quantale.

Keywords: dissimilarity, similarity, back diagonal, diagonal, quantale, quantaloid

References

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