TRUNCATIONS AND BLAKERS-MASSEY IN AN ELEMENTARY HIGHER TOPOS

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Abstract. We study truncated and connected objects in an elementary higher topos. In particular, we show they have the same behavior as in spaces, construct a universal truncation functor using natural number objects and show all modalities, and in particular connected maps, satisfy the statement of the Blakers-Massey Theorem.

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Introduction

0.1 Motivation. One of the powerful tools in algebraic topology are truncations. They allow us to simplify the structure of a space and enable us to focus on certain aspects of a space. An elementary higher topos is a higher category which generalizes an elementary topos. One central thesis about an elementary higher topos is that we can employ techniques from algebraic topology. The goal of this paper is to advance this thesis by proving classical results from algebraic topology in an arbitrary elementary higher topos.

In particular, we will study n-truncated and n-connected maps in an elementary higher topos and prove that we can factor every map into an n-connected map followed by an n-truncated map. Moreover, we study more general factorization systems, so called modalities, in an elementary higher topos and prove that it satisfies the Blakers-Massey theorem. This implies in particular that the

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Blakers-Massey theorem holds for connected maps giving us the classical Blakers-Massey theorem as well as the Freudenthal suspension theorem.

0.2 Outline. In Section 1 we introduce truncated and connected objects and give various ways to classify them. In particular, we give certain closure properties (under base change and cobase change) and study the analogue of injections and surjections of spaces, namely (-1)-truncated and (-1)-connected maps.

In Section 2 we first introduce the join construction. Then we use the join construction to construct universal (-1)-truncations (Theorem 2.25). The approach in that section is largely built on work of Rijke [Ri17] and Rezk [Re05].

In Section 3 we use universal (-1)-truncations to construct n-truncations (Theorem 3.25). Again the work in this section is motivated by the work of Rijke in [Ri17].

In Section 4 we change the tune and show that all modalities in an elementary higher topos satisfy the Blakers-Massey Theorem (Theorem 4.18). This was already proven in [ABFJ17] for Grothendieck higher toposes and we show that the same proof holds for elementary higher toposes. Finally, we also show that this general result gives us the classical Blakers-Massey theorem for n-truncated and n-connected maps, as well as the Freudenthal suspension theorem.

0.3 Background. The work here is in the context of $(\infty, 1)$-categories and we use the model of complete Segal spaces [Ra18a], [Re01]. However, the results here are not model-specific and hold in the exact same way for quasi-categories. We use elementary higher toposes as defined in [Ra18b]. Moreover, we will make use of the fact that every elementary higher topos has a natural number object, which was proven in [Ra18c]. Many of the definitions and results from the two first sections are motivated by, or come directly from, [Re05]. The constructions from the second and third sections are motivated primarily by [Ri17]. Finally, all of Section 4 is direct adaptation of [ABFJ17].

0.4 Notation. Throughout this whole paper $\mathcal{E}$ is a fixed elementary higher topos. We denote the final object in $\mathcal{E}$ with 1. On the other hand, we denote the final object in spaces as $*$. Moreover, we denote the natural number object in $\mathcal{E}$ as $\mathbb{N}$. To avoid confusion, we denote the n-sphere in spaces by S^n, but the internal n-sphere in $\mathcal{E}$ by $S^n_{\mathcal{E}}$ (Definition 1.14).

0.5 Acknowledgements. I want to thank Georg Biedermann for helpful conversations about their work in [ABFJ17].

Truncated and Connected Objects

In this section we will introduce truncated and connected objects. We end this section by looking at some basic connectivity results.

1.1 Truncated Objects. In this subsection we define and study truncated objects. Let us first review truncated spaces.

Definition 1.1. Let $S^{-1} = \emptyset$, the empty space, and S^n $(n \geq 0)$ be defined inductively by the pushout square

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Definition 1.2. Let $n \geq -2$. A space X is n-truncated if

$$Map(*, X) \to Map(S^{n+1}, X)$$

is an equivalence of spaces.

Example 1.3. A space is (-2)-truncated if it is contractible. Moreover, it is (-1)-truncated if it is empty or contractible.

There is a relative version of this definition.

Definition 1.4. A map of spaces $f : Y \to X$ is n-truncated if every fiber is n-truncated.

The goal is to generalize this concept to an arbitrary $(\infty, 1)$-category.

Definition 1.5. An object X in $\mathcal{E}$ is n-truncated if for every object Y the mapping object $Map_{\mathcal{E}}(Y, X)$ is an n-truncated space.

Remark 1.6. This definition holds for any $(\infty, 1)$-category, but here we focus only on the case of elementary higher toposes.

Example 1.7. An object X is (-2)-truncated if $Map(Y, X)$ is contractible. This is means X is equivalent to the final object 1.

The definition has a relative version.

Definition 1.8. A map $f : Y \to X$ is n-truncated if it is n-truncated as an object in $\mathcal{E}_{/X}$.

We have following alternative characterization of n-truncated maps.

Lemma 1.9. *A map $f : Y \to X$ is n-truncated if for every object Z the map of spaces $f_* : Map(Z, Y) \to Map(Z, X)$ is n-truncated.*

Example 1.10. A map f is (-2)-truncated if and only if it is an equivalence in $\mathcal{E}$.

The two definitions are related by the following lemma.

Lemma 1.11. *An object X is n-truncated if and only if $X \to 1$ is n-truncated.*

Notice this definition also implies following important result.

Lemma 1.12. *n-truncated maps are closed under base change.*

Remark 1.13. This does not hold for cobase change. In the pushout diagram of spaces

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^2 \end{tikzcd}$$

the map $S^1 \to 1$ is 1-truncated. But the map $1 \to S^2$ is clearly not 1-truncated as the homotopy fiber over each point in S^2 is ΩS^2, which is not 1-truncated.

Our next goal is to give an alternative characterization of n-truncated maps.

Definition 1.14. Let $S^{-1}_{\mathcal{E}} = \emptyset_{\mathcal{E}}$, the initial object, and $S^n_{\mathcal{E}}$ ($n \geq 0$) be defined inductively by the pushout square

$$\begin{tikzcd} S^{n-1}_{\mathcal{E}} \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & S^n_{\mathcal{E}} \end{tikzcd}$$

We call these objects the *n-spheres in $\mathcal{E}$*. Notice they come with a well-defined basepoint $1 \to S^n_{\mathcal{E}}$.

Lemma 1.15. *Let Z and X be two objects in $\mathcal{E}$ and $n \geq -1$. Then we have an equivalence*

$$Map_{\mathcal{E}}(Z \times S^n_{\mathcal{E}}, X) \simeq Map_{\mathcal{S}paces}(S^n, Map_{\mathcal{E}}(Z, X))$$

Proof. We will prove it by induction. If $n = -1$ then the result is clear. Let us assume it holds for n. As $\mathcal{E}$ satisfies descent the object $Z \times S^{n+1}$ is the pushout of the following diagram

$$\begin{tikzcd} Z \times S^n_{\mathcal{E}} \arrow[r] \arrow[d] & Z \arrow[d] \\ Z \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Z \times S^{n+1}_{\mathcal{E}} \end{tikzcd}$$

Taking mapping spaces gives us following pullback square.

$$\begin{tikzcd} Map_{\mathcal{E}}(Z \times S^{n+1}_{\mathcal{E}}, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Map_{\mathcal{E}}(Z, X) \arrow[d] \\ Map_{\mathcal{E}}(Z, X) \arrow[r] & Map_{\mathcal{E}}(Z \times S^{n}_{\mathcal{E}}, X) \end{tikzcd}$$

Using our induction assumption we can rewrite this pullback square as follows.

$$\begin{tikzcd} Map_{\mathcal{E}}(Z \times S^{n+1}_{\mathcal{E}}, X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Map_{\mathcal{S}\mathrm{paces}}(*, Map_{\mathcal{E}}(Z, X)) \arrow[d] \\ Map_{\mathcal{S}\mathrm{paces}}(*, Map_{\mathcal{E}}(Z, X)) \arrow[r] & Map_{\mathcal{S}\mathrm{paces}}(S^n, Map_{\mathcal{E}}(Z, X)) \end{tikzcd}$$

This implies that

$$Map_{\mathcal{E}}(Z \times S^{n+1}_{\mathcal{E}}, X) \xrightarrow{\simeq} Map_{\mathcal{S}\mathrm{paces}}(\Sigma S^n, Map_{\mathcal{E}}(Z, X)) \xrightarrow{\simeq} Map_{\mathcal{S}\mathrm{paces}}(S^{n+1}, Map_{\mathcal{E}}(Z, X))$$

This finishes our induction step. □

Lemma 1.16. *An object X in $\mathcal{E}$ is n-truncated if and only if for every object Z the induced map*

$$Map_{\mathcal{E}}(Z \times S^{n+1}_{\mathcal{E}}, X) \to Map_{\mathcal{E}}(Z, X)$$

is an equivalence of spaces.

Proof. By the previous lemma we have a diagram where the vertical map is always an equivalence.

$$\begin{tikzcd} Map_{\mathcal{S}\mathrm{paces}}(*, Map_{\mathcal{E}}(Z \times S^{n+1}_{\mathcal{E}}, X)) \arrow[d, "\simeq"] \arrow[dr] & \\ & Map_{\mathcal{S}\mathrm{paces}}(*, Map_{\mathcal{E}}(Z, X)) \\ Map_{\mathcal{S}\mathrm{paces}}(S^{n+1}, Map_{\mathcal{E}}(Z, X)) \arrow[ur] & \end{tikzcd}$$

So the top map is an equivalence if and only if the bottom map is an equivalence. However, the bottom map is an equivalence if and only if $Map_{\mathcal{E}}(Z, X)$ is n-truncated. □

Corollary 1.17. *An object X is n-truncated if and only if the map $X^{S^{n+1}_{\mathcal{E}}} \to X$ is an equivalence in $\mathcal{E}$.*

Example 1.18. X is (-1)-truncated if and only if the diagonal map $\Delta : X \to X^{S^0_{\mathcal{E}}} \simeq X \times X$ is an equivalence.

This example generalizes in the appropriate way.

Proposition 1.19. *X is n-truncated if and only if $\Delta : X \to X \times X$ is $(n-1)$-truncated.*

Proof. Let Z be an arbitrary object. Then we get the map of spaces

$$Map_{\mathcal{E}}(Z, X) \to Map_{\mathcal{E}}(Z, X) \times Map_{\mathcal{E}}(Z, X)$$

The fiber over each point (f, g) is the space of paths in $Map_{\mathcal{E}}(Z, X)$ from f to g. Thus $X \to X \times X$ is $(n-1)$-truncated if and only if each fiber is $(n-1)$-truncated which can only be true if $Map(Z, X)$ is n-truncated for each Z. This is equivalent to X being n-truncated. □

These results have relative versions which can be proven analogously.

Lemma 1.20. *A map $Y \to X$ is n-truncated if and only if for every map $p : Z \to X$ the induced map*

$$Map_{/X}(Z \times S^{n+1}_{\mathcal{E}}, Y) \to Map_{/X}(Z, Y)$$

is an equivalence of spaces. Here we take the map $p\pi_1 : Z \times S^{n+1}_{\mathcal{E}} \to X$.

Corollary 1.21. *A map $Y \to X$ is n-truncated if and only if the map*

$$[Y \to X]^{[X \times S^{n+1}_{\mathcal{E}} \to X]} \to [Y \to X]$$

is an equivalence in $\mathcal{E}_{/X}$. Here the left hand side is the internal mapping object in $\mathcal{E}_{/X}$.

Lemma 1.22. *A map $A \to B$ is n-truncated if and only if the map $\Delta : A \to A \times_B A$ is $(n-1)$-truncated. In particular, a map $A \to B$ is (-1)-truncated if and only if $A \to A \times_B A$ is an equivalence.*

One implication of these results is the comparison between (-1)-truncated maps and monos.

Proposition 1.23. *A map $f : A \to B$ is mono if and only if it is (-1)-truncated.*

Proof. Assume f is (-1)-truncated and fix an object C. Then we have following pullback diagram

$$\begin{tikzcd}
Map(C, A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Map(C, A) \arrow[d] \\
Map(C, A) \arrow[r] & Map(C, B)
\end{tikzcd}$$

Let us now fix a map $f : C \to B$ and the denote the fiber over g by Fib_g. Then the pullback gives us an equivalence

$$Fib_g \xrightarrow{\simeq} Fib_g \times Fib_g$$

This implies that Fib_g is (-1)-truncated which exactly implies that f is mono.

On the other hand if f is mono, then for any map $g : C \to B$ the space Fib_g is either empty or contractible which implies that that the square above is a pullback square. □

1.2 Connected Objects. In this subsection we define and study connected objects in an elementary higher topos. The goal is to show how they complement truncated objects. First we again review the concept of connected spaces.

Definition 1.24. A space X is n-connected if its n-truncation is contractible. This means that $\pi_k(X)$ is trivial for $k \leq n$.

This definition also has a relative version

Definition 1.25. A map of spaces $Y \to X$ is n-connected if the homotopy fiber over each point $x \in X$ is n-connected.

Right now we cannot generalize this definition to an arbitrary elementary higher topos as we have not yet proven that we have an n-truncation functor. We will present the analogue to the definition above in Proposition 3.11 after we discussed truncation functors. Thus we will now give another formulation of connected maps of spaces.

Lemma 1.26. *A map of spaces $Y \to X$ is n-connected if and only if for every n-truncated map $Z \to X$ the induced map*

$$Map_{/X}(X, Z) \to Map_{/X}(Y, Z)$$

is an equivalence of spaces.

This equivalent definition can be easily generalized to a definition for connected maps in a topos.

Definition 1.27. A map $Y \to X$ in a topos $\mathcal{E}$ is n-connected if for every n-truncated map $Z \to X$ the induced map

$$Map_{/X}(X, Z) \to Map_{/X}(Y, Z)$$

is an equivalence of spaces.

The definition has following special case.

Definition 1.28. An object Y is n-connected if for every n-truncated object Z the induced map

$$Map(1, Z) \to Map(Y, Z)$$

is an equivalence of spaces.

Example 1.29. Lemma 1.16 tells us that the object $S^n_{\mathcal{E}}$ is n-connected.

In particular we can generalize Lemma 1.20 using n-connected maps.

Lemma 1.30. *A map $Y \to X$ is n-truncated if for every map $Z \to X$ and n-connected object W the induced map*

$$Map_{/X}(Z \times W, Y) \to Map_{/X}(Z, Y)$$

is an equivalence.

We showed that n-truncated maps are preserved by base change. Similarly we can show that n-connected maps are preserved by cobase change.

Proposition 1.31. *Let $X \to Y$ be n-connected. Then for any map $X \to A$ the induced map $A \to Y \coprod_X A$ is also n-connected.*

Proof. Let $Z \to A \coprod_X Y$ be an n-truncated map. A pushout diagram in $\mathcal{E}$ is still a pushout diagram in the over category. This gives us following equivalence

$$Map_{/A \coprod_X Y}(A \coprod_X Y, Z) \simeq Map_{/A \coprod_X Y}(A, Z) \underset{Map_{/A \coprod_X Y}(X,Z)}{\times} Map_{/A \coprod_X Y}(Y, Z)$$

Now we have following commutative square.

$$\begin{tikzcd}
Map_{/A \coprod_X Y}(Y, Z) \arrow[r, "\simeq"] \arrow[d] & Map_Y(Y, Z \underset{A \coprod_X Y}{\times} Y) \arrow[d, "\simeq"] \\
Map_{/A \coprod_X Y}(X, Z) \arrow[r, "\simeq"] & Map_Y(X, Z \underset{A \coprod_X Y}{\times} Y)
\end{tikzcd}$$

The horizontal maps are equivalences because of the adjunction. The right vertical map is an equivalence because $X \to Y$ is n-connected and $Z \times_{A \coprod_X Y} Y$ is n-truncated, by Lemma 1.12. This implies that the left hand map is also an equivalence. Finally, we have following pullback square.

$$\begin{tikzcd}
Map_{/A \coprod_X Y}(A, Z) \underset{Map_{/A \coprod_X Y}(X,Z)}{\times} Map_{/A \coprod_X Y}(Y, Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Map_{/A \coprod_X Y}(Y, Z) \arrow[d, "\simeq"] \\
Map_{/A \coprod_X Y}(A, Z) \arrow[r] & Map_{/A \coprod_X Y}(X, Z)
\end{tikzcd}$$

The fact that the right hand vertical map is an equivalence implies that the left hand vertical map is also an equivalence, which is exactly the result we wanted. $\square$

Remark 1.32. In fact n-connected maps are also preserved by base change. We will prove it in Corollary 3.13, but before that we need to develop more theory and define truncation functors.

Connected maps also behave well with respect to composition.

Proposition 1.33. *Let $f : Y \to X$ and $g : Z \to Y$.*

(1) If f and g are n-connected then fg is n-connected.
(2) If g and fg are n-connected then f is n-connected.

Proof. Let W be a n-truncated object over X. We have following diagram

$$
\begin{tikzcd}
 & & Map_{/X}(X,W) \arrow[dl, "f^*"'] \arrow[dr, "(fg)^*"] & & \\
 & Map_{/X}(Y,W) \arrow[dl, "\simeq"'] & & Map_{/X}(Z,W) \arrow[dr, "\simeq"] & \\
Map_{/Y}(Y, W \underset{X}{\times} Y) \arrow[rrrr, "g^*", "\simeq"'] & & & & Map_{/Y}(Z, W \underset{X}{\times} Y)
\end{tikzcd}
$$

If we assume that f and g are n-connected then f^* and g^* are equivalences which implies that $(fg)^*$ is an equivalence which means that fg is n-connected. On the other hand if we assume that g and fg are n-connected then g^* and $(fg)^*$ are equivalences which implies that f^* is an equivalence which means that f is n-connected. □

Truncated and connected objects do not intersect in a non-trivial way.

Lemma 1.34. *Let Y be an object that is n-truncated and n-connected. Then Y is the final object.*

Proof. As Y is n-connected we know that we have an equivalence

$$Map(1, Z) \simeq Map(Y, Z)$$

for every n-truncated object Z. But Y itself is n-truncated, which means we have an equivalence.

$$Map(1, Y) \simeq Map(Y, Y)$$

This means that every map $Y \to Y$ will factor through the final map $Y \to 1$. If we apply this to the identity map we get a retract

$$Y \to 1 \to Y$$

which proves that Y is equivalent to the final object. □

This result also has an interesting relative version.

Corollary 1.35. *Let $f : Y \to X$ be a map that is n-truncated and n-connected. Then f is an equivalence.*

Remark 1.36. For the case $n = -1$ this generalizes the classical result for elementary toposes, where every map that is epi and mono is an isomorphism. That is known as being a *balanced category*.

(-1)-Connected Maps: (-1)-connected maps are special and thus deserve our special attention.

Remark 1.37. A (-1)-connected map is also called an *effective epimorphism* or a *cover*. However, in order to avoid additional notation, we will simply call them (-1)-connected maps.

First of all we present a nice way to characterize (-1)-connected maps. It was proven for Grothendieck higher toposes in [Re05, Lemma 7.9] and [Lu09, Proposition 6.2.3.10]. We generalize it here to elementary higher toposes.

Lemma 1.38. *A map $f : Y \to X$ is (-1)-connected if and only if the induced map on the set of subobjects, $f^* : Sub(X) \to Sub(Y)$, is injective.*

Proof. First notice that f^* is injective if and only if $f^*(A) = f^*(B)$ implies $A = B$ for all $A \leq B$. That is because f^* preserves the meet of two subobjects. Thus we will always assume $A \leq B$ throughout the proof.

In order to prove the result we first have to gain a better understanding of (-1)-connected maps. By definition $Y \to X$ is (-1)-connected if we have an equivalence

$$Map_{/X}(X, Z) \to Map_{/X}(Y, Z)$$

where $Z \to X$ is (-1)-truncated. However, the fact that $Z \to X$ is (-1)-truncated means that Z is a subobject of X. Thus the space $Map_{/X}(X, Z)$ is contractible if and only if $Z = X$, otherwise it is empty. The equivalence then implies that $Map_{/X}(Y, Z)$ has to be empty except if $Z = X$.

We can summarize this analysis by saying that in the diagram below:

$$\begin{tikzcd} & Z \arrow[d, hook] \\ Y \arrow[ur, dashed] \arrow[r, two heads, "f"] & X \end{tikzcd}$$

f being (-1)-connected is equivalent to a lift existing if and only if Z is the maximal subobject. We will use this statement to prove the lemma.

First assume that f is (-1)-connected. We want to prove that $f^* : Sub(X) \to Sub(Y)$ is injective. Let A and B be two subobjects of X such that $f^*(A) = f^*(B)$ and $B \leq A$ (using the realization from the first paragraph). Then this gives us following diagram

$$\begin{tikzcd} f^*(B) \arrow[r, two heads] \arrow[d, hook, "\simeq"] & B \arrow[d, hook] \\ f^*(A) \arrow[r, two heads] \arrow[d, hook] & A \arrow[d, hook] \\ Y \arrow[r, two heads, "f"] & X \end{tikzcd}$$

We already know that (-1)-connected maps are stable under base change. This implies that the map $f^*(A) \to A$ is also (-1)-connected. By the previous paragraph then $f^*(A)$ has a lift to B if and only if $A = B$.

On the other side assume that $f^* : Sub(X) \to Sub(Y)$ is injective. We will prove that $f : Y \to X$ is (-1)-connected. By the previous paragraphs it suffices to prove that f lifts to a subobject Z of X if and only if $Z = X$. Thus let $Z \hookrightarrow X$ be a subobject and assume a lift $Y \to Z$ exists. This means we have the pullback diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

This means that $f^*(Z) = f^*(X) = Y$. By injectivity of f^* this means that $Z = X$, which finishes the proof. $\square$

Example 1.39. Using this lemma we can immediately deduce that a (-1)-connected map of spaces is a map that is surjective on path components.

We can use the previous lemma to find many interesting basic results about (-1)-connected maps.

Lemma 1.40. *$f : C \to A$ and $g : C \to B$ be two maps. Then the natural map*

$$A \coprod B \to A \coprod_C B$$

is (-1)-connected.

Proof. We will show the induced map on subobjects in injective. First notice that Sub is represented by the subobject classifier Ω. This gives us isomorphisms

$$Sub(A \coprod_C B) \cong Hom(A \coprod_C B, \Omega) \cong Hom(A, \Omega) \underset{Hom(C,\Omega)}{\times} Hom(B, \Omega)$$

$$Sub(A \coprod B) \cong Hom(A \coprod B, \Omega) \cong Hom(A, \Omega) \times Hom(B, \Omega)$$

Thus we need to prove that the map

$$Hom(A, \Omega) \underset{Hom(C,\Omega)}{\times} Hom(B, \Omega) \to Hom(A, \Omega) \times Hom(B, \Omega)$$

is injective map of sets, which is clearly true by the definition of pullback of sets. $\square$

Remark 1.41. In fact, if we assume that all colimits exist, then this lemma also holds in a more general setting. For a proof in a higher Grothendieck topos (which by definition has all colimits) see [Lu09, Lemma 6.2.3.13]

Remark 1.42. Note the analogous statement for pullbacks and (-1)-truncated maps does not hold. In other words the map

$$A \underset{C}{\times} B \to A \times B$$

is generally not in any way truncated. For an example in the category of spaces let $A = B = *$, the one point space, and C be any space. Then the diagram above is the map $\Omega C \to *$ from the loop space of C, which is generally not in any way truncated (just take $C = S^2$).

Notice epi and (-1)-connected are not related the same way that (-1)-truncated maps and monos are related (Proposition 1.23).

Definition 1.43. A map $f : X \to Y$ is an *epimorphism* if for any object Z the induced map

$$Map(Y, Z) \to Map(X, Z)$$

is a (-1)-truncated map of spaces.

Example 1.44. The map of spaces $S^1 \to *$ is (-1)-connected. However, it is clearly not epi. Indeed, if the map $Map(*, Z) \to Map(S^1, Z)$ is mono for any space Z then by the pullback below

$$\begin{tikzcd} Map(S^2, Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Map(*, Z) \arrow[d] \\ Map(*, Z) \arrow[r] & Map(S^1, Z) \end{tikzcd}$$

$Map(S^2, Z)$ would be equivalent to $Map(*, Z)$.

1.3 Connectivity Theorems.

In this subsection we state some basic connectivity theorems. The theorems have already been proven in the setting of an elementary higher topos that is presentable [Re05, Subsection 8.9]. Fortunately the theorems and their proofs do not actually make use of the presentability condition, but rather only rely on the definition of truncation and connection and basic topos theoretic properties such as descent. Thus we will not repeat the proofs but rather simply state the theorems and provide the necessary references.

Proposition 1.45. [Re05, Proposition 8.10] *Let $f : X \to Y$ be a map. The following are equivalent.*

(1) For all n-truncated maps $g : W \to Z$ the diagram

$$\begin{tikzcd} Map(Y, W) \arrow[r, "g_*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\ulcorner", very near start] & Map(Y, Z) \arrow[d, "f^*"] \\ Map(X, W) \arrow[r, "g_*"'] & Map(X, Z) \end{tikzcd}$$

is a homotopy pullback diagram of spaces.

(2) f is n-connected.

Remark 1.46. The proposition is basically saying that if we take f is n-connected and g is n-truncated then the space of lifts of the square below is contractible.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & W \arrow[d, "g"] \\ Y \arrow[r] \arrow[ur, dashed] & Z \end{tikzcd}$$

Proposition 1.47. [Re05, Proposition 8.11] *A map $f : Y \to X$ is n-connected if and only if for every n-truncated map $g : Z \to X$ the map of internal mapping objects*

$$[g : Z \to X]^{[id_X : X \to X]} \to [g : Z \to X]^{[f : Y \to X]}$$

is an equivalence in $\mathcal{E}_{/X}$.

Remark 1.48. This proposition is just the internal version of the original Definition 1.27.

Definition 1.49. Let $f : X \to Y$ and $g : W \to Z$ be two maps. We define the *gap map* $u_{f,g}$ as the following map.

$$\begin{tikzcd}
W^Y \arrow[dr, dashed, "u_{f,g}"] \arrow[drr, bend left, "f^*"] \arrow[ddr, bend right, "g_*"'] & & \\
& Z^Y \underset{Z^X}{\times} W^X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & W^X \arrow[d, "g_*"] \\
& Z^Y \arrow[r, "f^*"] & Z^X
\end{tikzcd}$$

Proposition 1.50. [Re05, Proposition 8.13] *Let $f : X \to Y$ be a map and $n \geq -2$. Then the following are equivalent:*

(1) For all $m \geq -2$ and all m-truncated maps $g : W \to Z$ the induced map $u_{f,g}$ is $(m-n-2)$-truncated if $m \geq n$ and an equivalence if $m \leq n$.
(2) For all n-truncated maps $g : W \to Z$ the map $u_{f,g}$ is an equivalence.
(3) f is n-connected.

In order to further study truncated and connected maps we need to be able to truncate objects internally. In a presentable higher topos we can construct truncated objects via the existence of infinite colimits and the small object argument. However, an elementary higher topos is not presentable and thus we cannot use such techniques.

Fortunately, there is a way to get around that. For that we need to make a technical digression and study the join construction.

The Join Construction

The goal of this section is to construct (-1)-truncations. There is a concrete way to construct (-1)-truncations in a higher Grothendieck topos which is motivated by the Cech nerve [Re05, Proposition 7.8] [Lu09, Proposition 6.2.3.4]. However, the Cech nerve is a simplicial object and so in order to construct the (-1)-truncation we need colimits of simplicial diagrams, which do not exist in an elementary higher topos.

On the other hand every elementary higher topos has a natural number object [Ra18c]. Combining this with the fact that a topos has universes, we can define and compute sequential colimits

[Ra18c, Subsection 4.4]. The goal is thus to find a way to replace the simplicial diagram by a sequential diagram. This has been done successfully by Rijke in the context of homotopy type theory where he replaces a simplicial diagram of products with a sequential diagram of joins [Ri17].

The goal of the coming two sections is to adapt the approach of Rijke from homotopy type theory to elementary higher toposes. Thus, in this section we define joins and show how the analogous join construction gives us (-1)-truncations. Then, in the next section we will induct on (-1)-truncations to prove the existence of n-truncations.

2.1 Join.

In this subsection we define the join of two objects and give some basic properties.

Definition 2.1. Let A and B be two objects in $\mathcal{E}$. We define the join $A * B$ in $\mathcal{E}$ by the following diagram.

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A * B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Notation 2.2. As $\mathcal{E}_{/X}$ is also a topos this definition directly generalizes to the join of two maps $f * g$ for two maps $f : A \to X$ and $g : B \to X$.

We now want to establish some basic results about the join construction that will be necessary later on.

Theorem 2.3. *Let $p : Y \to X$ be an object in $\mathcal{E}_{/X}$ and assume we have two maps $f : A \to X$ and $g : B \to X$. Then we have an equivalence*

$$p^* f * p^* g \simeq p^*(f * g).$$

Proof. If we pullback the diagram above by the map $p : Y \to X$ we get

$$\begin{tikzcd} p^*A \underset{Y}{\times} p^*B \arrow[r] \arrow[d] & p^*B \arrow[d] \arrow[ddr, bend left, "p^*g"] & \\ p^*A \arrow[r] \arrow[drr, bend right, "p^*f"'] & p^*A \underset{Y}{*} p^*B \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "p^*(f*g)"] & \\ & & Y \end{tikzcd}$$

By the descent condition, taking pullback preserves pushout diagrams which means that $p^*(f * g)$ is the join of $p^* f$ and $p^* g$ giving us the desired result. $\square$

Lemma 2.4. *The join of two* (-1)*-truncated objects is* (-1)*-truncated.*

Proof. We know that $A \times B$ is also (-1)-truncated, and so the result follows from the fact that subobjects of 1 are closed under pushouts. □

In fact we have a certain inverse.

Lemma 2.5. *An object* A *is* (-1)*-truncated if and only if the map* $A \to A * A$ *is an equivalence.*

Proof. If A is (-1)-truncated then $A \times A \simeq A$ and so the result follows. On the other hand if $A * A \simeq A$ then we have pushout diagram

$$\begin{tikzcd} A \times A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & A \arrow[d, "id_A"] \\ A \arrow[r, "id_A"] & A \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

First this implies that $\pi_1 \simeq \pi_2$. Now for any object B we thus have following commutative diagram (which is not necessarily a pushout diagram):

$$\begin{tikzcd} Map(B, A) \times Map(B, A) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Map(B, A) \arrow[d, "id"] \\ Map(B, A) \arrow[r, "id"] & Map(B, A) \end{tikzcd}$$

If $Map(B, A)$ is empty then there is nothing to prove. However, if not then let $f \in Map(B, A)$ be any map. The fact that $\pi_1 \simeq \pi_2$ implies that the following is a commutative diagram

$$\begin{tikzcd} Map(B, A) \arrow[dr, "id \times f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "id"'] & & \\ & Map(B, A) \times Map(B, A) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Map(B, A) \arrow[d, "id"] \\ & Map(B, A) \arrow[r, "id"] & Map(B, A) \end{tikzcd}$$

where $f : Map(B, A) \to Map(B, A)$ is the map that takes everything to $f \in Map(B, A)$. The fact that this diagram commutes implies that the identity is homotopic to the constant map which means $Map(B, A)$ is contractible. □

Finally notice that colimits are universal with respect to joins (the same way they are universal with respect to pullbacks).

Lemma 2.6. *Let $f : B \to C$ be any map. Then $Coeq(f * id_A, id) \simeq Coeq(f, id) * A$.*

Proof. By descent, we have $Coeq(f \times id_A, id) \simeq Coeq(f, id) \times A$. The result then follows by taking the pushout. □

2.2 Join Sequence and Truncations. Having established some basic facts about joins, we will now use them to construct (-1)-truncations.

Definition 2.7. Let X be an object in $\mathcal{E}$. The (-1)-truncation is a choice of object Y and map $i : X \to Y$ such that following two conditions hold:

(1) Y is (-1)-truncated.
(2) For any other (-1)-truncated object Z the map $Map(Y, Z) \to Map(X, Z)$ is an equivalence.

It is valuable to see the relative definition which gives us the *image*.

Definition 2.8. Let $f : Y \to X$ be a map in $\mathcal{E}$. The image is a diagram of the following form

$$\begin{tikzcd} Y \arrow[r, "q_f"] \arrow[dr, "f"'] & Im(f) \arrow[r, dashed] \arrow[d, hook, "i_f"] & Z \arrow[dl, hook, "i"] \\ & X & \end{tikzcd}$$

where i_f is (-1)-truncated and with the following universal property. For any (-1)-truncated map $i : Z \to X$, the induced map

$$(q_f)^* : Map_{/X}(Im(f), Z) \to Map_{/X}(Y, Z)$$

is an equivalence.

The existence of (-1)-truncations in $\mathcal{E}$ will imply that every map has an image, as it will simply be the (-1)-truncation of an object $Y \to X$ in the topos $\mathcal{E}_{/X}$. Our goal is to actually construct a functorial factorization, which we will express using an adjunction (Theorem 2.27).

The goal is to show that we can build a sequence of objects which converges to the universal (-1)-truncation. In order to do that we need understand sequences and sequential colimits in an elementary higher topos. This has been done carefully in [Ra18c]. Here we will review some basic definitions that we will need in this section and we refer the reader to the main source for more details.

Definition 2.9. A *natural number object* is an object $\mathbb{N}$ in $\mathcal{E}$ along with two maps

$$1 \xrightarrow{o} \mathbb{N} \xrightarrow{s} \mathbb{N}$$

such that $(\mathbb{N}, o, s)$ is initial.

Remark 2.10. We are in particular interested in constructing maps $\mathbb{N} \to \mathcal{U}$, where $\mathcal{U}$ is a universe. The initiality condition stated above implies that we can construct maps $\mathbb{N} \to \mathcal{U}$ by giving a map $1 \to \mathcal{U}$ and a map $\mathcal{U} \to \mathcal{U}$. However, by representability, a map $1 \to \mathcal{U}$ is just a choice of object in $\mathcal{E}$ that is classified by $\mathcal{U}$ and a map $\mathcal{U} \to \mathcal{U}$ is a choice of map $\mathcal{E} \to \mathcal{E}$.

Example 2.11. Let us give an example of the previous remark. Let X be an object in $\mathcal{E}$. Using the fact that we have coproducts we get a functor $- \coprod X : \mathcal{E} \to \mathcal{E}$. By the initiality of the natural number object we thus get a map $\coprod_n X : \mathbb{N} \to \mathcal{U}$.

We can capture this notion in following definition.

Definition 2.12. A sequence of objects $\{A_n\}_{n:\mathbb{N}}$ is a map $\{A_n\}_{n:\mathbb{N}} : \mathbb{N} \to \mathcal{U}$.

Notation 2.13. We will adopt following notational convention. Let $X : 1 \to \mathcal{U}$ and $p : \mathcal{U} \to \mathcal{U}$ be two maps. Then we get a map $\mathbb{N} \to \mathcal{U}$ that we will denote as a sequence

$$X, p(X), p^2(X), p^3(X), ...$$

The next goal is to use natural number objects to construct sequential colimits. For that we need internal coproducts.

Definition 2.14. Let A_n be a sequence of objects. We define the *internal coproduct*, $\sum\limits_{n:\mathbb{N}} A_n$ as the pullback

$$\begin{tikzcd}
\sum\limits_{n:\mathbb{N}} A_n \arrow[r] \arrow[d, "p_A"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_* \arrow[d] \\
\mathbb{N} \arrow[r] & \mathcal{U}
\end{tikzcd}$$

Definition 2.15. A *sequential diagram* $\{f_n : A_n \to A_{n+1}\}_{n:\mathbb{N}}$ is a sequence of objects $\{A_n\}_{n:\mathbb{N}} : \mathbb{N} \to \mathcal{U}$ as well as a choice of map

$$\begin{tikzcd}
\sum\limits_{n:\mathbb{N}} A_n \arrow[rr, "\{f_n\}_{n:\mathbb{N}}"] \arrow[dr] & & \sum\limits_{n:\mathbb{N}} A_{n+1} \arrow[dl] \\
& \mathbb{N} &
\end{tikzcd}$$

Notation 2.16. For any $n : \mathbb{N}$ we get a map $f_n : A_n \to A_{n+1}$. Thus we will use following notation for a sequential diagram

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots .$$

Definition 2.17. Let $\{f_n\}_{n:\mathbb{N}}$ be a sequential diagram of the sequence of objects A. Then the *sequential colimit* of f is the coequalizer

$$\sum_{n:\mathbb{N}} A_n \underset{id_{\sum_{n:\mathbb{N}} A_n}}{\overset{f}{\rightrightarrows}} \sum_{n:\mathbb{N}} A_n \longrightarrow A_\infty$$

We want to construct a sequential diagram using joins, analogous to Remark [Ra18c, Remark 4.26].

Remark 2.18. Let $\mathcal{U}_1$ be the morphism classifier [Ra18b, Definition 3.1]. Then the data of the map $\emptyset \to A : 1_{\mathcal{E}} \to \mathcal{U}_1$ and $- * id_A : \mathcal{U}_1 \to \mathcal{U}_1$ gives us the commutative diagram

$$\begin{tikzcd} \sum_{n:\mathbb{N}} A^{*n} \arrow[rr, "inl"] \arrow[dr] & & \sum_{n:\mathbb{N}} A^{*n+1} \arrow[dl] \\ & \mathbb{N} & \end{tikzcd}$$

Notation 2.19. Following our previous notation convention, we will depict this sequential diagram as

$$A \to A * A \to (A * A) * A \to ...$$

The goal is to prove that the sequential colimit of this sequence is the actual (-1)-truncation of A, which is the content of Theorem 2.25. However, before that we first need to prove several lemmas.

Lemma 2.20. *Let A and A' be two objects and $i : A \to A'$ be a map such that for every (-1)-truncated object B the induced map*

$$i^* : Map(A', B) \to Map(A, B)$$

is an equivalence. Then the induced map

$$\iota_0^* : Map(A * A', B) \to Map(A, B)$$

is an equivalence as well.

Proof. As B is (-1)-truncated, the space $Map(A, B)$ is either empty or contractible. If it is empty the result is obvious. If it is contractible then we have to prove that $Map(A * A', B)$ is non-empty as well. As we know it is (-1)-truncated this implies that $Map(A * A', B)$ contractible which will give us the desired result.

Let $j : A \to B$ be a map. Then by assumption that i^* is an equivalence we thus have a map $h : A' \to B$ such that $hi \simeq j$. This in particular implies that $j\pi_1 \simeq k\pi_2$ which induces a map $j * h : A * A' \to B$. □

Remark 2.21. The statement of the previous lemma and the proof is an adaptation of [Ri17, Lemma 3.1].

Lemma 2.22. *Let (A_n, f_n) be a sequential diagram. Moreover, let $i_n : A \to A_n$ be a map between the sequences, where we take A to be the constant sequence.*

$$\begin{tikzcd} & & & A \arrow[dlll, bend right, "i_0"'] \arrow[dll, bend right, "i_1"'] \arrow[dl, "i_2"] \arrow[d, bend left] \\ A_0 \arrow[r, "f_0"] & A_1 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & \ldots \end{tikzcd}$$

Moreover, for every (-1)-truncated object B we have an equivalence

$$i_n^* : Map(A_n, B) \to Map(A, B)$$

Then we have an equivalence $i_\infty^* : Map(A_\infty, B) \to Map(A, B)$.

Proof. The space $Map(A, B)$ is (-1)-truncated. If it is empty the result follows trivially. Let us assume $Map(A, B)$ is contractible. Then by the assumption $Map(A_n, B)$ is also contractible, which means there exist maps $h_n : A_n \to B$. This gives us a diagram where we take B to be the constant sequence.

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[drrr, bend right, "h_0"'] & A_1 \arrow[r, "f_1"] \arrow[drr, bend right, "h_1"] & A_2 \arrow[r, "f_2"] \arrow[dr, "h_2"] & \ldots \arrow[d, bend left] \\ & & & B \end{tikzcd}$$

This map induces a map $h_\infty : A_\infty \to B$, which implies $Map(A_\infty, B)$ is non-empty giving us the desired result. ☐

Remark 2.23. This statement and proof is an adaptation of [Ri17, Lemma 3.2] from homotopy type theory.

Finally, we also need to understand how the join operation affects a sequential colimit.

Lemma 2.24. *Let (A_n, f_n) be a sequence. Let B be another object, which gives us a sequence $(A_n * B, f_n * id_B)$. Then we have an equivalence $A_\infty * B \simeq (A_n * B)_\infty$.*

Proof. This follows from the fact that the join operation commutes with coequalizers, by Lemma 2.6. ☐

Theorem 2.25. *Let A be any object and let $\{inl : A^{*n} \to A^{*n+1}\}_{n:\mathbb{N}}$ be the sequential diagram described in Remark 2.18. Then the sequential colimit $A^{*\infty}$ is the (-1)-truncation of A.*

Proof. The two lemmas above already imply that the colimit satisfies the universal property. Thus all that is left is to prove that $A^{*\infty}$ is (-1)-truncated. According to Lemma 2.5 all we need is that the map $A^{*\infty} \to A^{*\infty} * A^{*\infty}$ is an equivalence. For that we need several steps.

First, take the sequence

$$A * A \to (A * A) * A \to ....$$

We can think of this sequence in two different ways. On the one side it is $id_A * inl$. Thus by Lemma 2.24 the sequential colimit of this sequence is $A * A^{*\infty}$. On the other side it is the sequence $inl(s)$. Thus by [Ra18c, Theorem 4.30] it has sequential colimit $A^{*\infty}$. This implies that the map $A \to A * A^{*\infty}$ is an equivalence. Using a similar argument we deduce that $A \to A^{*n} * A^{*\infty}$ is an equivalence.

Now we have following maps of sequences

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\simeq"] & A * A \arrow[r] \arrow[d, "\simeq"] & A * A * A \arrow[r] \arrow[d, "\simeq"] & ... \arrow[r] & A^{*\infty} \arrow[d, "\simeq"] \\
A * A^{*\infty} \arrow[r] & A * A * A^{*\infty} \arrow[r] & A * A * A * A^{*\infty} \arrow[r] & ... \arrow[r] & A^{*\infty} * A^{*\infty}
\end{tikzcd}$$

The first row has sequential colimit $A^{*\infty}$, whereas by Lemma 2.6 the bottom row has sequential colimit $A^{*\infty} * A^{*\infty}$. However, as the sequences are equivalent it follows that $A^{*\infty} \to A^{*\infty} * A^{*\infty}$ is an equivalence, which proves that $A^{*\infty}$ is (-1)-truncated. □

Notice the construction is a colimit construction and thus functorial, which means we have following theorem.

Notation 2.26. Let $\tau_{-1}\mathcal{E}$ be the subcategory of (-1)-truncated objects in $\mathcal{E}$.

Theorem 2.27. *There is an adjunction*

$$\mathcal{E} \underset{i}{\overset{\tau_{-1}}{\rightleftarrows}} \tau_{-1}\mathcal{E}$$

Where $i : \tau_{-1}\mathcal{E} \to \mathcal{E}$ is the inclusion map. We call τ_{-1} the truncation functor.

Proof. The functoriality of the colimit construction implies that sending A to $A^{*\infty}$ is functorial. We have already proven that $A^{*\infty}$ is (-1)-truncated and that there is an adjunction. □

Notation 2.28. Henceforth we will denote the (-1)-truncation of A by $\tau_{-1}(A)$ and the sequential colimit map $inl^{\circ\infty} : A \to A^{*\infty} = \tau_{-1}(A)$ by $\eta_A : A \to \tau_{-1}(A)$ and we notice that η_A is the unit map of the adjunction.

There is also a relative version, which we will state for the sake of notation.

Notation 2.29. For a map $Y \to X$ we denote the (-1)-truncation by $\tau_{-1}^X(Y) \to X$.

Remark 2.30. We can think of the truncation $\tau_{-1}A$ as the biggest subobject of 1 that has a map to A. Thus, in particular, if there exists a map $1 \to A$ then $\tau_{-1}A = 1$. For example, let Ω be the subobject classifier and $\mathcal{U}$ be a universe. Then we have $\tau_{-1}\Omega = \tau_{-1}\mathcal{U} = 1$.

Truncation Functors

In the previous section we constructed a (-1)-truncation functor. We want to generalize it to an n-truncation functor for all n. We want to proceed by induction starting from the case $n = -2$.

The intuition comes from the situation in spaces. Assuming we have an n-truncation functor for spaces, we can construct an $(n+1)$-truncation of a space X simply by building a space that has the same points as X but where we n-truncate the loop spaces $\tau_n(\Omega_x X)$. We want to repeat this approach in an arbitrary elementary topos.

The key realization is that we have a Yoneda lemma for elementary higher toposes.

Theorem 3.1. [Ra18d, Theorem 3.3] *Let A be a fixed object and $\mathcal{U}$ be a universe that classifies the diagonal map $\Delta : A \to A \times A$. Then the induced map $\mathcal{Y}_A : A \to \mathcal{U}^A$ is (-1)-truncated.*

We can take A and embed it in $\mathcal{U}^A$. Then we can take the n-truncation of the image of the Yoneda functor. The resulting object will be our $(n+1)$-truncation, $\tau_{n+1}(A)$. The goal of this section is to make this intuition precise.

Remark 3.2. The intuition outlined here has already been developed in homotopy type theory [Ri17] and serves as motivation for the steps.

3.1 Properties of Trunction Functors.

In this subsection we fix an integer $n \geq -2$ and assume there exists a truncation functor $\tau_n : \mathcal{E} \to \tau_n\mathcal{E}$ and analyze it.

Notation 3.3. We denote the subcategory of n-truncated objects in $\mathcal{E}$ as $\tau_n\mathcal{E}$. Notice $\tau_n\mathcal{E}$ is a weak $(n+1,1)$-category. In particular, $(\tau_n\mathcal{E})^{core}$ is an $(n+1)$-truncated space as every loop space is n-truncated.

Remark 3.4. The adjunction gives us an equivalence

$$Map(\tau_n X, \tau_n X) \xrightarrow{\simeq} Map(X, \tau_n X)$$

This gives us a canonical map $\eta_X : X \to \tau_n X$ (the image of the identity), which is the unit of the adjunction.

It is worth pointing out the one case that is trivial.

Remark 3.5. The only (-2)-truncated object in $\mathcal{E}$ is the final object and so the (-2)-truncation of every object is the final object. Thus τ_{-2} trivially exists.

Remark 3.6. Whenever we have a higher category then we can define an $(n+1,1)$-category by externally truncating the mapping spaces. However, this external truncation does not generally coincide with an internal truncation. For an example notice that the following spaces are not equivalent

$$\tau_0(Map(S^1, S^1)) \not\simeq Map(S^1, \tau_0(S^1)).$$

Here we focus exclusively on the internal truncations as discussed before and not the external truncation.

Let us study some basic facts about truncations.

Lemma 3.7. *Let $f : X \to Y$ be a map in $\mathcal{E}$. Then $\tau_n f$ is an equivalence if and only if for all n-truncated Z the map*

$$Map(Y, Z) \to Map(X, Z)$$

is an equivalence of spaces.

Proof. By the Yoneda lemma, the map $\tau_n f : \tau_n X \to \tau_n Y$ in $\tau_n \mathcal{E}$ is an equivalence if and only if the map $(\tau_n f)^* : Map(\tau_n Y, Z) \to Map(\tau_n X, Z)$ is an equivalence of spaces for all n-truncated Z.

Now we have following commutative diagram

$$\begin{tikzcd} Map(\tau_n Y, Z) \arrow[r] \arrow[d, "\simeq"] & Map(\tau_n X, Z) \arrow[d, "\simeq"] \\ Map(Y, Z) \arrow[r] & Map(X, Z) \end{tikzcd}$$

By adjunction the vertical maps are equivalences. Thus the top map is an equivalence if and only if the bottom map is an equivalence, which gives us the desired result. □

Lemma 3.8. [Re05, Proposition 8.5] *Let X be an object in $\mathcal{E}$. Then the unit of the adjunction $\eta_X : X \to \tau_n(X)$ is n-connected.*

Proof. We have to prove that for every n-truncated map $g : Z \to \tau_n(X)$ the induced map

$$Map_{/\tau_n(X)}(\tau_n X, Z) \to Map_{/\tau_n(X)}(X, Z)$$

is an equivalence. We have following commutative square

$$\begin{tikzcd} Map(\tau_n X, Z) \arrow[r, "\simeq"] \arrow[d, "g_*"'] & Map(X, Z) \arrow[d, "g_*"] \\ Map(\tau_n X, \tau_n X) \arrow[r, "\simeq"] & Map(X, \tau_n X) \end{tikzcd}$$

As $\tau_n X$ is n-truncated and $Z \to \tau_n X$ is also n-truncated, Z is also n-truncated. Thus by the previous lemma the horizontal maps are equivalences of spaces. Take the point $id : * \to Map(\tau_n X, \tau_n X)$ and $\eta_X : * \to Map(X, \tau_n X)$. Then we can pullback the square above to the following square:

$$\begin{tikzcd} Map_{/\tau_n(X)}(\tau_n X, Z) \arrow[r, "\simeq"] \arrow[d] & Map_{/\tau_n(X)}(X, Z) \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

As the pullback of an equivalence is still an equivalence, the top map is still an equivalence and this gives us the desired result. □

This lemma has following interesting corollary.

Corollary 3.9. *Let X be an object with an n-connected map $g : X \to Y$ such that Y is n-truncated. Then $Y \simeq \tau_n(X)$.*

This corollary also gives us a stability of truncations

Proposition 3.10. *Let $f : Y \to X$ and $g : Z \to X$ be two maps. Moreover, let $Y \to \tau^X(f) \to X$ be the factorization of f. Then the pullback of the factorization along g, as depicted in the diagram below, is the factorization of g^*f.*

$$\begin{tikzcd}
g^*Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] \arrow[rr, bend right, "g^*f"] & g^*(\tau_n^X(Y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "g"] \\
Y \arrow[r] \arrow[rr, bend right, "f"'] & \tau_n^X(Y) \arrow[r] & X
\end{tikzcd}$$

Proof. This follows from the previous corollary combined with the fact that connected and truncated maps are stable under base change. □

It also implies following proposition.

Proposition 3.11. *An object X in $\mathcal{E}$ is n-connected if and only if $\tau_n(X)$ is equivalent to the final object.*

Proof. Let us assume $\tau_n(X)$ is equivalent to the final object. By the proposition above $X \to \tau_n(X) \simeq 1$ is n-connected, which implies that X is n-connected.

On the other hand, assume X is n-connected. Then the map $X \to 1$ is n-connected and the map $1 \to 1$ is n-truncated. Thus by the corollary $1 \simeq \tau_n(X)$. □

We can also use it to show that truncations commute with pullbacks.

Proposition 3.12. *Let*

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "f"] \\
Y \arrow[r, "p"'] & X
\end{tikzcd}$$

be a pullback square. Then

$$\begin{tikzcd}
\tau_n^Y W \arrow[r] \arrow[d, "\tau_n^Y f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & \tau_n^X Z \arrow[d, "\tau_n^X f"] \\
Y \arrow[r, "p"'] & X
\end{tikzcd}$$

is also a pullback square.

Proof. We know that $X \to \tau_n^X Z \to Z$ is the factorization of f. This means that $p^*X \to p^*\tau_n^X Z \to Y$ is a factorization of p^*f, according to Proposition 3.10. However, by assumption $p^*f \simeq f'$ and $p^*X \simeq W$. This implies that $W \to p^*\tau_n^X Z \to Y$ is the factorization of $f' : W \to Y$ finishing the proof. □

This result has following immediate corollary.

Corollary 3.13. *Let*

$$\begin{tikzcd}
W \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "f"] \\
Y \arrow[r, "p"'] & X
\end{tikzcd}$$

be a pullback square.Then if f is n-connected, f' is also n-connected.

Proof. If f is n-connected then $\tau_n^X f$ is an equivalence (by Proposition 3.11). By the previous proposition, the pullback, namely $\tau_n^Y f'$, is an equivalence as well. This, again by Proposition 3.11 implies that f' is n-connected. □

Remark 3.14. Proposition 1.31 combined with Corollary 3.13 implies that n-connected maps are stable under base change and cobase change. This will be quite important later on, when we study modalities in a topos (Proposition 4.20).

Having a truncation functor we could reprove many of the results we showed in the first section in a much simpler manner, which is exactly what has been done in [Re05]. On the other hand we can also prove some additional results about truncated and connected objects that we could not prove before.

Proposition 3.15. *In the pullback diagram below where p is a (-1)-connected map*

$$\begin{tikzcd}
Y' \arrow[r, two heads] \arrow[d, "f'"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "f"] \\
X' \arrow[r, two heads, "p"] & X
\end{tikzcd}$$

we have the following. f is a weak equivalence/n-truncated map/(-1)-connected map if and only if f' is a weak equivalence/n-truncated map/(-1)-connected map.

Proof. Clearly if f satisfies any of those conditions then f' does as well as they are stable under pullback. Thus we will assume f' satisfies the conditions and show that f satisfies them as well.

First we assume that f' is an equivalence and we will prove that f is an equivalence. Here we make explicit use of the construction of the (-1)-truncation. The commutative diagram above then results in a diagram

$$\begin{tikzcd} X' \arrow[r, "inl"] \arrow[d, "f'", "\simeq"'] & X' \underset{X}{*} X' \arrow[r, "inl"] \arrow[d, "f'*f'", "\simeq"'] & (X' \underset{X}{*} X') \underset{X}{*} X' \arrow[r] \arrow[d, "(f'*f')*f'", "\simeq"'] & ... \arrow[r] & X \arrow[d, "f"] \\ S' \arrow[r, "inl"] & S' \underset{S}{*} S' \arrow[r, "inl"] & (S' \underset{S}{*} S') \underset{S}{*} S' \arrow[r] & ... \arrow[r] & S \end{tikzcd}$$

The horizontal maps are sequential diagrams. By Theorem 2.25 the sequential colimits of those diagrams construct the (-1)-truncations of g and g'. However, the maps g and g' are (-1)-connected and so by Corollary 3.9 they are the actual truncations which means the horizontal sequential diagram are sequential colimit diagrams. Moreover, the vertical maps f', $f' * f'$, $(f' * f') * f'$... are all equivalences and so by the homotopy invariance of sequential colimits, the colimit of the maps, namely f, is also an equivalence.

We now want to prove that if f' is n-truncated then f is n-truncated. Notice the pullback diagram above gives us following pullback diagram.

$$\begin{tikzcd} Y' \arrow[r, two heads] \arrow[d, "\Delta'"] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "\Delta"] \\ Y' \underset{X'}{\times} Y' \arrow[r, two heads, "(p,p)"] & Y \underset{X}{\times} Y \end{tikzcd}$$

Here the map (p, p) is also a (-1)-connected map. We will use this and induction to prove the result.

(-1) If f' is (-1)-truncated then Δ' is an equivalence which implies that Δ is an equivalence (by the first part) which means f is (-1)-truncated.
(n-1) Let us assume if f' is $(n-1)$-truncated then f is $(n-1)$-truncated.
(n) Assume f' is n-truncated. Then Δ' is $(n-1)$-truncated, which means that Δ is $(n-1)$-truncated (by the induction assumption. Hence, f is also n-truncated.

Finally we have to prove that if f' is (-1)-connected then f is (-1)-connected. By Proposition 1.33 the composition pf' is (-1)-connected. However we have $pf' = f(f^*p)$ and so by Proposition 1.33 f is (-1)-connected. $\square$

Truncations can be used to study universes. For that we will need following lemma.

Lemma 3.16. *Let A be (-1)-connected and $A \to A \times A$ be n-connected. Then A is $(n+1)$-connected.*

Proof. Fix an $(n+1)$-truncated object Z. We have to prove that

$$Map(1, Z) \to Map(A, Z)$$

is an equivalence. As A is (-1)-connected we have a map $a : 1 \to A$. As the map $1 \to 1$ is the identity it thus suffices to prove that the map

$$Map(A, Z) \to Map(1, Z)$$

is an equivalence. Clearly it is a surjection as the map has a section. It thus suffices to prove the map is an embedding. Fix a map $g : A \to Z$. Then we get a map of loop spaces

$$\Omega_g Map(A, Z) \to \Omega_{g(a)} Map(1, Z)$$

We have to show this map is an equivalence, but for that it suffices to prove that g factors through the constant map $a : 1 \to A$.

We have following diagram.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\Delta"'] & Z \arrow[d, "\Delta"] \\ A \times A \arrow[r, "g \times g"] \arrow[ur, dashed] & Z \times Z \end{tikzcd}$$

Because $A \to A \times A$ is n-connected and $Z \to Z \times Z$ is n-truncated (which follows from Z being $(n+1)$-truncated and Proposition 1.19) there exists a lift $h : A \times A \to Z$, which proves that g is a constant map. $\square$

Remark 3.17. Here is a more local explanation of the last step in the previous proof. Let $a, b : 1 \to A$ be two maps. Then we can pullback the square above to the square

$$\begin{tikzcd} Path_A(a, b) \arrow[r, "g"] \arrow[d] & Path_Z(g(a), g(b)) \arrow[d] \\ 1 \arrow[r, "1"] \arrow[ur, dashed] & 1 \end{tikzcd}$$

where $Path_A(a, b)$ and $Path_Z(g(a), g(b))$ are the pullbacks of the diagonal maps along the maps (a, b) and $(g(a), g(b))$. The fact that the diagonal map $A \to A \times A$ is n-connected implies that

$Path_A(a,b)$ is n-connected. Moreover, the fact that the diagonal map $Z \to Z \times Z$ is n-truncated implies that $Path_Z(g(a), g(b))$ is n-truncated. Thus the map g factors through 1.

We can now study truncations of universes. Fix a complete Segal universe $\mathcal{U}$. The composition of the truncation functor with the inclusion functor $i \circ \tau_n : \mathcal{E} \to \mathcal{E}$ induces an internal map $i \circ \tau_n : \mathcal{U} \to \mathcal{U}$. Using the fact that we can factor the map to get a factorization $\mathcal{U} \to \mathcal{U}^{\leq n} \to \mathcal{U}$, where the first map is (-1)-connected and the second is (-1)-truncated. This object has following important universal property.

Lemma 3.18. *For each object Z we have an equivalence*

$$\tau_n(\mathcal{E}_{/Z}) \simeq Map(Z, \mathcal{U}^{\leq n})$$

Proof. We can factor every map of categories uniquely into an essential surjection followed by an embedding. We can apply this factorization to the map $i\tau_n : \mathcal{E} \to \mathcal{E}$. However, we know that i is an embedding and τ_n is essentially surjective and so by uniqueness (i, τ_n) is the unique factorization. Thus we get a diagram

$$\begin{tikzcd}
\mathcal{E}_{/Z} \arrow[r, two heads, "\tau_n"] \arrow[d, "\simeq"] & \tau_n(\mathcal{E}_{/Z}) \arrow[r, hook, "i"] \arrow[d] & \mathcal{E}_{/Z} \arrow[d, "\simeq"] \\
Map(Z, \mathcal{U}) \arrow[r, two heads] & Map(Z, \mathcal{U}^{\leq n}) \arrow[r, hook] & Map(Z, \mathcal{U})
\end{tikzcd}$$

By the uniqueness of epi mono factorizations we thus get the desired equivalence. $\square$

Remark 3.19. Notice that any universe $\mathcal{U}$ is itself an object in $\mathcal{E}$. This means that it also has a truncation $\tau_n(\mathcal{U})$. However, this differs from the object $\mathcal{U}^{\leq n}$.

Example 3.20. Let $\mathcal{E}$ be the category of (not necessarily small) spaces. In that case the core of the subcategory of small spaces $\mathcal{U}$ is a universe in $\mathcal{E}$. Then $\mathcal{U}^{\leq n}$ is the core of the subcategory of n-truncated small spaces. However, on the other hand $\tau_n(\mathcal{U})$ is just the truncation. So, for example $\tau_{-1}(\mathcal{U}) = *$ and $\tau_0(\mathcal{U})$ is the set of homotopy equivalence classes of small spaces. On the other hand $\mathcal{U}^{\leq -1} = \{\emptyset, *\}$ and $\mathcal{U}^{\leq 0}$ is the groupoid of small sets.

The truncation of the universe gives us an alternative way to construct truncations.

Proposition 3.21. *Let $p : Y \to X$ be a map classified by $\ulcorner p \urcorner : X \to \mathcal{U}$. Then the composition $\tau_n \circ \ulcorner p \urcorner : X \to \mathcal{U}^{\leq n}$ classifies the truncation $\tau^X(Y) \to X$*

Proof. This follows immediately from the commutative diagram

$$\begin{tikzcd}
\mathcal{E}_{/X} \arrow[r, "\tau_n"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\ulcorner", very near start] & \tau_n(\mathcal{E}_{/X}) \arrow[d, "\simeq"] \\
Map(X, \mathcal{U}) \arrow[r, "\tau_n"'] & Map(X, \mathcal{U}^{\leq n})
\end{tikzcd}$$

□

One direct implication of the lemma is the following

Corollary 3.22. *The object $\mathcal{U}^{\leq n}$ is $(n+1)$-truncated.*

It is interesting how this manifests in the context of (-1)-truncations.

Lemma 3.23. *Let $\mathcal{U}$ be a universe classifying the class of maps S. Then $\mathcal{U}^{\leq -1}$ classifies all (-1)-truncated maps that are in S.*

This basic results gives us an alternative way to characterize a subobject classifier in an elementary higher topos.

Proposition 3.24. *Let $\mathcal{E}$ be a category with finite limits and colimits and with complete Segal universes. The following are equivalent:*

1. *$\mathcal{E}$ is an elementary higher topos.*
2. *There exists a universe $\mathcal{U}$ that classifies a class of maps S that includes all (-1)-truncated maps.*

Proof. $\mathcal{E}$ is a topos if and only $\mathcal{E}$ has a subobject classifier. If $\mathcal{E}$ has a subobject classifier Ω then any universe that classifies $1 \to \Omega$ also classifies all (-1)-truncated maps. On the other hand, if $\mathcal{U}$ classifies all (-1)-truncated maps, then $\mathcal{U}^{\leq -1}$ is a subobject classifier. □

3.2 Inductive Construction of Truncations. In this subsection we construct an n-truncation functor for every n. Concretely we want to prove following theorem.

Theorem 3.25. *There exists an n-truncation functor*

$$\mathcal{E} \underset{i}{\overset{\tau_n}{\rightleftarrows}} \tau_n\mathcal{E}$$

Notice we have already shown that truncations exist for $n = -2$ (Remark 3.5) and $n = -1$ (Theorem 2.27). We will takes these as our base cases and then prove the general result using induction. Thus, assume an n-truncation functor τ_n exists. We will use it to construct τ_{n+1}. For the construction we will make use of the results in the previous section.

Fix an object A. We construct the $(n+1)$-truncation τ_{n+1} as follows. The map $\Delta : A \to A \times A$ is classified by a map $\ulcorner \Delta \urcorner : A \times A \to \mathcal{U}$. By adjunction this gives us a map $\mathcal{Y} : A \to \mathcal{U}^A$. Post-composing it with $(\tau_n)^A$ we get the map $\mathcal{Y}_n : A \to (\mathcal{U}^{\leq n})^A$. Finally we can decompose this map using the fact that we have a factorization into a (-1)-truncated map and a (-1)-connected map. We will prove that the image of this factorization is the desired $(n+1)$-truncation. We can depict the construction in a diagram as follows:

$$\begin{tikzcd} A \arrow[r, "y"] \arrow[d, two heads] & \mathcal{U}^A \arrow[r, "\tau_n"] & (\mathcal{U}^{\leq n})^A \\ \tau_{n+1}(A) \arrow[urr, hook] & & \end{tikzcd}$$

We will prove that $\tau_{n+1}(A)$ is the $(n+1)$-truncation. For that we have to show two things

(1) $\tau_{n+1}(A)$ is $(n+1)$-truncated.
(2) $A \to \tau_{n+1}(A)$ is $(n+1)$-connected.

(1) $\mathcal{U}^{\leq n}$ is $(n+1)$-truncated. This means that $(\mathcal{U}^{\leq n})^A$ is also $(n+1)$-truncated. Thus the subobject $\tau_{n+1}(A)$ is also $(n+1)$-truncated.

(2) In order to show that $A \to \tau_{n+1}(A)$ is $(n+1)$-connected we will prove that $A \to A \times_{\tau_{n+1}(A)} A$ is n-connected as we already know that $A \to \tau_{n+1}(A)$ is (-1)-connected. The result then follows from Proposition 3.11.

By Lemma 3.16 it suffices to show that $\varphi : \tau_n^{[A \times_{\tau_{n+1}(A)} A]}(A) \to \tau_{n+1}(A)$ is an equivalence. The map φ is n-truncated and so we get a diagram

$$\begin{tikzcd} \tau_n^{[A \times_{\tau_{n+1}(A)} A]}(A) \arrow[rr] \arrow[dd] \arrow[dr, bend right] & & \mathcal{U}_*^{\leq n} \arrow[dd] \\ & \tau_{n+1}(A) \arrow[ur, bend right] \arrow[dd] & \\ A \times_{\tau_{n+1}(A)} A \arrow[rr, "\ulcorner \varphi \urcorner"] \arrow[dr, bend right] & & \mathcal{U}^{\leq n} \\ & \tau_{n+1}(A) \arrow[ur, bend right, "!"] & \end{tikzcd}$$

By the definition of the pullback the map $\ulcorner \varphi \urcorner$ factors through $\tau_{n+1}(A)$. But the map $! : \tau_{n+1}(A) \to \mathcal{U}^{\leq n}$ is exactly the map classifying the subobject $\tau_{n+1}(A)$ of $(\mathcal{U}^{\leq n})^A$, which means that pulling back along ! gives us the identity map. Thus $A \to \tau_{n+1}(A)$ is $(n+1)$-connected.

The assignment $A \to \tau_{n+1}(A)$ thus gives us the $(n+1)$-truncation of an object. As the argument was inductive this implies that we have a truncation for every n and gives us the desired result.

Blakers Massey Theorem for Modalities

In this section we prove that every modality in an elementary higher topos satisfies the Blakers-Massey Theorem. Then we use that for the modality of n-truncated and n-connected maps. The result has already been proven for Grothendieck higher toposes in [ABFJ17]. The goal of this section is to show that the same proof holds for elementary higher toposes as well.

4.1 Modalities. In this subsection we review the important definitions and establish some basic propositions regarding modalities that we need in the next subsection. The notation here will closely follow [ABFJ17].

Definition 4.1. A *weak factorization system* on a category is a choice of functors $\mathcal{L}$ and $\mathcal{R}$ such that for all morphisms we have $f \simeq \mathcal{L}(f) \circ \mathcal{R}(f)$.

Notation 4.2. We will denote the factorization of $f : X \to Y$ as $X \to ||f|| \to Y$.

Definition 4.3. Two maps $f : A \to B$ and $g : X \to Y$ are orthogonal if the square

$$\begin{tikzcd} Map_{\mathcal{E}}(B, X) \arrow[r, "g_*"] \arrow[d, "f^*"'] \arrow[dr, phantom, very near start, "\ulcorner"] & Map_{\mathcal{E}}(B, Y) \arrow[d, "f^*"] \\ Map_{\mathcal{E}}(A, X) \arrow[r, "g_*"'] & Map_{\mathcal{E}}(A, Y) \end{tikzcd}$$

is a homotopy pullback square of spaces.

Definition 4.4. A *factorization system* is a weak factorization system such that for any map f $\mathcal{L}(f)$ is orthogonal to $\mathcal{R}(f)$.

Notation 4.5. We denote the subcategory of the arrow category $Arr(\mathcal{E})$ consisting of maps in the image of $\mathcal{R}$ by $Arr(\mathcal{R})$. Similarly, $Arr(\mathcal{L})$ is the subcategory generated by the image of $\mathcal{L}$.

We have following facts about factorization systems:

Lemma 4.6. [ABFJ17, Lemma 3.1.5] *The map $\mathcal{L} : Arr(\mathcal{E}) \to Arr(\mathcal{L})$ ($\mathcal{R} : Arr(\mathcal{E}) \to Arr(\mathcal{R})$) is the right (left) adjoint to the inclusion.*

Lemma 4.7. [ABFJ17, Lemma 3.1.6]

(1) If $fg \in \mathcal{L}$ and $f \in \mathcal{L}$ then $g \in \mathcal{L}$. Also, if $fg \in \mathcal{R}$ and $g \in \mathcal{R}$ then $f \in \mathcal{R}$.
(2) $\mathcal{L}(f)$ is an equivalence if and only if $f \in \mathcal{R}$. Similarly, $\mathcal{R}(f)$ is an equivalence if and only if $f \in \mathcal{L}$.

Definition 4.8. A *modality* is a a factorization system such that $\mathcal{L}$ is closed under pullback.

Definition 4.9. Let us assume we have following commutative square.

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "k"] \\ X \arrow[r, "h"'] & W \end{tikzcd}$$

Then we define the *cogap map* as the map $\lfloor h, k \rfloor : X \coprod_Z Y \to W$ and the *gap map* $(f, g) : Z \to X \times_W Y$.

Definition 4.10. Let $f : A \to B$ and $f' : A' \to B'$ be two maps in $\mathcal{E}$. We define $f \Box f'$, called the *pushout product*, in the following diagram

$$\begin{tikzcd}
A \times A' \arrow[r, "id_A \times f'"] \arrow[d, "f \times id_{A'}"] & B \times A' \arrow[d] \arrow[ddr, bend left, "id_B \times f'"] & \\
A \times B' \arrow[r] \arrow[drr, bend right, "f \times id_{B'}"] & B \times A' \coprod_{A \times A'} A \times B' \arrow[dr, "f \Box f'"] & \\
 & & B \times B'
\end{tikzcd}$$

Notation 4.11. We will denote the relative pushout product construction in $\mathcal{E}_{/Z}$ as $\Box_Z$.

Definition 4.12. A class of maps $\mathcal{M}$ is *local* if in the pullback square where g is (-1)-connected

$$\begin{tikzcd}
X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\
Y' \arrow[r, two heads, "g"] & Y
\end{tikzcd}$$

we have $f' \in \mathcal{M}$ if and only if $f \in \mathcal{M}$.

Remark 4.13. The definition above differs from the one given [ABFJ17, Definition 3.7.1]. However, the two definitions are equivalent in a higher topos [ABFJ17, Remark 3.7.2] [Lu09, Proposition 6.2.3.14]. More importantly, we will only require the condition stated in Definition 4.12 for the later proofs.

Example 4.14. Notice Proposition 3.15 implies that equivalences and truncated maps are local.

Definition 4.15. Let $\mathcal{M}$ be a local class of maps and assume we have following commutative square.

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\
X \arrow[r, "h"] & W
\end{tikzcd}$$

Then we say the square is $\mathcal{M}$-Cartesian if the gap map (f, g) is in $\mathcal{M}$.

Proposition 4.16. *$\mathcal{L}$ and $\mathcal{R}$ are local.*

Proof. It suffices to prove that $\mathcal{L}$ is local. In the pullback square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

let g be (-1)-connected and $f' \in \mathcal{L}$. We have to show that $f \in \mathcal{L}$. Using the factorization of f and f' we get the diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\mathcal{L}(f')"] \arrow[dr, phantom, very near start, "\ulcorner"] & X \arrow[d, "\mathcal{L}(f)"] \\ {||f'||} \arrow[r] \arrow[d, "\mathcal{R}(f')", "\simeq"'] \arrow[dr, phantom, very near start, "\ulcorner"] & {||f||} \arrow[d, "\mathcal{R}(f)", "\simeq"'] \\ Y' \arrow[r, two heads, "g"] & Y \end{tikzcd}$$

As $f' \in \mathcal{L}$ we know that $\mathcal{R}(f')$ is an equivalence. As equivalences are local (Proposition 3.15) this means that $\mathcal{R}(f)$ is also an equivalence. However this means that $f \in \mathcal{L}$ which finishes the proof. □

Remark 4.17. This proposition is proven in [ABFJ17, Proposition 3.7.5], but the proof given there does not hold in an elementary higher topos and needed to be adjusted.

4.2 Blakers-Massey Theorem for Modalities. In this subsection we prove the Blakers-Massey theorem for any modality in an elementary higher topos. Let us first state the relevant theorems.

Theorem 4.18. [ABFJ17, Theorem 4.1.1] *(Blakers-Massey Theorem) Let $(\mathcal{L}, \mathcal{R})$ be a modality in an elementary higher topos $\mathcal{E}$. Consider the pushout square:*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] \arrow[ur, phantom, very near end, "\ulcorner"] & W \end{tikzcd}$$

Suppose that $\Delta f \square_Z \Delta g \in \mathcal{L}$. Then the square is $\mathcal{L}$-Cartesian.

Theorem 4.19. [ABFJ17, Theorem 3.6.1] *(Dual Blakers-Massey Theorem) Let $(\mathcal{L}, \mathcal{R})$ be a modality in an elementary higher topos $\mathcal{E}$. Suppose we are given a pullback square*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[d, "g"] \\ Y \arrow[r, "k"'] & W \end{tikzcd}$$

and suppose that the map $k \square g \in \mathcal{L}$. Then the cogap map $\lfloor k, g \rfloor : Y \coprod_X Z \to W$ is in $\mathcal{L}$.

As there is already a detailed proof in [ABFJ17] we will simply show that an elementary higher topos satisfies all the conditions mentioned in the steps of that proof.

Before we do so let us review the structure of the paper [ABFJ17]. The first two sections focus on reviewing concepts about higher topos theory. Most of the third section focuses on certain conditions under which modalities exists, which are not relevant to our discussion. The only results we need from this section are in [ABFJ17, Subsection 3.7, Subsection 3.8], where the authors prove a descent condition for modalities. The fourth section then gives us the proofs of the Blakers-Massey theorem. The proofs in [ABFJ17, Section 4] generalize in a straightforward way, but some proofs in [ABFJ17, Subsection 3.7, Subsection 3.8] need adjustments.

Thus we will not discuss the proofs in [ABFJ17, Section 4] any further. However, we will analyze the steps of the proofs in [ABFJ17, Subsection 3.7, Subsection 3.8] and, when need be, explain how they can be adjusted to hold in an elementary higher topos.

(1) [ABFJ17, Definition 3.7.1]: This definition of a local class of maps does not hold in an elementary higher topos as we do not have arbitrary coproducts. We instead use [ABFJ17, Remark 3.7.2] as the definition of local maps (Definition 4.12).
(2) [ABFJ17, Proposition 3.7.5]: This proof does not hold in an elementary higher topos as we do not have arbitrary coproducts. We presented an alternative proof in Proposition 4.16.
(3) [ABFJ17, Lemma 3.8.5]: In this lemma we use the fact that $\mathcal{L}$ is local. Notice we only need the locality condition in the sense of Definition 4.12 and not [ABFJ17, Definition 3.7.1].
(4) [ABFJ17, Lemma 3.8.6] This proof depends on [ABFJ17, Remark 3.1.4 (5)] which states under which conditions an infinite colimit diagram gives us a (-1)-connected map. It does not hold in an elementary higher topos in the form stated as we do not have infinite colimits. However, the remark is only applied to the specific pushout $G \coprod_E F$ for which the condition does hold in an elementary higher topos (as proven in Lemma 1.40).
(5) [ABFJ17, Lemma 3.8.7] The proof follows by the same argument. Notice that although the proof states that $\mathcal{L}$ has all colimits it suffices to have all finite colimits as we are only looking at pushout squares.

All the remaining results that we need for our proof from [ABFJ17, Section 4] hold and do not need any adjustments. This finishes the proof in the setting of an elementary higher topos.

4.3 Blakers-Massey Theorem for Truncations. Having proven Blakers-Massey Theorem for any modality we can finally show the case for truncations.

Proposition 4.20. *Let $n \geq -2$. Then the class of n-truncated maps and n-connected maps form a modality.*

Proof. By Theorem 3.25 it forms a weak factorization system. By Proposition 1.45 it is a strong factorization system. Finally, by Corollary 3.13 it is a modality. □

Using the Blakers-Massey Theorem in a topos we can recover the classical results.

Corollary 4.21. *(Classical Blakers-Massey Theorem) Let us assume we have a pushout square such that f is m-connected and g is n-connected.*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[d, "k"] \\ X \arrow[r, "h"] & W \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Then, the gap map $(f,g) : Z \to X \times_W Y$ is $(m+n)$-connected.

Corollary 4.22. *(Join Theorem) We have following diagram*

$$\begin{tikzcd} X \underset{B}{\times} Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left, "g"] & \\ X \arrow[r] \arrow[drr, bend right, "f"] & X \coprod\limits_{X \underset{B}{\times} Y} Y \arrow[dr, "h"] & \\ & & B \end{tikzcd}$$

If f is m-connected and g is n-connected then h is $(m+n+2)$-connected.

Remark 4.23. This is a direct implication of the Dual Blakers-Massey Theorem, but is called the *join theorem* in [Re05, Proposition 8.15], because when B is the final object then $X \coprod_{X \underset{B}{\times} Y} Y$ is simply the join $X * Y$ (Definition 2.1). The join theorem then proves that taking successive joins of one object raises the connectivity. This is the intuition we used in Theorem 2.25 to construct truncation functors.

This also gives us the Freudenthal suspension theorem.

Corollary 4.24. *(Freudenthal Suspension Theorem) Let X be an n-connected object. Then the map $X \to \Omega\Sigma X$ is $2n$-connected.*

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