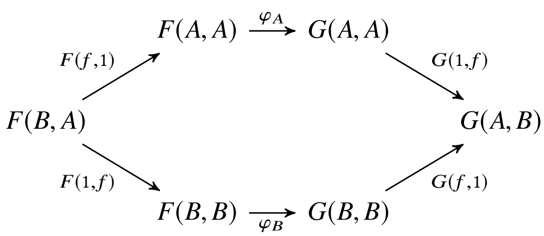
A Solution for the Compositionality Problem of Dinatural Transformations

Guy McCusker[*] Alessio Santamaria[†]

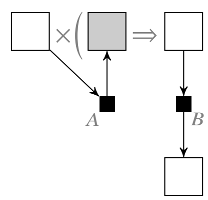
Dinatural transformations are a generalisation of the well-known *natural transformations*, as such they are ubiquitous in Mathematics and Computer Science. They appeared for the first time in [18] in the context of Algebraic Topology, where the notion of *(co)end* of a functor was introduced; lately they were formally defined in [3]. Given functors $F, G \colon \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$, a dinatural transformation $\varphi \colon F \to G$ is a family $\big(\varphi_A \colon F(A,A) \to G(A,A)\big)_{A \in \mathbb{C}}$ such that for all $f \colon A \to B$ in $\mathbb{C}$ the following hexagon commutes:



A classical example is the family of evaluation maps $\big(eval_{A,B} \colon A \times (A \Rightarrow B) \to B\big)$ in any cartesian closed category $\mathbb{C}$: the transformation *eval* is natural in B and dinatural in A.

Dinatural transformations, however, suffer from a troublesome shortcoming: they do not compose. This remarkable problem was already known to their discoverers: many studies have been conducted about them [1, 2, 5, 6, 9, 11, 12, 14, 15, 16, 17], and many attempts have been made to find a proper calculus for dinatural transformations, but only *ad hoc* solutions have been found and, ultimately, they have remained poorly understood. We present a sufficient and essentially necessary condition for two arbitrary, consecutive dinatural transformations φ and ψ for the composite $\psi \circ \varphi$ to be dinatural, thus solving the compositionality problem of dinatural transformations in its full generality [10]. We were inspired by the work of Eilenberg and Kelly on *extranatural transformations* [4], which are less general than dinaturals and also fail to compose: the authors associated to each extranatural a graph, the archetype of a *string diagram*, that captures their naturality properties. We extended such graphical calculus to dinatural transformations; for example, consider the transformation *eval* as above: its domain is the functor $T \colon \mathbb{C} \times \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ where $T(X,Y,Z) = X \times (Y \Rightarrow Z)$, while the codomain is $id_{\mathbb{C}}$. The graph of *eval* is:

$\Gamma(eval) =$



The three upper boxes correspond to the arguments of T, while the lower one to $id_{\mathbb{C}}$. Graphs of consecutive dinatural transformations $\varphi \colon F \to G$ and $\psi \colon G \to H$ can be composed by "glueing" them together along the G-boxes. Our result asserts that if the composite graph $\Gamma(\psi) \circ \Gamma(\varphi)$ is *acyclic*, then $\psi \circ \varphi$ is indeed dinatural. The proof exploits the theory of *Petri Nets* [13], of which these graphs are a particular example, by translating the dinaturality property of $\psi \circ \varphi$ into a reachability problem for the Petri Net $\Gamma(\psi) \circ \Gamma(\varphi)$. We can now finally define a generalised functor category $\{\mathbb{C}, \mathbb{D}\}$ of mixed-variance functors and (partially) dinatural transformations; this is the first step towards the formalisation of a generalised Godement calculus as sought by Kelly in [7] in order to describe coherence problems abstractly [8].

[*]University of Bath, G.A.McCusker@bath.ac.uk
[†]University of Bath, A.Santamaria@bath.ac.uk

References

[1] E. S. Bainbridge, P. J. Freyd, A. Scedrov and P. J. Scott. 'Functorial polymorphism'. In: *Theoretical Computer Science*. Special Issue Fourth Workshop on Mathematical Foundations of Programming Semantics, Boulder, CO, May 1988 70.1 (15th Jan. 1990), pp. 35–64. ISSN: 0304-3975. DOI: 10.1016/0304-3975(90)90151-7.

[2] R. Blute. 'Linear logic, coherence and dinaturality'. In: *Theoretical Computer Science* 115.1 (5th July 1993), pp. 3–41. ISSN: 0304-3975. DOI: 10.1016/0304-3975(93)90053-V.

[3] E. Dubuc and R. Street. 'Dinatural transformations'. In: *Reports of the Midwest Category Seminar IV*. Ed. by S. Mac Lane, H. Applegate, M. Barr, B. Day, E. Dubuc, A. P. Phreilambud, R. Street, M. Tierney and S. Swierczkowski. Vol. 137. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1970, pp. 126–137. ISBN: 978-3-540-04926-5. DOI: 10.1007/BFb0060443.

[4] S. Eilenberg and G. M. Kelly. 'A generalization of the functorial calculus'. In: *Journal of Algebra* 3.3 (May 1966), pp. 366–375. ISSN: 0021-8693. DOI: 10.1016/0021-8693(66)90006-8.

[5] P. J. Freyd, E. P. Robinson and G. Rosolini. 'Dinaturality for free'. In: *Applications of Categories in Computer Science: Proceedings of the London Mathematical Society Symposium, Durham 1991*. Ed. by A. M. Pitts, M. P. Fourman and P. T. Johnstone. London Mathematical Society Lecture Note Series. Cambridge: Cambridge University Press, 1992, pp. 107–118. ISBN: 978-0-521-42726-5. DOI: 10.1017/CBO9780511525902.007.

[6] J.-Y. Girard, A. Scedrov and P. J. Scott. 'Normal Forms and Cut-Free Proofs as Natural Transformations'. In: *Logic from Computer Science*. Ed. by N. M. Yiannis. Vol. 21. Mathematical Sciences Research Institute Publications. Springer, New York, NY, 1992, pp. 217–241. ISBN: 978-1-4612-7685-2 978-1-4612-2822-6. DOI: 10.1007/978-1-4612-2822-6_8.

[7] G. M. Kelly. 'Many-variable functorial calculus. I.' In: *Coherence in Categories*. Ed. by G. M. Kelly, M. Laplaza, G. Lewis and S. Mac Lane. Vol. 281. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1972, pp. 66–105. ISBN: 978-3-540-05963-9. DOI: 10.1007/BFb0059556.

[8] G. M. Kelly. 'An abstract approach to coherence'. In: *Coherence in Categories*. Ed. by G. M. Kelly, M. Laplaza, G. Lewis and S. Mac Lane. Vol. 281. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1972, pp. 106–147. ISBN: 978-3-540-05963-9 978-3-540-37958-4. DOI: 10.1007/BFb0059557.

[9] J. d. Lataillade. 'Dinatural Terms in System F'. In: *2009 24th Annual IEEE Symposium on Logic In Computer Science*. 2009 24th Annual IEEE Symposium on Logic In Computer Science. Aug. 2009, pp. 267–276. DOI: 10.1109/LICS.2009.30.

[10] G. McCusker and A. Santamaria. 'On Compositionality of Dinatural Transformations'. In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by D. Ghica and A. Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018, 33:1–33:22. ISBN: 978-3-95977-088-0. DOI: 10.4230/LIPIcs.CSL.2018.33.

[11] P. S. Mulry. 'Categorical fixed point semantics'. In: *Theoretical Computer Science*. Special Issue Fourth Workshop on Mathematical Foundations of Programming Semantics, Boulder, CO, May 1988 70.1 (15th Jan. 1990), pp. 85–97. ISSN: 0304-3975. DOI: 10.1016/0304-3975(90)90154-A.

[12] R. Paré and L. Román. 'Dinatural numbers'. In: *Journal of Pure and Applied Algebra* 128.1 (2nd June 1998), pp. 33–92. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(97)00036-4.

[13] C. A. Petri. 'Kommunikation mit Automaten'. OCLC: 258511501. PhD thesis. Bonn: Mathematisches Institut der Universität Bonn, 1962.

[14] P. Pistone. 'On Dinaturality, Typability and beta-eta-Stable Models'. In: *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017)*. Ed. by D. Miller. Vol. 84. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2017, 29:1–29:17. ISBN: 978-3-95977-047-7. DOI: 10.4230/LIPIcs.FSCD.2017.29.

[15] G. Plotkin and M. Abadi. 'A logic for parametric polymorphism'. In: *Typed Lambda Calculi and Applications*. Ed. by M. Bezem and J. F. Groote. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 1993, pp. 361–375. ISBN: 978-3-540-47586-6.

[16] A. K. Simpson. 'A characterisation of the least-fixed-point operator by dinaturality'. In: *Theoretical Computer Science* 118.2 (27th Sept. 1993), pp. 301–314. ISSN: 0304-3975. DOI: 10.1016/0304-3975(93)90112-7.

[17] P. Wadler. 'Theorems for Free!' In: *Proceedings of the Fourth International Conference on Functional Programming Languages and Computer Architecture*. FPCA '89. New York, NY, USA: ACM, 1989, pp. 347–359. ISBN: 978-0-89791-328-7. DOI: 10.1145/99370.99404.

[18] N. Yoneda. 'On Ext and Exact Sequences'. In: *Journal of the Faculty of Science, University of Tokyo, Setion 1, Mathematics, asronomy, physics, chemistry.* 8 (1960), pp. 507–576. ISSN: 0368-2269.