The Existential Completion

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Abstract

We determine the existential completion of a primary doctrine, and we prove that the 2-monad obtained from it is lax-idempotent, and that the 2-category of existential doctrines is isomorphic to the 2-category of algebras for this 2-monad. We also show that the existential completion of an elementary doctrine is again elementary. Finally we extend the notion of exact completion of an elementary existential doctrine to an arbitrary elementary doctrine.

1 Introduction

In recent years, many relevant logical completions have been extensively studied in category theory. The main instance is the exact completion, see [3, 4, 5], which is the universal extension of a category with finite limits to an exact category. In [16, 17, 18], Maietti and Rosolini introduce a categorical version of quotient for an equivalence relation, and they study that in a doctrine equipped with a sufficient logical structure to describe the notion of an equivalence relation. In [18] they show that both the exact completion of a regular category and the exact completion of a category with binary products, a weak terminal object and weak pullbacks can be seen as instances of a more general completion with respect to an elementary existential doctrine.

In this paper we present the existential completion of a primary doctrine, and we give an explicit description of the 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ constructed from the 2-adjunction, where **PD** is the 2-category of primary doctrines.

It is well known that pseudo-monads can express uniformly and elegantly many algebraic structure; we refer the reader to [25, 24, 9] for a detailed description of these topics. We show that every existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ admits an action $a\colon \mathrm{T}_e P \longrightarrow P$ such that (P,a) is a T_e-algebra, and that if (R,b) is T_e-algebra then the doctrine is existential, and this gives an equivalence between the 2-category T_e-**Alg** and the 2-category **ED** whose objects are existential doctrines.

Here the action encodes the existential structure for a doctrine, and we prove that this structure is uniquely determined to within appropriate isomorphism and that the 2-monad T_e is property-like and lax-idempotent in the sense of Kelly and Lack [9].

We conclude proving that the existential completion preserves elementary doctrines, and then we generalize the bi-adjunction **EED** $\rightarrow$ **Xct** presented in [18, 15] to a bi-adjunction from the 2-category **PED** of elementary doctrine to the 2-category of exact categories **Xct**.

In the first two sections we recall some definitions and results about the theory of pseudo-monads, primary and existential doctrine which are needed for the rest of the paper.

In section 3 we present the existential completion. We introduce a functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{PED}$ from the 2-category of primary doctrines to the 2-category of existential doctrines, and we prove that it is a left 2-adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{PED} \longrightarrow \mathbf{PD}$.

In sections 4 we prove that the 2-monad T_e constructed from the 2-adjunction is lax-idempotent and, in section 5, that the category T_{e}-**Alg** is 2-equivalent to the 2-category of existential doctrine.

In section 6 we show that the existential completion of an elementary doctrine is elementary, and we use this fact to extend the notion of exact completion to elementary doctrines.

2 A brief recap of two-dimensional monad theory

This section is devoted to the formal definition of 2-monad on a 2-category and a characterization of the definitions. We use 2-categorical pasting notation freely, following the usual convention of the topic as used extensively in [1, 24, 25].

You can find all the details of the main results of this section in the works of Kelly and Lack [9]. For a more general and complete description of these topics, and a generalization for the case of pseudo-monad, you can see the Ph.D thesis of Tanaka [23], the articles of Marmolejo [20, 19] and the work of Kelly [10]. Moreover we refer to [2, 14] for all the standard results and notions about 2-category theory.

A ***2-monad*** (T, μ, η) on a 2-category $\mathcal{A}$ is a 2-functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$ together 2-natural transformations $\mu\colon \mathrm{T}^2 \longrightarrow \mathrm{T}$ and $\eta\colon 1_{\mathcal{A}} \longrightarrow \mathrm{T}$ such that the following diagrams

$$\begin{tikzcd}
\mathrm{T}^3 \arrow[r, "\mathrm{T}\mu"] \arrow[d, "\mu\mathrm{T}"'] & \mathrm{T}^2 \arrow[d, "\mu"] \\
\mathrm{T}^2 \arrow[r, "\mu"'] & \mathrm{T}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{T} \arrow[r, "\eta\mathrm{T}"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^2 \arrow[d, "\mu"] & \mathrm{T} \arrow[l, "\mathrm{T}\eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T} &
\end{tikzcd}$$

commute. Let (T, μ, η) be a 2-monad on a 2-category $\mathcal{A}$. A T-***algebra*** is a pair (A, a) where, A is an object of $\mathcal{A}$ and $a\colon \mathrm{T}A \longrightarrow A$ is a 1-cell such that the following diagrams commute:

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}a"] \arrow[d, "\mu_A"'] & \mathrm{T}A \arrow[d, "a"] \\
\mathrm{T}A \arrow[r, "a"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "1_A"'] & \mathrm{T}A \arrow[d, "a"] \\
& A
\end{tikzcd}$$

A ***lax*** T-***morphism*** from a T-algebra (A, a) to a T-algebra (B, b) is a pair $(f, \overline{f})$ where f is a 1-cell $f\colon A \longrightarrow B$ and $\overline{f}$ is a 2-cell

$$\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

which satisfies the following ***coherence*** conditions:

$$\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mu_A"'] & TB \arrow[d, "\mu_B"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}^2A \arrow[r, "\mathrm{T}^2f"] \arrow[d, "\mathrm{T}a"'] \arrow[dr, phantom, "\Downarrow\mathrm{T}\overline{f}"] & \mathrm{T}B \arrow[d, "\mathrm{T}b"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1_A"'] & B \arrow[d, "1_B"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

The regions in which no 2-cell is written always commute by the naturality of η and μ, and are deemed to contain the identity 2-cell.
If $(f, \overline{f})$ is a lax morphism and $\overline{f}$ is invertible, then it is said T-***morphism*** and if $\overline{f}$ is the identity it is said ***strict***-T-***morphism***.
The category of T-algebras and lax T-morphisms becomes a 2-category T-$\mathbf{Alg}_{\mathrm{l}}$, when we introduce as 2-cells the T-***transformations***, where a T-transformation from a 1-cell $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ to $(g, \overline{g})\colon (A, a) \longrightarrow (B, b)$ is a 2-cell $\alpha\colon f \Longrightarrow g$ in $\mathcal{A}$ satisfies the following coherence condition

$$
\begin{tikzcd}
\mathrm{T}A \arrow[r, bend left, "\mathrm{T}f"] \arrow[r, bend right, "\mathrm{T}g"'] \arrow[r, phantom, "\Downarrow\mathrm{T}\alpha"] \arrow[d, "a"'] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] \arrow[ur, phantom, "\Downarrow\overline{g}"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] \arrow[ur, phantom, "\Downarrow\overline{f}"] & B
\end{tikzcd}
$$

expressing compatibility of α with $\overline{f}$ and $\overline{g}$.
It is observed in [9] that using this notion of T-morphism, we can express more precisely what it might mean to say that an action of a monad T on an object A is ***unique to within a unique isomorphism***. We shall mean that, given two action $a, a'\colon \mathrm{T}A \longrightarrow A$ there is a unique invertible 2-cell $\alpha\colon a \Longrightarrow a'$ such that $(1_A, \alpha)\colon (A, a) \longrightarrow (A, a')$ is a morphism of T-algebras (in particular it is an isomorphism of T-algebras). In this case we will say that the T-***algebra structure is essentially unique***. More precisely a 2-monad (T, μ, η) is said ***property-like***, if it satisfies the following conditions:

- for every T-algebras (A, a) and (B, b), and for every invertible 1-cell $f\colon A \longrightarrow B$ there exists a unique invertible 2-cell $\overline{f}$

$$
\begin{tikzcd}
\mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] & \mathrm{T}B \arrow[d, "b"] \\
A \arrow[r, "f"'] \arrow[ur, phantom, "\Downarrow\overline{f}"] & B
\end{tikzcd}
$$

  such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a morphism of T-algebras;

- for every T-algebras (A, a) and (B, b), and for every 1-cell $f\colon A \longrightarrow B$ if there exists a 2-cell $\overline{f}$

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras, then it is the unique 2-cell with such property.

We conclude this section recalling a stronger property on a 2-monads (T, μ, η) on $\mathcal{A}$ which implies that T is property-like: a 2-monad (T, μ, η) is said ***lax-idempotent***, if for every T-algebras (A, a) and (B, b), and for every 1-cell $f\colon A \longrightarrow B$ there exists a unique 2-cell $\overline{f}$

$$\begin{tikzcd} \mathrm{T}A \arrow[r, "\mathrm{T}f"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\overline{f}"] & \mathrm{T}B \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras. In particular every lax-idempotent monad is property like. See [9, Proposition 6.1].

3 Primary and existential doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in series of seminal papers, together with the more general notion of existential elementary doctrine. These were studied in [11, 12, 13]. We recall from *loc. cit.* some definitions which will be useful in the following. The reader can find all the details about the theory of elementary and existential doctrine in [16, 17, 18].

Definition 3.1. Let $\mathcal{C}$ be a category with finite products. A ***primary doctrine*** if a functor $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** from the opposite of the category $\mathcal{C}$ to the category of inf-semilattices.

Definition 3.2. A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is ***elementary*** if for every A in $\mathcal{C}$ there exists an object δ_A in $P(A \times A)$ such that

1. the assignment
$$\exists_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}(\alpha) := P_{\mathrm{pr}_1}(\alpha) \wedge \delta_A$$
for α in PA determines a left adjoint to $P_{\langle \mathrm{id}_A, \mathrm{id}_A \rangle}\colon P(A \times A) \longrightarrow PA$;

2. for every morphism e of the form $\langle \mathrm{pr}_1, \mathrm{pr}_2, \mathrm{pr}_3\rangle\colon X \times A \longrightarrow X \times A \times A$ in $\mathcal{C}$, the assignment

$$\exists_e(\alpha) := P_{\langle \mathrm{pr}_1, \mathrm{pr}_2\rangle}(\alpha) \wedge P_{\langle \mathrm{pr}_2, \mathrm{pr}_3\rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \longrightarrow P(X \times A)$.

Definition 3.3. A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is ***existential*** if, for every A_1 and A_2 in $\mathcal{C}$, for any projection $\mathrm{pr}_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor

$$P_{\mathrm{pr}_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$

has a left adjoint $\exists_{\mathrm{pr}_i}$, and these satisfy:

1. ***Beck-Chevalley condition:*** for any pullback diagram

$$\begin{tikzcd} X' \arrow[r, "\mathrm{pr}'"] \arrow[d, "f'"'] & A' \arrow[d, "f"] \\ X \arrow[r, "\mathrm{pr}"'] & A \end{tikzcd}$$

with pr and pr' projections, for any β in $P(X)$ the canonical arrow

$$\exists_{\mathrm{pr}'} P_{f'}(\beta) \le P_f \exists_{\mathrm{pr}}(\beta)$$

is an isomorphism;

2. ***Frobenius reciprocity:*** for any projection $\mathrm{pr}\colon X \longrightarrow A$, α in $P(A)$ and β in $P(X)$, the canonical arrow

$$\exists_{\mathrm{pr}}(P_{\mathrm{pr}}(\alpha) \wedge \beta) \le \alpha \wedge \exists_{\mathrm{pr}}(\beta)$$

in $P(A)$ is an isomorphism.

Remark 3.4. In an existential elementary doctrine, for every map $f\colon A \longrightarrow B$ in $\mathcal{C}$ the functor P_f has a left adjoint $\exists_f$ that can be computed as

$$\exists_{\mathrm{pr}_2}(P_{f \times \mathrm{id}_B}(\delta_B) \wedge P_{\mathrm{pr}_1}(\alpha))$$

for α in $P(A)$, where pr_1 and pr_2 are the projections from $A \times B$.

Examples 3.5. The following examples are discussed in [11].

1. Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathrm{Sub}_{\mathcal{C}}\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathrm{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $\mathrm{Sub}_{\mathcal{C}}(f)\colon \mathrm{Sub}_{\mathcal{C}}(A) \longrightarrow \mathrm{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f. The fiber equalities are the diagonal arrows. This is an existential elementary doctrine if and only if the category $\mathcal{C}$ has a stable, proper factorization system $\langle \mathcal{E}, \mathcal{M} \rangle$. See [7].

2. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

$$\Psi_{\mathcal{D}}\colon \mathcal{D}^{op} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}(f)\colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is existential, and the existential left adjoint are given by the post-composition.

3. Let $\mathcal{T}$ be a theory in a first order language **Sg**. We define a primary doctrine

$$LT\colon \mathcal{C}_{\mathcal{T}}^{op} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{C}_{\mathcal{T}}$ is the category of lists of variables and term substitutions:

- ***objects*** of $\mathcal{C}_{\mathcal{T}}$ are finite lists of variables $\vec{x} := (x_1, \dots, x_n)$, and we include the empty list $()$;
- a ***morphisms*** from $(x_1, \dots, x_n)$ into $(y_1, \dots, y_m)$ is a substitution $[t_1/y_1, \dots, t_m/y_m]$ where the terms t_i are built in **Sg** on the variable $x_1, \dots, x_n$;
- the ***composition*** of two morphisms $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}]\colon \vec{y} \longrightarrow \vec{z}$ is given by the substitution

$$[s_1[\vec{t}/\vec{y}]/z_k, \dots, s_k[\vec{t}/\vec{y}]/z_k]\colon \vec{x} \longrightarrow \vec{z}\,.$$

The functor $LT\colon \mathcal{C}_{\mathcal{T}}^{op} \longrightarrow \mathbf{InfSL}$ sends a list $(x_1, \dots, x_n)$ in the class $LT(x_1, \dots, x_n)$ of all well formed formulas in the context $(x_1, \dots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in LT(x_1, \dots, x_n)$ if $\psi \vdash_{\mathcal{T}} \phi$, and then we quotient in the usual way to obtain a partial order on $LT(x_1, \dots, x_n)$. Given a morphism of $\mathcal{C}_{\mathcal{T}}$

$$[t_1/y_1, \dots, t_m/y_m]\colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

then the functor $LT_{[\vec{t}/\vec{y}]}$ acts as the substitution $LT_{[\vec{t}/\vec{y}]}(\psi(y_1, \dots, y_m)) = \psi[\vec{t}/\vec{y}]$.
The doctrine $LT\colon \mathcal{C}_{\mathcal{T}}^{op} \longrightarrow \mathbf{InfSL}$ is elementary exactly when $\mathcal{T}$ has an equality predicate and it is existential. For all the detail we refer to [17], and for the case of a many sorted first order theory we refer to [21].

4 Existential completion

In this section we construct an existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, starting from a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$. Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a fixed primary doctrine for the rest of the section, and let $a \subset \mathcal{C}_1$ be a subset of morphisms closed under pullbacks, compositions and such that it contains the identity morphisms.
For every object A of $\mathcal{C}$ consider the following preorder:

- the objects are pairs $(B \xrightarrow{g\in a} A\,,\ \alpha \in PB)$;
- $(B \xrightarrow{h\in a} A\,,\ \alpha \in PB) \leq (D \xrightarrow{f\in a} A\,,\ \gamma \in PD)$ if there exists $w\colon B \longrightarrow D$ such that

$$\begin{tikzcd} & B \arrow[dl, "w"'] \arrow[d, "h"] \\ D \arrow[r, "f"'] & A \end{tikzcd}$$

  commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder. Let $P^e(A)$ be the partial order obtained by identifying two objects when $(B \xrightarrow{h\in a} A\,,\ \alpha \in PB) \gtreqless (D \xrightarrow{f\in a} A\,,\ \gamma \in PD)$ in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.
Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, let $P^e_f(C \xrightarrow{g\in a} B\,,\ \beta \in PC)$ be the object

$$(D \xrightarrow{g^*f} A\,,\ P_{f^*g}(\beta) \in PD)$$

where

$$\begin{tikzcd} D \arrow[r, "g^*f"] \arrow[d, "f^*g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback because $g \in a$. Note that P^e_f is well defined, because isomorphisms are stable under pullback.

Proposition 4.1. *Let $P\colon \mathcal{C}^{op} \longrightarrow \textbf{InfSL}$ be a primary doctrine. Then $P^e\colon \mathcal{C}^{op} \longrightarrow \textbf{InfSL}$ is a primary doctrine, in particular:*

(i) for every object A in $\mathcal{C}$, $P^e(A)$ is a inf-semilattice;

(ii) for every morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, P^e_f is an homomorphism of inf-semilattices.

Proof. (i) For every A we have the top element $(\,A \xrightarrow{\mathrm{id}_A} A\,,\ \top_A)$. Consider $(\,A_1 \xrightarrow{h_1} A\,,\ \alpha_1 \in PA_1)$ and $(\,A_2 \xrightarrow{h_2} A\,,\ \alpha_2 \in PA_2)$. In order to define the greatest lower bound of the two objects consider a pullback

$$\begin{tikzcd} A_1 \wedge A_2 \arrow[r, "h_1^* h_2"] \arrow[d, "h_2^* h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_2 \arrow[d, "h_2"] \\ A_1 \arrow[r, "h_1"'] & A \end{tikzcd}$$

which exists because $h_1 \in a$ (and $h_2 \in a$). We claim that

$$(\,A_1 \wedge A_2 \xrightarrow{h_1 h_2^* h_1} A\,, P_{h_2^* h_1}(\alpha_1) \wedge P_{h_1^* h_2}(\alpha_2))$$

is such an infimum. It is easy to check that

$$(\,A_1 \wedge A_2 \xrightarrow{h_1 h_2^* h_1} A\,, P_{h_2^* h_1}(\alpha_1) \wedge P_{h_1^* h_2}(\alpha_2)) \leq (\,A_i \xrightarrow{h_i} A\,,\ \alpha_i \in PA_i)$$

for $i = 1, 2$. Next consider $(\,B \xrightarrow{g} A\,,\ \beta \in PB) \leq (\,A_i \xrightarrow{h_i} A\,,\ \alpha_i \in PA_i)$ for $i = 1, 2$ and $g = h_i w_i$. Then there is a morphism $w\colon C \longrightarrow A_1 \wedge A_2$ such that

$$\begin{tikzcd} B \arrow[dr, dashed, "w"] \arrow[drr, bend left, "w_2"] \arrow[ddr, bend right, "w_1"'] & & \\ & A_1 \wedge A_2 \arrow[r, "h_1^* h_2"] \arrow[d, "h_2^* h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_2 \arrow[d, "h_2"] \\ & A_1 \arrow[r, "h_1"'] & A \end{tikzcd}$$

commutes and $P_w(P_{h_2^* h_1}(\alpha_1) \wedge P_{h_1^* h_2}(\alpha_2)) = P_{w_1}(\alpha_1) \wedge P_{w_2}(\alpha_2) \geq \beta$.
(ii) We first prove that for every morphism $f\colon A \longrightarrow B$ the P_f^e preserves the order. Consider $(\,C_1 \xrightarrow{g_1 \in a} B\,,\ \alpha_1 \in PC_1) \leq (\,C_2 \xrightarrow{g_2 \in a} B\,,\ \alpha_2 \in PC_2)$ with $g_2 w = g_1$ and $P_w(\alpha_2) \geq \alpha_1$. We want to prove that

$$(\,D_1 \xrightarrow{g_1^* f} A\,,\ P_{f^* g_1}(\alpha_1) \in PD_1) \leq (\,D_2 \xrightarrow{g_2^* f} A\,,\ P_{f^* g_2}(\alpha_2) \in PD_1)$$

We can observe that $g_2 w f^* g_1 = g_1 f^* g_1 = f g_1^* f$. Then there exists a unique $\overline{w}\colon D_1 \longrightarrow D_2$

such that the following diagram commutes

$$\begin{tikzcd}
D_1 \arrow[dr, dashed, "\overline{w}"] \arrow[drr, bend left, "g_1^* f"] \arrow[ddr, bend right, "w f^* g_1"'] & & \\
& D_2 \arrow[r, "g_2^* f"] \arrow[d, "f^* g_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
& C_2 \arrow[r, "g_2"'] & B
\end{tikzcd}$$

Moreover $P_{\overline{w}}(P_{f^*g_2}(\alpha_2)) = P_{f^*g_1}(P_w(\alpha_2)) \geq P_{f^*g_1}(\alpha_1)$, and it is easy to see that P^e_f preserves top elements. Finally it is straightforward to prove that $P^e_f(\alpha \wedge \beta) = P^e_f(\alpha) \wedge P^e_f(\beta)$. Moreover it is straightforward to prove that $P^e_f(\alpha \wedge \beta) = P^e_f(\alpha) \wedge P^e_f(\beta)$. $\square$

Proposition 4.2. *Given a morphism* $f \colon A \longrightarrow B$ *of a, let*

$$\exists^e_f(C \xrightarrow{h} A , \alpha \in PC) := (C \xrightarrow{fh} B , \alpha \in PC)$$

when $(C \xrightarrow{h} A , \alpha \in PC)$ *is in* $P^e(A)$*. Then* $\exists^e_f$ *is left adjoint to* P^e_f*.*

Proof. Let $\alpha := (C_1 \xrightarrow{g_1} B ,\ \alpha_1 \in PC_1)$ and $\beta := (D_2 \xrightarrow{f_2} A ,\ \beta_2 \in PD_2)$. Now we assume that $\beta \leq P^e_f(\alpha)$. This means that

$$\begin{tikzcd}
& D_2 \arrow[dl, "w"'] \arrow[d, "f_2"] \\
D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
C_1 \arrow[r, "g_1"'] & B
\end{tikzcd}$$

and $P_w(P_{f^*g_1}(\alpha_1)) \geq \beta_2$. Then we have

$$\begin{tikzcd}
& D_2 \arrow[dl, "f^* g_1 w"'] \arrow[d, "\Delta_A f_2"] \\
C_1 \arrow[r, "g"'] & B
\end{tikzcd}$$

and $P_{wf^*g_1}(\alpha_1) \geq \beta$. Then $\exists^e_f(\beta) \leq \alpha$.
Now assume $\exists^e_f(\beta) \leq \alpha$

$$\begin{tikzcd} & D_2 \arrow[dl, "\overline{w}"'] \arrow[d, "ff_2"] \\ C_1 \arrow[r, "g_1"'] & B \end{tikzcd}$$

with $P_{\overline{w}}(\alpha_1) \geq \beta_2$ Then there exists $w \colon D_2 \longrightarrow D_1$ such that the following diagram commutes

$$\begin{tikzcd} D_2 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "\overline{w}"'] & & \\ & D_1 \arrow[r, "g_1^* f"] \arrow[d, "f^* g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ & C_1 \arrow[r, "g_1"'] & B \end{tikzcd}$$

an $P_w(P_{f^*g_1}(\alpha_1) = P_{\overline{w}}(\alpha_1) \geq \beta_1$. Then we can conclude that $\beta \leq P^e_f(\alpha)$. □

Theorem 4.3. *For every primary doctrine* $P \colon \mathcal{C}^{op} \longrightarrow$ ***InfSL***, $P^e \colon \mathcal{C}^{op} \longrightarrow$ ***InfSL*** *satisfies:*

(i) **Beck-Chevalley Condition***: for any pullback*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

with $g \in a$ *(hence also* $g' \in a$*), for any* $\beta \in P^e(X)$ *the following equality holds*

$$\exists^e_{g'} P^e_{f'}(\beta) = P^e_f \exists^e_g(\beta)$$

(ii) **Frobenius Reciprocity***: for every morphism* $f \colon X \longrightarrow A$ *of* a*, for every* $\alpha \in P^e(A)$ *and* $\beta \in P^e(X)$*, the following equality holds*

$$\exists^e_f(P^e_f(\alpha) \wedge \beta) = \alpha \wedge \exists^e_f(\beta)$$

Proof. (i) Consider the following pullback square

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

where $g, g' \in a$, and let $\beta := (\ C_1 \xrightarrow{h_1} X\ ,\ \beta_1 \in PC_1) \in P^e(X)$. Consider the following diagram

$$\begin{tikzcd} D_1 \arrow[r, "h_1^* f'"] \arrow[d, "f'^* h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ C_1 \arrow[r, "h_1"'] & X \arrow[r, "g"'] & A \end{tikzcd}$$

Since the two square are pullbacks, then the big square is a pullback, and then

$$(\ D_1 \xrightarrow{g' h_1^* f'} A\ , P_{f'^* h_1}(\beta_1)) = (\ D_1 \xrightarrow{g h_1^* f} A\ , P_{f^* g h_1}(\beta_1))$$

and these are by definition

$$\exists^e_{g'} P^e_{f'}(\beta) = P^e_f \exists^e_g(\beta).$$

Therefore the Beck-Chevalley Condition is satisfied.

(ii) Now consider a morphism $f\colon X \longrightarrow A$ of a, an element $\alpha := (\ C_1 \xrightarrow{h_1} A\ ,\ \alpha_1 \in PC_1)$ in $P^e(A)$, and an element $\beta = (\ D_2 \xrightarrow{h_2} X\ ,\ \beta_2 \in PD_2)$ in $P^e(X)$. Observe that the following diagram is a pullback

$$\begin{tikzcd} D_2 \wedge D_1 \arrow[r, "h_2^*(h_1^* f)"] \arrow[d, "(h_1^* f)^* h_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_1 \arrow[r, "f^* h_1"] \arrow[d, "h_1^* f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "h_1"] \\ D_2 \arrow[r, "h_2"'] & X \arrow[r, "f"'] & A \end{tikzcd}$$

and this means that

$$\exists^e_f(P^e_f(\alpha) \wedge \beta) = \alpha \wedge \exists^e_f(\beta).$$

Therefore the Frobenius Reciprocity is satisfied. $\square$

Corollary 4.4. *Let* $P\colon \mathcal{C}^{op} \longrightarrow$ ***InfSL*** *be a primary doctrine, then the doctrine* $P^e\colon \mathcal{C}^{op} \longrightarrow$ ***InfSL*** *is existential.*

Remark 4.5. In the case that a is the class of the projections, then from primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ it can be constructed an existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ in the sense of Definition 3.3. Therefore the notion of existential doctrine can be generalized in the sense that an existential doctrine can be defined as a pair

$$(P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL} , a)$$

where $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is a primary doctrine and a is a class of morphisms of $\mathcal{C}$ closed by pullbacks, composition and identities, which satisfies the conditions of Theorem 4.3.

Remark 4.6. Let $P\colon \mathcal{C} \longrightarrow \mathbf{PosTop}$ be a functor where **PosTop** is the category of posets with top element. We can apply the existential completion since we have not used the hypothesis that PA has infimum during the proofs; we have proved that if it has a infimum it is preserved by the completion. Moreover we can express the Frobenius condition without using infima, and also this condition is preserved by completion.
The requirement of top element for the posets of the category **PosTop** is due to the fact that we want an injection from the doctrine $P\colon \mathcal{C} \longrightarrow \mathbf{PosTop}$ into $P^e\colon \mathcal{C} \longrightarrow \mathbf{PosTop}$, since from a logical point of view, we are extending a theory where formulas have no occurrences of the symbol $\exists$, to an existential one, and we require that the theorems of the previous theory are preserved.

For the rest of this section we assume that the morphisms of a are projections. We define a 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{PED}$ sending a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ into the existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$. For all the standard notions about 2-category theory we refer to [2] and [14].

Proposition 4.7. *Consider the category* $\boldsymbol{PD}(P,R)$*. We define*

$$\mathrm{E}_{P,R}\colon \boldsymbol{PD}(P,R) \longrightarrow \boldsymbol{PED}(P^e,R^e)$$

as follow:

- *for every 1-cell* (F,b), $\mathrm{E}_{P,R}(F,b) := (F,b^e)$, *where* $b^e_A\colon P^eA \longrightarrow R^eFA$ *sends an object* $(C \xrightarrow{g} A \ ,\ \alpha)$ *in the object* $(FC \xrightarrow{Fg} FA \ ,\ b_C(\alpha))$*;*
- *for every 2-cell* $\theta\colon (F,b) \Longrightarrow (G,c)$, $\mathrm{E}_{P,R}\theta$ *is essentially the same.*

With the previous assignment E *is a 2-functor.*

Proof. We prove that $(F,b^e)\colon P^e \longrightarrow R^e$ is a 1-cell of $\mathbf{PED}(P^e,R^e)$. We first prove that for every $A \in \mathcal{C}$, b^e_A preserves the order.
If $(C_1 \xrightarrow{g_1} A \ ,\ \alpha_1) \le (C_2 \xrightarrow{g_2} A \ ,\ \alpha_2)$, we have a morphism $w\colon C_1 \longrightarrow C_2$ such that the

following diagram commutes

$$\begin{tikzcd}
 & C_1 \arrow[dl, "w"'] \arrow[d, "g_1"] \\
C_2 \arrow[r, "g_2"'] & A
\end{tikzcd}$$

and $\alpha_1 \leq P_w(\alpha_2)$. Since b is a natural transformation, we have that $b_{C_1}P_w = R_{Fw}b_{C_2}$. Then we can conclude that $(\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1)) \leq (\ FC_2 \xrightarrow{Fg_2} FA\ ,\ b_{C_2}(\alpha_2))$ because $Fg_2Fw = Fg_1$ and $R_{Fw}(b_{C_2}\alpha_2) = b_{C_1}P_w(\alpha_2) \geq b_{C_1}(\alpha_1)$. Moreover, since F preserves products, we can conclude that b^e_A preserves inf.

One can prove that $b^e\colon P^e \longrightarrow R^eF^{\mathrm{op}}$ is a natural transformation using the facts that F preserves products. Moreover we can easily see that b^e preserves the left adjoints along projections. Then (F, b^e) is a 1-cell of **PED**.

Now consider a 2-cell $\theta\colon (F, b) \Longrightarrow (G, c)$, and let $\alpha := (\ C_1 \xrightarrow{g_1} A\ ,\ \alpha_1)$ be an object of $P^e(A)$. Then

$$b^e_A(\alpha) = (\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1))$$

and

$$R^e_{\theta_A}c^e_A(\alpha) = (\ D_1 \xrightarrow{Gg_1^*\theta_A} FA\ ,\ R_{\theta^*_A Gg_1}c_{C_1}(\alpha_1))$$

where

$$\begin{tikzcd}
D_1 \arrow[r, "Gg_1^*\theta_A"] \arrow[d, "\theta^*_A Gg_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FA \arrow[d, "\theta_A"] \\
GC_1 \arrow[r, "Gg_1"'] & GA
\end{tikzcd}$$

Now observe that since $\theta\colon F \longrightarrow G$ is a natural transformation, there exists a unique $w\colon FC_1 \longrightarrow D_1$ such that the diagram

$$\begin{tikzcd}
FC_1 \arrow[dr, dashed, "w"] \arrow[drr, bend left, "Fg_1"] \arrow[ddr, bend right, "\theta_{C_1}"'] & & \\
 & D_1 \arrow[r, "Gg_1^*\theta_A"] \arrow[d, "\theta^*_A Gg_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & FA \arrow[d, "\theta_A"] \\
 & GC_1 \arrow[r, "Gg_1"'] & GA
\end{tikzcd}$$

commutes and then $R_wR_{\theta^*_A Gg_1}c_{C_1}(\alpha_1) = R_{\theta_{C_1}}c_{C_1}(\alpha_1) \geq b_{C_1}(\alpha_1)$. Therefore we can conclude that $b^e_A(\alpha) \leq R^e_{\theta_A}c^e_A(\alpha)$, and then $\theta\colon F \longrightarrow G$ can is a 2-cell $\theta\colon (F, b^e) \Longrightarrow (G, c^e)$, and

$\mathrm{E}_{P,R}(\theta\gamma) = \mathrm{E}_{P,R}(\theta)\mathrm{E}_{P,R}(\gamma)$.
Finally one can prove that the following diagram commutes observing that for every $(F, b) \in$ $\mathbf{PD}(P, R)$ and $(G, c) \in \mathbf{PD}(R, D)$, $(GF, c^e \star b^e) = (GF, (c \star b)^e)$.

$$\begin{array}{ccc} \mathbf{PD}(P,R) \times \mathbf{PD}(R,D) & \xrightarrow{c_{PRD}} & \mathbf{PD}(P,D) \\ {\scriptstyle \mathrm{E}_{PR}\times\mathrm{E}_{RD}}\downarrow & & \downarrow{\scriptstyle \mathrm{E}_{PD}} \\ \mathbf{PED}(P^e,R^e) \times \mathbf{PED}(R^e,D^e) & \xrightarrow[c_{P^eR^eD^e}]{} & \mathbf{PED}(P^e,D^e) \end{array}$$

and the same for the unit diagram.
Therefore we can conclude that E is a 2-functor. □

Now we prove the 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{PED}$ is left adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{PED} \longrightarrow \mathbf{PD}$.

Proposition 4.8. *Let $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** be an elementary doctrine. Then*

$$(\mathrm{id}_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^e$$

where $\iota_{PA}\colon PA \longrightarrow P^eA$ sends α into $(A \xrightarrow{\mathrm{id}_A} A\ ,\ \alpha)$ is a 1-cell. Moreover the assignment

$$\eta\colon\ \mathrm{id}_{\boldsymbol{ED}} \longrightarrow \mathrm{UE}$$

where $\eta_P := (\mathrm{id}_{\mathcal{C}}, \iota_P)$, is a 2-natural transformation.

Proof. It is easy to prove that $\iota_{PA}\colon PA \longrightarrow P^eA$ preserves all the structures. For every morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, it one can see that the following diagram commutes

$$\begin{array}{ccc} PB & \xrightarrow{P_f} & PA \\ {\scriptstyle \iota_{PB}}\downarrow & & \downarrow{\scriptstyle \iota_{PA}} \\ P^eB & \xrightarrow[P^e_f]{} & P^eA \end{array}$$

Then we can conclude that $(\mathrm{id}_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^e$ is a 1-cell of **ED** and it is a direct verification the proof the η is a 2-natural transformation. □

Proposition 4.9. *Let $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** be an existential doctrine. Then*

$$(\mathrm{id}_{\mathcal{C}}, \zeta_P)\colon P^e \longrightarrow P$$

where $\zeta_{PA}\colon P^eA \longrightarrow PA$ sends $(C \xrightarrow{f} A\ ,\ \alpha)$ in $\exists_f(\alpha)$ is a 1-cell. Moreover the assignment

$$\varepsilon\colon \mathrm{EU} \longrightarrow \mathrm{id}_{\boldsymbol{EED}}$$

where $\varepsilon_P = (\mathrm{id}_{\mathcal{C}}, \zeta_P)$, is a 2-natural transformation.

Proof. Suppose ($C_1 \xrightarrow{g_1} A$, α_1) $\leq$ ($C_2 \xrightarrow{g_2} A$, α_2), with $w\colon C_1 \longrightarrow C_2$, $g_2 w = g_1$ and $P_w(\alpha_2) \geq \alpha_1$. Then by Beck-Chevalley we have the equality

$$\exists_{g_2^* g_1} P_{g_1^* g_2}(\alpha_2) = P_{g_1} \exists_{g_2}(\alpha_2)$$

and

$$P_{g_1} \exists_{g_2}(\alpha_2) = P_w P_{g_2} \exists_{g_2}(\alpha_2) \geq P_w(\alpha_2) \geq \alpha_1$$

Then

$$\exists_{g_1}(\alpha_1) \leq \exists_{g_1} \exists_{g_2^* g_1} P_{g_1^* g_2}(\alpha_2) = \exists_{g_2} \exists_{g_1^* g_2} P_{g_1^* g_2}(\alpha_2) \leq \exists_{g_2}(\alpha_2)$$

and $\delta_A = \zeta_A(A \xrightarrow{\mathrm{id}_A} A$, $\top_A)$. Now we prove the naturality of ζ_P. Let $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$. Then the following diagram commutes

$$\begin{tikzcd} P^e B \arrow[r, "P^e_f"] \arrow[d, "\zeta_B"'] & P^e A \arrow[d, "\zeta_A"] \\ PB \arrow[r, "P_f"'] & PA \end{tikzcd}$$

because for every ($C \xrightarrow{g} B$, $\beta \in PC$) we have $\exists_{g^* f} P_{f^* g}(\beta) = P_f \exists_g(\beta)$ by Beck-Chevalley. Moreover it is easy to see that ζ_P preserves left-adjoints. Then we an conclude that for every elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** , ζ_P is a 1-cell of **EED**.
The proof of the naturality of ε is a routine verification. One must use the fact that we are working in **EED**, and then for every 1-cell (F, b), b preserves left-adjoints along the projections.
□

Proposition 4.10. *For every primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** we have*

$$\varepsilon_{P^e} \circ \eta_P{}^e = \mathrm{id}_P$$

Proof. Consider the following diagram

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[d, "\mathrm{id}_{\mathcal{C}}^{op}"'] \arrow[drr, "P^e"] & & \\ \mathcal{C}^{op} \arrow[rr, "(P^e)^e"'] \arrow[d, "\mathrm{id}_{\mathcal{C}}^{op}"'] & \arrow[u, phantom, "\Downarrow \iota^e"] \arrow[d, phantom, "\Downarrow \zeta_{P^e}"] & \textbf{InfSL} \\ \mathcal{C} \arrow[urr, "P^e"'] & & \end{tikzcd}$$

and let $(C \xrightarrow{g} A \ , \ \alpha \in PA)$ be an element of P^eA. Then

$$\iota_{P}{}^{e}_{A}(C \xrightarrow{g} A \ , \ \alpha \in PC) = (A \xrightarrow{\mathrm{id}_A} A \ , \ (C \xrightarrow{g} A \ , \ \alpha \in PC) \in P^eA)$$

and

$$\zeta_{P^e A}(A \xrightarrow{\mathrm{id}_A} A \ , \ (C \xrightarrow{g} A \ , \ \alpha \in PC) \in P^eA) = \exists^e_{\mathrm{id}_A}(C \xrightarrow{g} A \ , \ \alpha \in PC)$$

By definition of $\exists^e$ we have

$$\exists^e_{\mathrm{id}_A}(C \xrightarrow{g} A \ , \ \alpha \in PC) = (C \xrightarrow{g} A \ , \ \alpha \in PC)$$

Then we can conclude that for every $P \colon \mathcal{C}^{op} \longrightarrow$ **InfSL** , we have $\varepsilon_{P^e} \circ \eta_{P}{}^{e} = \mathrm{id}_{P^e}$. $\square$

Corollary 4.11.

$$\varepsilon \mathrm{E} \circ \mathrm{E}\eta = \mathrm{id}_{\mathrm{E}}$$

Proposition 4.12. *For every existential doctrine* $P \colon \mathcal{C}^{op} \longrightarrow$ ***InfSL*** *we have*

$$\varepsilon_P \circ \eta_P = \mathrm{id}_P$$

Proof. One can check it directly. $\square$

Corollary 4.13.

$$\mathrm{U}\varepsilon \circ \eta \mathrm{U} = \mathrm{id}_{\mathrm{U}}$$

Theorem 4.14. *The 2-functor* E *is 2-adjoint to the 2-functor* U.

5 The 2-monad T_e

In this section we construct a 2-monad $\mathrm{T}_e \colon$ **PD** $\longrightarrow$ **PD** , and we prove that every existential doctrine can be seen as an algebra for this 2-monad. Finally we show that the 2-monad T_e is lax-idempotent.
We define:

- $\mathrm{T}_e \colon$ **ED** $\longrightarrow$ **ED** the 2-functor $\mathrm{T} = \mathrm{U} \circ \mathrm{E}$;
- $\eta \colon \ \mathrm{id}_{\mathbf{ED}} \longrightarrow \mathrm{T}_e$ is the 2-natural transformation defined in Proposition 4.8;
- $\mu \colon \mathrm{T}_e^2 \longrightarrow \mathrm{T}_e$ is the 2-natural transformation $\mu = \mathrm{U}\varepsilon\mathrm{E}$;

Proposition 5.1. T_e *is a 2-monad.*

Proof. One can easily check that the following diagrams commute

$$\begin{tikzcd}
\mathrm{T}_e^3 \arrow[r, "\mu\mathrm{T}_e"] \arrow[d, "T_e\mu"'] & \mathrm{T}_e^2 \arrow[d, "\mu"] \\
\mathrm{T}_e^2 \arrow[r, "\mu"'] & \mathrm{T}_e
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{id}_{\mathbf{ED}} \circ \mathrm{T}_e \arrow[r, "\eta\mathrm{T}_e"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}_e^2 \arrow[d, "\mu"] & \mathrm{T}_e \circ \mathrm{id}_{\mathbf{ED}} \arrow[l, "\mathrm{T}_e\eta"'] \arrow[dl, "\mathrm{id}"] \\
& \mathrm{T}_e &
\end{tikzcd}$$

$\square$

Remark 5.2. Observe that $\mathrm{T}_e^2 \cong \mathrm{T}_e$, and μ_P is an isomorphism.

Proposition 5.3. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \textbf{\textit{InfSL}}$ *be an existential doctrine. Then* (P, ε_P) *is a* T_e*-algebra.*

Proof. It is a direct verification. $\square$

Proposition 5.4. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \textbf{\textit{InfSL}}$ *be an primary doctrine, and let* $(P,(F,a))$ *be a* T_e*-algebra. Then* $P\colon \mathcal{C}^{op} \longrightarrow \textbf{\textit{InfSL}}$ *is existential,* $F = \mathrm{id}_{\mathcal{C}}$ *and* $a = \varepsilon_P$*.*

Proof. By the identity axiom for T_e-algebras, we know that F must be the identity functor, and $a_A\iota_A = \mathrm{id}_{PA}$.

$$\begin{tikzcd}
P \arrow[r, "\eta_P"] \arrow[dr, "\mathrm{id}_P"'] & P^e \arrow[d, "(F,a)"] \\
& P
\end{tikzcd}$$

Now for every morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, where f is a projection, we claim that

$$\exists_f(\alpha) := a_B \exists^e_f \iota_A(\alpha)$$

is left adjoint to P_f. Let $\alpha \in PA$ and $\beta \in PB$, and suppose that $\alpha \leq P_f(\beta)$. Then we have that

$$(\, A \xrightarrow{f} B \,,\ \alpha) \leq (\, B \xrightarrow{\mathrm{id}_B} B \,,\ \beta)$$

in P^eB and $(\, A \xrightarrow{f} B \,,\ \alpha) = \exists^e_f(\, A \xrightarrow{\mathrm{id}_A} A \,,\ \alpha)$. Therefore, by definition of ι, we have

$$\exists^e_f \iota_A(\alpha) \leq \iota_B(\beta)$$

and then

$$a_B \exists^e_f \iota_A(\alpha) \leq a_B \iota_B(\beta) = \beta$$

Now suppose that $\exists_f(\alpha) \leq \beta$. Then

$$a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq \beta$$

so

$$P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq P_f(\beta)$$

Now we use the naturality of a, and we have

$$P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) = a_A P^e_f(\ A \xrightarrow{f} B\ ,\ \alpha)$$

Now observe that $P^e_f(\ A \xrightarrow{f} B\ ,\ \alpha) \geq (\ A \xrightarrow{\mathrm{id}_A} A\ ,\ \alpha) = \iota_A(\alpha)$. Therefore we can conclude that

$$\alpha = a_A \iota_A(\alpha) \leq P_f a_B(\ A \xrightarrow{f} B\ ,\ \alpha) \leq P_f(\beta)$$

Now we prove that Bech-Chevalley holds. Consider the following pullback

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A' \arrow[d, "f"] \\ X \arrow[r, "g"'] & A \end{tikzcd}$$

and $\alpha \in PX$. Then we have

$$\exists_{g'} P_{f'}(\alpha) = a_{A'} \exists^e_{g'} \iota_{X'}(P_{f'}\alpha) = a_{A'}(\ X' \xrightarrow{g'} A'\ , P_{f'}(\alpha))$$

Observe that

$$(\ X' \xrightarrow{g'} A'\ , P_{f'}(\alpha)) = P^e_f(\ X \xrightarrow{g} A\ ,\ \alpha)$$

and since a is a natural transformation, we have

$$a_{A'} P^e_f(\ X \xrightarrow{g} A\ ,\ \alpha) = P_f a_A(\ X \xrightarrow{g} A\ ,\ \alpha)$$

Finally we can conclude that Bech-Chevalley holds because

$$P_f \exists_g(\alpha) = P_f a_A \exists^e_g \iota_X(\alpha) = P_f a_A(\ X \xrightarrow{g} A\ ,\ \alpha)$$

and then

$$\exists_{g'} P_{f'}(\alpha) = P_f \exists_g(\alpha)$$

Now consider a projection $f\colon A \longrightarrow B$, an two elements $\beta \in PB$ and $\alpha \in PA$. We want to prove that the Frobenius reciprocity holds.

$$\exists_f(P_f(\beta)\wedge\alpha) = a_B\exists^e_f\iota_A(P_f(\beta)\wedge\alpha) = a_B(\, A \xrightarrow{f} B\ ,\ P_f(\beta)\wedge\alpha)$$

and

$$\beta\wedge\exists_f(\alpha) = a_B\iota_B(\beta)\wedge a_B(\, A \xrightarrow{f} B\ ,\ \alpha)$$

and

$$a_B\iota_B(\beta)\wedge a_B(\, A \xrightarrow{f} B\ ,\ \alpha) = a_B((\, B \xrightarrow{\mathrm{id}_B} B\ ,\ \beta)\wedge(\, A \xrightarrow{f} B\ ,\ \alpha))$$

We can observe that

$$a_B((\, B \xrightarrow{\mathrm{id}_B} B\ ,\ \beta)\wedge(\, A \xrightarrow{f} B\ ,\ \alpha)) = a_B(\, A \xrightarrow{f} B\ ,\ P_f(\beta)\wedge\alpha)$$

Then we can conclude that

$$\exists_f(P_f(\beta)\wedge\alpha) = \beta\wedge\exists_f(\alpha)$$

Therefore the primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is existential. Finally we can observe that

$$a_A(\, C \xrightarrow{g} A\ ,\alpha) = a_A\exists^e_g(\, C \xrightarrow{\mathrm{id}_C} C\ ,\ \alpha) = a_A\exists^e_g\iota_C(\alpha) = \exists_g(\alpha)$$

□

Proposition 5.5. *Let (P,ε_P) and (R,ε_R) be two T_e-algebras. If $(F,b)\colon(P,\varepsilon_P)\longrightarrow(R,\varepsilon_R)$ is a morphism of T_e-algebras, then (F,b) is a 1-cell of **PED**. Moreover every 1-cell of **PED** induces a morphism of T_e algebras.*

Proof. By definition of morphism of T_e-algebras, the following diagram commutes

$$\begin{tikzcd} P^e \arrow[r, "(F{,}b^e)"] \arrow[d, "\varepsilon_P"'] & R^e \arrow[d, "\varepsilon_R"] \\ P \arrow[r, "(F{,}b)"'] & R \end{tikzcd}$$

then for every object $(\, C \xrightarrow{g} A\ ,\ \alpha\in PC)$ of P^eA we have

$$\exists^R_{Fg}b_C(\alpha) = b_A\exists^P_g(\alpha)$$

and this means that for every projection $g\colon C \longrightarrow A$ the following diagram commutes

$$\begin{tikzcd} PC \arrow[r, "\exists^P_g"] \arrow[d, "b_C"'] & PA \arrow[d, "b_A"] \\ RFC \arrow[r, "\exists^R_{Fg}"'] & RFA \end{tikzcd}$$

We can prove the converse using the same arguments. $\square$

Corollary 5.6. *We have the following isomorphism*

$$\mathrm{T_e}\text{-}\boldsymbol{Alg} \cong \boldsymbol{PED}$$

Proof. It follows from Proposition 5.5 and Proposition 5.4. $\square$

Proposition 5.7. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \boldsymbol{InfSL}$ *be a primary doctrine, and let* $(P,(F,a))$ *be a pseudo-*T_e*-algebra. Then* $P\colon \mathcal{C}^{op} \longrightarrow \boldsymbol{InfSL}$ *is existential.*

Proof. Let $(P,(F,a))$ be a pseudo-algebra, then there exists an invertible 2-cell

$$\begin{tikzcd}
P \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_P"'] & P^e \arrow[d, "(F,a)"] \arrow[dl, phantom, "\not\Downarrow a_\eta"] \\
 & P
\end{tikzcd}$$

and by definition, it is a natural transformation $a_\eta\colon F \longrightarrow \mathrm{id}_{\mathcal{C}}$, and for every $A \in \mathcal{C}$ and $\alpha \in PA$ we have $a_A\iota_A(\alpha) = P_{a_{\eta_A}}(\alpha)$.
Now consider a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ and $\alpha \in PA$. We define

$$\exists_f(\alpha) := P_{a_{\eta_A}{}^{-1}} a_B \exists^e_f \iota_A(\alpha)$$

Using the same argument of Proposition 5.4 we can conclude that the elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential. $\square$

Proposition 5.8. $\lambda_P\colon \mathrm{id}_{P^{ee}} \Longrightarrow \eta_{P^e}\mu_P$ *defined as* $\lambda_P := \mathrm{id}_{\mathcal{C}}$ *is a 2-cell in* $\boldsymbol{ED}$*.*

Proof. It is clearly a natural transformation. We must check that for every $\alpha \in (P^e)^e A$

$$\alpha \leq \iota_{P^e A}\zeta_{P^e A}(\alpha)$$

Let $\alpha := (\, C \xrightarrow{g} A\,,\ (\, D \xrightarrow{f} C\,,\ \beta \in PD))$. Then we have

$$\iota_{P^e A}\zeta_{P^e A}(\alpha) = \iota_{P^e A}(\, D \xrightarrow{gf} A\,,\ \beta \in PD) = (\, A \xrightarrow{\mathrm{id}_A} A\,, (\, D \xrightarrow{gf} A\,,\ \beta \in PD))$$

Now we want to prove that

$$P^e_g(\, D \xrightarrow{gf} A\,,\ \beta \in PD) \geq (\, D \xrightarrow{f} C\,,\ \beta \in PD)$$

To see this inequality we can observe that the following diagram commutes, since every square is a pullback

$$
\begin{tikzcd}
D_2 \arrow[drrr, bend left, "f"] \arrow[dr, dashed, "w"] \arrow[ddr, bend right, "\mathrm{id}_D"'] & & & \\
& L \arrow[r, "m_2"] \arrow[d, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[r, "h_2"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "g"] \\
& D \arrow[r, "f"'] & C \arrow[r, "g"'] & A
\end{tikzcd}
$$

and then $P_w(P_{m_1}(\beta)) = \beta$. $\square$

Corollary 5.9. *We can define a modification* $\lambda\colon\ \mathrm{id}_{T_e^2} \longrightarrow \eta T_e \mu$ *, where λ_P is defined as above.*

Theorem 5.10. *The 2-natural transformation μ is left adjoint to $\eta \mathrm{T}_e$, where the unit of the adjunction is λ and the counit is the identity.*

Proof. It follows from the fact that for every $P\colon \mathcal{C}^{op} \longrightarrow \textbf{InfSL}$, the first component of the 1-cells μ_P, $\eta \mathrm{T}_e$ are the identity functor, and since λ_P is the identity natural transformation, when we look at the conditions of adjoint 1-cell in the 2-category **Cat**, we can observe that all the components are identities. $\square$

Corollary 5.11. *The 2-monad* $\mathrm{T}_e\colon \textbf{\textit{PD}} \longrightarrow \textbf{\textit{PD}}$ *is lax-idempotent.*

Proof. It is a direct consequence of [9, Theorem 6.2] and Theorem 5.10 $\square$

Observe that we can prove that the 2-monad T_e is lax-idempotent directly.

Proposition 5.12. *Let (P, ε_P) and (R, ε_R) be T_e algebras, and let* $(F, b)\colon P \longrightarrow R$ *be a 1-cell of* ***PD****. Then $((F, b), \mathrm{id}_F)$ is lax-morphism of algebras, and* $\mathrm{id}_F\colon \varepsilon_R(F, b^e) \Longrightarrow (F, b)\varepsilon_P$ *is the unique 2-cell making $(\mathrm{id}_F, (F, b))$ a lax-morphism.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
P^e \arrow[r, "{(F,b^e)}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow \mathrm{id}_F"] & R^e \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}
$$

We must prove that for every object A of $\mathcal{C}$ and every ($C \xrightarrow{f} A$, α) in $P^e A$

$$\exists^R_{Ff} b_C(\alpha) \leq b_A \exists^P_f(\alpha)$$

but the previous holds if and only if

$$b_C(\alpha) \leq R_{Ff} b_A \exists_f^P(\alpha) = b_C P_f \exists_f^P(\alpha)$$

and this holds since $\alpha \leq P_f \exists_f^P(\alpha)$.
Finally it is easy to see that $\mathrm{id}_F\colon \varepsilon_R(F,b^e) \Longrightarrow (F,b)\varepsilon_P$ satisfies the coherence conditions for lax-T_e-morphisms.
Now suppose there exists another 2-cell $\theta\colon \varepsilon_R(F,b^e) \Longrightarrow (F,b)\varepsilon_P$ such that $((F,b),\theta)$ is a lax-morphism

$$\begin{tikzcd}
P^e \arrow[r, "{(F,b^e)}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow\theta"] & R^e \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}$$

Then it must satisfy the following condition

$$\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[d, "\eta_A"'] & R \arrow[d, "\eta_B"] \\
P^e \arrow[r, "{(F,b^e)}"] \arrow[d, "\varepsilon_P"'] \arrow[dr, phantom, "\Downarrow\theta"] & R^e \arrow[d, "\varepsilon_R"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
P \arrow[r, "{(F,b)}"] \arrow[d, "1_P"'] & R \arrow[d, "1_B"] \\
P \arrow[r, "{(F,b)}"'] & R
\end{tikzcd}$$

and this means that $\theta = \mathrm{id}_F$. $\square$

6 Exact completion for elementary doctrine

It is proved in [18] that there is a bi-adjunction between the category **EED** $\to$ **Xct** given by the composition of the following 2-functors: the first is the left biadjoint to the inclusion of **CEED** into **EED**, see [18, Theorem 3.1]. The second is the biequivalence between **CEED** and the 2-category **LFS** of categories with finite limits and a proper stable factorization system, see [7]. The third is provided in [8], where it is proved that the inclusion of the 2-category **Reg** of regular categories (with exact functors) into **LFS** has a left bi-adjoint. The last functor is the bi-adjoint to the forgetful functor from the 2-category **Xct** into **Reg**, see [5].
In this section we combine these results with the existential completion for elementary doctrine, proving that the completion presented in Section 4 preserves the elementary doctrines, in the sense that if $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is an elementary doctrine, then $P^e\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is an

elementary existential doctrine.
Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider its existential completion $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$. Given two objects A and C of $\mathcal{C}$ we define

$$\exists^e_{\Delta_A\times \mathrm{id}_C}\colon P^e(A\times C)\longrightarrow P^e(A\times A\times C)$$

on $\alpha := (\ A\times C\times D \xrightarrow{\mathrm{pr}} A\times C\ ,\ \alpha\in P(A\times C\times D))$ as

$$\exists^e_{\Delta_A\times \mathrm{id}_C}(\alpha) := (\ A\times A\times C\times D \xrightarrow{\mathrm{pr}} A\times A\times C\ ,\ \exists_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha)\in P(A\times A\times C\times D))$$

Remark 6.1. We can prove that $\exists^e_{\Delta_A\times \mathrm{id}_C}$ is a well defined functor for every A and C: consider two elements of $P^e(A\times C)$

$$\overline{\alpha} := (\ A\times C\times D \xrightarrow{\mathrm{pr}} A\times C\ ,\ \alpha\in P(A\times C\times D))$$

and

$$\overline{\beta} = (\ A\times C\times B \xrightarrow{\mathrm{pr}'} A\times C\ ,\ \beta\in P(A\times C\times B))$$

and suppose that $\overline{\alpha}\leq\overline{\beta}$. By definition there exists a morphism $f\colon A\times C\times D\longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd}
& A\times C\times D \arrow[dl, "\langle \mathrm{pr}_{A\times C}, f\rangle"'] \arrow[d, "\mathrm{pr}_{A\times C}"] \\
A\times C\times B \arrow[r, "\mathrm{pr}'_{A\times C}"'] & A\times C
\end{tikzcd}$$

and $P_{\langle \mathrm{pr}_{A\times C}, f\rangle}(\beta)\geq\alpha$. Since the doctrine $P\colon \mathcal{C}^{op}\longrightarrow \mathbf{InfSL}$ is elementary we have

$$\beta\leq P_{\Delta_A\times \mathrm{id}_{C\times B}}\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta)$$

and then

$$\alpha\leq P_{\langle \mathrm{pr}_{A\times C}, f\rangle}(P_{\Delta_A\times \mathrm{id}_{C\times B}}\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta))$$

Now observe that $(\Delta_A\times \mathrm{id}_{C\times B})(\langle \mathrm{pr}_{A\times C}, f\rangle) = (\langle \mathrm{pr}_{A\times A\times C}, f\,\mathrm{pr}_{A\times C\times D}\rangle)(\Delta_A\times \mathrm{id}_{C\times D})$, and this implies

$$\alpha\leq P_{\Delta_A\times \mathrm{id}_{C\times D}}(P_{\langle \mathrm{pr}_{A\times A\times C}, f\,\mathrm{pr}_{A\times C\times D}\rangle}\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta))$$

Therefore we conclude

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha)\leq P_{\langle \mathrm{pr}_{A\times A\times C}, f\,\mathrm{pr}_{A\times C\times D}\rangle}\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\beta).$$

It is easy to observe that the last inequality implies

$$\exists^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\leq\exists^e_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$$

Proposition 6.2. *With the notation used before the functor*

$$\exists^e_{\Delta_A\times \mathrm{id}_C}\colon P^e(A\times C)\longrightarrow P^e(A\times A\times C)$$

is left adjoint to the functor

$$P^e_{\Delta_A\times \mathrm{id}_C}\colon P^e(A\times A\times C)\longrightarrow P^e(A\times C)$$

Proof. Consider an element $\overline{\alpha}\in P^e(A\times C)$,

$$\overline{\alpha} := (\ A\times C\times B \xrightarrow{\mathrm{pr}} A\times C\ ,\ \alpha\in P(A\times C\times B))$$

and an element $\overline{\beta}\in P^e(A\times A\times C)$,

$$\overline{\beta} := (\ A\times A\times C\times D \xrightarrow{\mathrm{pr}'} A\times A\times C\ ,\ \beta\in P(A\times A\times C\times D))$$

and suppose that

$$\exists^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\leq \overline{\beta}$$

which means that there exists $f\colon A\times A\times C\times B\longrightarrow D$

$$\begin{tikzcd} & A\times A\times C\times B \arrow[dl, "\langle \mathrm{pr}_{A\times A\times C}, f\rangle"'] \arrow[d, "\mathrm{pr}_{A\times A\times C}"] \\ A\times A\times C\times D \arrow[r, "\mathrm{pr}_{A\times A\times C}"'] & A\times A\times C \end{tikzcd}$$

such that $\exists_{\Delta_A\times \mathrm{id}_{C\times B}}(\alpha)\leq P_{\langle \mathrm{pr}_{A\times A\times C}, f\rangle}(\beta)$. Therefore we have

$$\alpha\leq P_{\Delta_A\times \mathrm{id}_{C\times B}}P_{\langle \mathrm{pr}_{A\times A\times C}, f\rangle}(\beta)$$

and since

$$(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B}) = (\Delta_A\times \mathrm{id}_{C\times D})\,\mathrm{pr}_{A\times C\times D}(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B})$$

we can conclude that

$$\alpha\leq P_{\mathrm{pr}_{A\times C\times D}(\langle \mathrm{pr}_{A\times A\times C}, f\rangle)(\Delta_A\times \mathrm{id}_{C\times B})}(P_{\Delta_A\times \mathrm{id}_{C\times D}}(\beta))$$

and then

$$\overline{\alpha}\leq P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$$

because

$$P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\beta}) = (\ A\times C\times D \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\Delta_A\times \mathrm{id}_{C\times D}}(\beta))$$

In the same way we can prove that $\overline{\alpha}\leq P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\beta})$ implies $\exists^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\leq \overline{\beta}$. $\square$

Proposition 6.3. *For every A and C, $\exists^e_{\Delta_A\times \mathrm{id}_C}$ satisfies the Frobenius condition.*

Proof. Consider $\overline{\alpha}\in P^e(A\times A\times C)$,

$$\overline{\alpha} := (\ A\times A\times C\times D \xrightarrow{\mathrm{pr}_{A\times A\times C}} A\times A\times C\ ,\ \alpha\in P(A\times A\times C\times D))$$

and $\overline{\beta}\in P^e(A\times C)$,

$$\overline{\beta} := (\ A\times C\times B \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ \beta\in P(A\times C\times B))$$

We can observe that

$$P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha}) = (\ A\times C\times D \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha))$$

and

$$P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\wedge\beta = (\ A\times C\times D\times B \xrightarrow{\mathrm{pr}_{A\times C}} A\times C\ ,\ P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle}P_{\Delta_A\times \mathrm{id}_{C\times D}}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta))$$

Moreover we can observe that $(\Delta_A\times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle = \mathrm{pr}_{A\times A\times C\times D}(\Delta_A\times \mathrm{id}_{C\times D\times B})$.
Therefore we have

$$\exists^e_{\Delta_A\times \mathrm{id}_C}(P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\wedge\overline{\beta})$$

is equal to

$$(\ A\times A\times C\times D\times B \xrightarrow{\mathrm{pr}} A\times A\times C\ ,\ \exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{(\Delta_A\times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)))$$

Now we can observe that

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{(\Delta_A\times \mathrm{id}_{C\times D})\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_D\rangle}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta))$$

is by definition

$$\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}(P_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha)\wedge P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)).$$

Since the doctrine $P\colon \mathcal{C}^{op}\longrightarrow \mathbf{InfSL}$ is elementary, it holds the Frobenius reciprocity for $\exists$, and therefore the previous is equal to

$$P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha)\wedge \exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta).$$

Then we have that

$$\exists^e_{\Delta_A\times \mathrm{id}_C}(P^e_{\Delta_A\times \mathrm{id}_C}(\overline{\alpha})\wedge\overline{\beta})$$

is equal to

$$(\ A\times A\times C\times D\times B \xrightarrow{\mathrm{pr}_{A\times A\times C}} A\times A\times C\ ,\ P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha)\wedge\exists_{\Delta_A\times \mathrm{id}_{C\times D\times B}}P_{\langle \mathrm{pr}_A,\mathrm{pr}_C,\mathrm{pr}_B\rangle}(\beta)).$$

Now we look for $\overline{\alpha} \wedge \exists^e_{\Delta_A \times \mathrm{id}_C}(\overline{\beta})$. It is straightforward to prove that the previous is equal to

$$(A \times A \times C \times D \times B \xrightarrow{\mathrm{pr}_{A\times A\times C}} A \times A \times C \ , \ P_{\mathrm{pr}_{A\times A\times C\times D}}(\alpha) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle} \exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta))$$

Since $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is elementary we know that

$$\exists_{\Delta_A \times \mathrm{id}_{C\times B}}(\beta) = P_{\langle \mathrm{pr}'_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)$$

where $\mathrm{pr}'_A\colon A \times A \times C \times B \longrightarrow A$ is the projection on the second component. By a direct computation we have

$$P_{\langle \mathrm{pr}_A, \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(P_{\langle \mathrm{pr}'_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)) = P_{\langle \mathrm{pr}'_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)$$

and

$$\exists_{\Delta_A \times \mathrm{id}_{C\times D\times B}}(P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta)) = P_{\langle \mathrm{pr}'_A, \mathrm{pr}_{C\times D\times B}\rangle}(P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta)) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)).$$

It is direct to verify that

$$P_{\langle \mathrm{pr}'_A, \mathrm{pr}_{C\times D\times B}\rangle}(P_{\langle \mathrm{pr}_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta)) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A)) = P_{\langle \mathrm{pr}'_A, \mathrm{pr}_C, \mathrm{pr}_B\rangle}(\beta) \wedge P_{\langle \mathrm{pr}_A, \mathrm{pr}'_A\rangle}(\delta_A).$$

Therefore the Frobenius condition is satisfied. $\square$

Corollary 6.4. *For every elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL**, the doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is elementary and existential.*

We combine the existential completion for elementary doctrines with the completions stated at the begin of this section, obtaining a general version of the exact completion described in [15, 18]. We can summarise this operation with the following diagram

$$\mathbf{PED} \longrightarrow \mathbf{EED} \longrightarrow \mathbf{CEED} \longrightarrow \mathbf{LFS} \longrightarrow \mathbf{Reg} \longrightarrow \mathbf{Xct}.$$

Given an elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL**, the completion **EED** $\rightarrow$ **Xct** produces an exact category denoted by $\mathcal{T}_P$ and this category is defined following the same idea used to define a topos from a tripos. See [6, 15, 18, 22].
We conclude giving a complete description of the exact category $\mathcal{T}_{P^e}$ obtained from an elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL**.

Given an elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL**, consider the category $\mathcal{T}_{P^e}$, called ***exact completion of the elementary doctrine*** P, whose
objects are pair (A, ρ) such that ρ is in $P(A \times A \times C)$ for some C and satisfies:

1. there exists a morphism $f\colon A \times A \times C \longrightarrow C$ such that

$$\rho \leq P_{\langle \mathrm{pr}_2, \mathrm{pr}_1, f\rangle}$$

in $P(A \times A \times C)$ where $\mathrm{pr}_1, \mathrm{pr}_2\colon A \times A \times C \longrightarrow A$;

2. there exists a morphism $g\colon A\times A\times A\times C \longrightarrow C$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_4\rangle}(\rho)\wedge P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,\mathrm{pr}_4\rangle}(\rho)\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,g\rangle}(\rho)$$

where $\mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_3\colon A\times A\times A\times C\longrightarrow A$;

a morphism $\phi\colon (A,\rho)\longrightarrow (B,\sigma)$, where $\rho\in P(A\times A\times C)$ and $\sigma\in P(B\times B\times D)$, is an object ϕ in $P(A\times B\times E)$ for some E such that

1. there exists a morphism $\langle f_1,f_2\rangle\colon A\times B\times E\longrightarrow C\times D$ such that

$$\phi\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_1,f_1\rangle}(\rho)\wedge P_{\langle \mathrm{pr}_2,\mathrm{pr}_2,f_2\rangle}(\sigma)$$

where the pr_i's are the projections from $A\times B\times E$;

2. there exists a morphism $h\colon A\times A\times B\times C\times E\longrightarrow E$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_4\rangle}(\rho)\wedge P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,\mathrm{pr}_5\rangle}(\phi)\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,h\rangle}(\phi)$$

where the pr_i's are the projections from $A\times A\times B\times C\times E$;

3. there exists a morphism $k\colon A\times B\times B\times D\times E\longrightarrow E$ such that

$$P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,\mathrm{pr}_4\rangle}(\sigma)\wedge P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_5\rangle}(\phi)\leq P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,k\rangle}(\phi)$$

where the pr_i's are the projections from $A\times B\times B\times D\times E$;

4. there exists a morphism $l\colon A\times B\times B\times E\longrightarrow D$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_2,\mathrm{pr}_4\rangle}(\phi)\wedge P_{\langle \mathrm{pr}_1,\mathrm{pr}_3,\mathrm{pr}_4\rangle}(\phi)\leq P_{\langle \mathrm{pr}_2,\mathrm{pr}_3,l\rangle}(\sigma)$$

where the pr_i's are the projections from $A\times B\times B\times E$;

5. there exists a morphism $\langle g_1,g_2\rangle\colon A\times C\longrightarrow B\times E$ such that

$$P_{\langle \mathrm{pr}_1,\mathrm{pr}_1,\mathrm{pr}_2\rangle}(\rho)\leq P_{\langle \mathrm{pr}_1,g_1,g_2\rangle}(\phi)$$

where the pr_i's are the projections from $A\times C$.

The composition of two morphisms is defined following the same structure of the tripos to topos.
Therefore we conclude with the following theorem which generalized the exact completion for an elementary existential doctrine to an arbitrary elementary doctrine.

Theorem 6.5. *The 2-functor* ***Xct*** $\to$ ***PED*** *that sends an exact category to the elementary doctrine of its subobjects has a left bi-adjoint which associates the exact category $\mathcal{T}_{P^e}$ to an elementary doctrine* $P\colon \mathcal{C}^{op}\longrightarrow$ ***InfSL*** .

References

[1] Blackwell, R., Kelly, G. M., and Power, J. (1989). Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59:1–41.

[2] Borceux, F. (1994). *Handbook of Categorical Algebra 1: Basic Category Theory*. Encyclopedia of Mathematics and its Applications 51. Cambridge University Press.

[3] Carboni, A. (1995). Some free constructions in realizability and proof theory. *Journal of Pure and Applied Algebra*, 103:117–148.

[4] Carboni, A. and Magno, R. C. (1982). The free exact category on a left exact one. *journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 33(3):295–301.

[5] Carboni, A. and Vitale, E. (1998). Regular and exact completions. *Journal of Pure and Applied Algebra*, 125:79–117.

[6] Hayland, M., Johnstone, P., and Pitts, A. (1980). Tripos theory. *Mathematical Proceedings of the Cambridge Philosophical Society*, 88:205–232.

[7] Hughes, J. and Jacobs, B. (2003). Factorization systems and fibrations: toward a fibered birkhoff variety theorem. *Electronic Notes in Theoretical Computer Science*, 69:156–182.

[8] Kelly, G. (1992). A note on relations relative to a factorisation system. In A. Carboni, I. C. P. and Rosolini, G., editors, *Category Theory '90*, volume 1488, pages 249–261. Springer-Verlag.

[9] Kelly, G. M. and Lack, S. (1997). On property-like structure. *Theory and Applications of Categories*, 3(9):213–250.

[10] Kelly, G. M. and Street, R. (1974). Review of the elements of 2-categories. In *Category Seminar. Lecture Notes in Mathematics*, volume 420, pages 75–103. Springer.

[11] Lawvere, F. (1969a). Adjointness in foundations. *Dialectica*, 23:281–296.

[12] Lawvere, F. (1969b). Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications*, volume 2, pages 134–145. Springer.

[13] Lawvere, F. (1970). Equality in hyperdoctrines and comprehension schema as an adjoint functor. In Heller, A., editor, *New York Symposium on Application of Categorical Algebra*, volume 2, pages 1–14. American Mathematical Society.

[14] Leinster, T. (2003). *Higher Operads, Higher Categories*. London Mathe-matical Society Lecture Notes Series. Cambridge University Press.

[15] Maietti, M., Pasquali, F., and Rosolini, G. (2017). Triposes, exact completions, and hilbert's ε-operator. *Tbilisi Mathematical Journal*, 10(3):141–166.

[16] Maietti, M. E. and Rosolini, G. (2013a). Elementary quotient completion. *Theory and Applications of Categories*, 27(17):445–463.

[17] Maietti, M. E. and Rosolini, G. (2013b). Quotient completion for the foundation of constructive mathematics. *Logica Universalis*, 7(3):371–402.

[18] Maietti, M. E. and Rosolini, G. (2013c). Unifying exact completions. *Applied Categorical Structures*, 23:43–52.

[19] Marmolejo, F. (1999). Distributive laws for pseudo monads. *Theory and Applications of Categories*, 5(5):91–147.

[20] Marmolejo, F. and Wood, R. (2008). Coherence for pseudodistributive laws revisited. *Theory and Applications of Categories*, 20(6):74–84.

[21] Pitts, A. M. (1995). Categorical logic. In S. Abramsky, D. M. G. and Maibaum, T. S. E., editors, *Handbook of Logic in Computer Science*, volume 6, pages 39–129. Oxford University Press.

[22] Pitts, A. M. (2002). Tripos theory in retrospect. *Mathematical Structures in Computer Science*, 12:265–279.

[23] Tanaka, M. (2005). *Pseudo-distributive laws and a unified framework for variable binding*. PhD thesis, University of Edinburgh.

[24] Tanaka, M. and Power, J. (2006a). Pseudo-distributive laws and axiomatics for variable binding. *Higher-Order and Symbolic Computation*, 19:305–337.

[25] Tanaka, M. and Power, J. (2006b). A unified category-theoretic semantics for binding signatures in substructural logics. *Oxford University Press*, 16(1).