Relative Partial Combinatory Algebras over Heyting Categories

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Abstract

A partial combinatory algebra (PCA) is an abstract model of computation that generalizes the classical notion of computability on the set of natural numbers. More precisely, it is a nonempty set equipped with a binary partial operation that satisfies an abstract version of the Smn-theorem. These models can be studied from the point of view of category theory. Every PCA A gives rise to a category of assemblies $\mathsf{Asm}(A)$, which may be viewed as the category of all data types that can be implemented in A. A category of the form $\mathsf{Asm}(A)$ is always a quasitopos, and quasitoposes are closed under slicing. However, categories of assemblies for a PCA are not in general closed under slicing. Therefore, we wish to investigate what categories of the form $\mathsf{Asm}(A)/X$, where X is an assembly, look like.

The answer to this question is provided by W. Stekelenburg's PhD thesis [Ste13], which generalizes the notion of a PCA in two ways. Firstly, the category of sets, which plays a crucial role in the construction of $\mathsf{Asm}(A)$, is replaced by a general Heyting category $\mathcal{H}$. Secondly, the notion of computability is relativized by selecting a set of privileged subobjects of A that count as 'realizing sets'. We therefore call these objects relative PCAs constructed over a Heyting category, or HPCA's for short. The construction of $\mathsf{Asm}(A)$ can be generalized to HPCAs A, and if X is an assembly, then $\mathsf{Asm}(A)/X$ is of the form $\mathsf{Asm}(A')$ for some HPCA A'. We describe this A' explicitly in terms of A and X and use this description to compute a number of examples of categories of the form $\mathsf{Asm}(A)/X$, where A is a PCA.

PCAs are the objects of a preorder-enriched category (see, e.g., [Lon94] and [vO08]). Similarly, HPCAs constructed over a given Heyting category $\mathcal{H}$ can be made into a preorder-enriched category $\mathsf{PCA}_{\mathcal{H}}$. In this talk, we construct a larger 2-category PCA, which is the total category of an opfibration over the category of Heyting categories and whose fiber above a Heyting category $\mathcal{H}$ is precisely $\mathsf{PCA}_{\mathcal{H}}$. We investigate the structure of these categories, showing that PCA has small products, and that each of the fibers $\mathsf{PCA}_{\mathcal{H}}$ has finite (pseudo-)coproducts. Moreover, we extend the construction of $\mathsf{Asm}(A)$ to a 2-functor from PCA into the category of categories. We characterize the image of this 2-functor, thereby generalizing Longley's work from [Lon94].

References

[Lon94] J. Longley. *Realizability Toposes and Language Semantics.* PhD thesis, University of Edinburgh, 1994.

[Ste13] W. P. Stekelenburg. *Realizability Categories.* PhD thesis, Utrecht University, 2013.

[vO08] J. van Oosten. *Realizability: An Introduction to its Categorical Side*, volume 152 of *Studies in Logic and the Foundations of Mathematics.* Elsevier, 2008.