Continuous complete categories enriched over quantales

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Ordered sets are often viewed as thin categories, and on the other hand, categories are regarded as generalized ordered structures. For instance, the complete distributivity [4, 5] and the continuity [1, 2] of categories are investigated, like that of ordered sets.

Form the viewpoint of Lawvere[3], categories enriched over a monoidal closed category, especially over a quantale, can also be regarded as ordered sets in the sense of "quantitative logics". Hence, quantale enriched categories are studied as quantitative ordered sets in the quantitative domain theory.

The main objective of this paper is to contribute to the study of "generalized" Scott's continuous lattices based on quantale enriched categories.

The characterization of continuous posets is concerned with the relation between a poset P and the posets $\mathsf{Idl}(P)$ of ideals of P. For all $p \in P$, $\downarrow p = \{x \in P : x \leq p\}$ defines an embedding

$$\downarrow : P \longrightarrow \mathsf{Idl}(P).$$

A poset P is directed complete if $\downarrow$ has a left adjoint

$$\sup \dashv \downarrow : P \longrightarrow \mathsf{Idl}(P)$$

and a directed complete poset P is continuous if there is a string of adjunctions

$$\twoheaddownarrow \dashv \sup \dashv \downarrow : P \longrightarrow \mathsf{Idl}(P).$$

In a locally small category $\mathcal{E}$, ind-objects, or equivalently, the presheaves generated by ind-objects, play the role of ideals in posets. Let $\mathsf{Ind}\text{-}\mathcal{E}$ be the category of all presheaves of $\mathcal{E}$ generated by ind-objects in $\mathcal{E}$, then $\mathcal{E}$ has small filtered colimits if the Yoneda embedding $\mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Ind}\text{-}\mathcal{E}$ has a left adjoint

$$\mathrm{colim} \dashv \mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Ind}\text{-}\mathcal{E}$$

and it is further continuous if there is a string of adjunctions

$$w \dashv \mathrm{colim} \dashv \mathsf{y} : \mathcal{E} \longrightarrow \mathsf{Ind}\text{-}\mathcal{E}.$$

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In quantale-enriched categories, forward Cauchy nets, or equivalently, presheaves generated by forward Cauchy nets, play the role of ind-objects. For a quantale Q, let $\mathcal{C}A$ be the collection of all presheaves generated by forward Cauchy nets in a Q-category A, then A is Yoneda complete if the Yoneda embedding $\mathbf{y} : A \longrightarrow \mathcal{C}A$ has a left adjoint

$$\sup \dashv \mathsf{y} : A \longrightarrow \mathcal{C}A.$$

Moreover, it is continuous if there is a string of adjunctions

$$t \dashv \sup \dashv \mathsf{y} : A \longrightarrow \mathcal{C}A.$$

In this paper, our main aim is to investigate the relationships between the completely distributivity, the completely co-distributivity and continuity of complete Q-categories. It is shown that, when the quantale Q is the unit interval $[0,1]$ equipped with a continuous triangular norm, the completely co-distributivity always implies the continuity, but completely distributivity implies the continuity if and only if the continuous triangular norm has an almost continuous implication.

References

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