Higher Modules and Directed Identity Types

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Virtual double categories provide an excellent framework for **formal category theory**. For many sorts of "category-like" object, there is a virtual double category of categories, profunctors and transformations between these things. The virtual double categories of interest typically contain canonical profunctors which behave like Hom-profunctors. The universal property characterizing these objects is an abstract version of the Yoneda Lemma. Equivalently, it says that we have **identity types** in a **directed type theory**.

The right adjoint of the functor which forgets such identity types is the monoids and modules construction on virtual double categories. Such constructions are ubiquitous throughout category theory and allow us to construct new virtual double categories of "category-like" objects. Ordinary, enriched and internal categories, together with the corresponding notions of profunctors and transformations, can be defined succinctly as the result of applying the monoids and modules construction to simpler virtual double categories. (See [3].) In this way we can construct all the data we need to "do category theory".

Our contribution is the extension of this discussion to **formal higher category theory**. Globular multicategories are to ω-categories what virtual double categories are to categories. Identity types can be defined in this setting. We call the right adjoint of the functor which forgets these identity types the **higher modules construction**. This allows us to construct notions of higher categories, higher profunctors, profunctors between profunctors, ... and transformations between these things.

At first we obtain strict notions of all these things. However, the Batanin-Leinster approach (see [1], [5], [4]) to weakening globular operads can be extended to this setting and in this way we obtain weak notions of all these objects. There is also a notion of composition of weak transformations.

Thus, globular multicategories should provide a good setting for "doing higher category theory". As a first result, we observe that, discarding everything except the trivial modules, we obtain a weak ω-category of weak ω-categories in the sense of Batanin. Globular multicategories should also provide a natural semantics for (directed) type theories with identity types. In fact [2] can be seen as a first result in this direction.

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References

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