A General Framework for Categorical Semantics of Type Theory

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Dybjer [4] introduced *categories with families* as a notion of a model of basic dependent type theory. Extending categories with families, one can define notions of models of dependent type theories such as Martin-Löf type theory [5], two-level type theory [1] and cubical type theory [3]. The way to define a model of a dependent type theory is by adding algebraic operations corresponding to type and term constructors, and it is a kind of routine. However, as far as the author knows, there are no general notions of a "type theory" and a "model of a type theory" that include all of these examples. In this talk, we propose abstract notions of a type theory and a model of a type theory to unify semantics of type theories based on categories with families.

Steve Awodey [2] pointed out that a category with families is the same thing as a *representable map* of presheaves and that type and term constructors are modeled by algebraic operations on presheaves. Inspired by this work, we define a type theory to be a category equipped with a class of morphisms called representable morphisms and a model of a type theory to be a functor into a presheaf category that carries representable morphisms to representable maps.

With these definitions, we establish basic properties of the semantics of type theory. We give a simple and uniform way to construct the *bi-initial model* of a type theory. We give a formal definition of the *internal language* of a model of a type theory $\mathbb{T}$, yielding a 2-functor from the 2-category of models of $\mathbb{T}$ to a suitable (locally discrete) 2-category of *theories over* $\mathbb{T}$. This 2-functor has a left bi-adjoint and induces a bi-equivalence between the 2-category of theories over $\mathbb{T}$ and a full sub-2-category of the 2-category of models of $\mathbb{T}$.

References

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