Codensity monads and D-ultrafilters

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Let $\mathcal{A}$ be a small full subcategory of a complete category $\mathcal{K}$. $\mathcal{A}$ is said to be codense in $\mathcal{K}$ if every object X of $\mathcal{K}$ is the limit of the canonical diagram $X/\mathcal{A} \to \mathcal{K}$ sending every $X \xrightarrow{a} A$ to its codomain. Recently, Tom Leinster has drawn attention to codensity monads [2]. As he observed, the codensity monad of the inclusion $E : \mathcal{A} \hookrightarrow \mathcal{K}$, which is the identity precisely when $\mathcal{A}$ is codense in $\mathcal{K}$, may be regarded as a measure of how 'far away' $\mathcal{A}$ is from being codense in $\mathcal{K}$.

Let $\mathcal{K}$ have a cogenerator D contained in $\mathcal{A}$. We describe the codensity monad of $E : \mathcal{A} \hookrightarrow \mathcal{K}$ as a submonad of the monad $\mathbb{S}$ induced by the adjunction $\mathcal{K}(-, D) \dashv D^{-} : \mathsf{Set}^{\mathrm{op}} \to \mathcal{K}$, in terms of an intersection of subobjects of SX, for every $X \in \mathcal{K}$. In the case where $\mathcal{K}$ is a symmetric, closed monoidal category, the codensity monad is characterized as a certain submonad of the double-dualization monad given by $(-)^{**} = [[-, D], D]$.

It is known that the codensity monad of the embedding of finite sets into the category of sets is the ultrafilter monad [1], and the codensity monad of the embedding of finite-dimensional vector spaces into the category of vector spaces over a field is the double-dualization monad [2]. We introduce the concept of D-ultrafilter on an object and prove that the codensity monad assigns to every object an object representing all D-ultrafilters on it. By taking the expected cogenerator D, we obtain the D-ultrafilters for several examples of embeddings of the full subcategory of finitely presentable objects into a locally finitely presentable category, including posets, semilattices and graphs. For the embedding of all finite sober spaces into the category of topological spaces the codensity monad is the prime filter monad.

References

[1] J. F. Kennison and D. Gildenhuys, Equational completion, model induced triples and pro-objects, *J. Pure Appl. Algebra* 1 (1971), 317-346.

[2] Tom Leinster, Codensity and the ultrafilter monad, *Theory Appl. Categories* 28 (2012), 332-370.

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