Hopf-Frobenius algebras

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In the monoidal categories approach to quantum theory [1, 6] Hopf algebras [14] have a central role in the formulation of complementary observables [5]. In this setting, a quantum observable is represented as special commutative †-Frobenius algebra; a pair of such observables are called *strongly complementary* if the algebra part of the first and the coalgebra part of the second jointly form a Hopf algebra. In abstract form, this combination of structures has been studied under the name "interacting Frobenius algebras" [8] where it is shown that relatively weak commutation rules between the two Frobenius algebras produce the Hopf algebra structure. From a different starting point Bonchi et al [3] showed that a distributive law between two Hopf algebras yields a pair of Frobenius structures, an approach which has been generalised to provide a model of Petri nets [2]. Given the similarity of the two structures it is appropriate to consider both as exemplars of a common family of *Hopf-Frobenius algebras*.

In the above settings, the algebras considered are both commutative and cocommutative. However more general Hopf algebras, perhaps not even symmetric, are a ubiquitous structure in mathematical physics, finding application in gauge theory [12], condensed matter theory [13], quantum field theory [4] and quantum gravity [11]. We take the first steps towards generalising the concept of Hopf-Frobenius algebra to the non-commutative case, and opening the door to applications of categorical quantum theory in other areas of physics.

Loosely speaking, a Hopf-Frobenius algebra consists of two monoids and two comonoids such that one way of pairing a monoid with a comonoid gives two Frobenius algebras, and the other pairing yields two Hopf algebras, with the additional condition that antipodes are constructed from the Frobenius forms. Fundamental to the concept of a Hopf-Frobenius algebra is a particular pair of morphisms called an integral and a cointegral. We show that when these morphisms are 'compatible' in a particular sense, they produce structure similar to a Hopf-Frobenius algebra. It is from this that we produce necessary and sufficient conditions to extend a Hopf algebra to a Hopf-Frobenius algebra in a symmetric monoidal category. It was previously known that in $\mathbf{FVect_k}$, the category of finite dimensional vector spaces, every Hopf algebra carries a Frobenius algebra on both its monoid [10] and its comonoid [7, 9]; in fact we show that every Hopf algebra in $\mathbf{FVect_k}$ is Hopf-Frobenius. We are therefore able to find many examples of Hopf-Frobenius algebras that are not commutative or cocommutative. Finally, due to the fact that every Frobenius algebra is self dual, in a compact closed category we may find a natural isomorphism between the algebra and its dual. We use this isomorphism to construct a Hopf algebra on $H \otimes H$ that is isomorphic to the Drinfeld double.

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