What is a monoid?

Paul Blain Levy, University of Birmingham

May 2, 2019

In many situations one encounters a notion that resembles that of a monoid. It consists of a carrier and two operations that resemble a unit and a multiplication, subject to three equations that resemble associativity and left and right unital laws. The question then arises whether this notion in fact that of a monoid in a suitable sense.

Category theorists have answered this question by providing a notion of monoid in a monoidal category. In many of the above situations, one chooses an appropriate monoidal category $\mathcal{C}$ and then the notion of interest is precisely that of monoid in $\mathcal{C}$. But sometimes the desired $\mathcal{C}$ is not a monoidal category but some kind of "generalized monoidal category".

For example, $\mathcal{C}$ may be a multicategory, where a morphism goes from a list of objects to an object. Hermida [2] showed that a multicategory with tensors corresponds to a monoidal category, but sometimes the multicategory we want does not have tensors, e.g. for size reasons. And sometimes the tensors exist but are complicated.

In other cases, $\mathcal{C}$ may be a left-skew monoidal category or a right-skew monoidal category, in the sense of Szlanchanyi [3]. Here—by contrast with the definition of monoidal category—the associator and unitors are not required to be isomorphisms. Bourke and Lack [1] have further generalized left-skew monoidal category to left-skew multicategory, and (symmetrically) right-skew monoidal category to right-skew multicategory.

However, there are situations where even these are not general enough. One example is the notion of "staggered category" that arose in studying a fragment of call-by-push-value (which is a kind of λ-calculus). In this talk we give a notion of *bi-skew multicategory* that subsumes both left-skew and right-skew multicategory. Then we define a notion of monoid in a bi-skew multicategory. By choosing appropriate bi-skew multicategories, we recover the notion of staggered category (with fixed object structure) and other examples.

References

[1] John Bourke and Stephen Lack. Skew monoidal categories and skew multicategories. *Journal of Algebra*, 506:237 – 266, 2018.

[2] Claudio Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164 – 225, 2000.

[3] Kornl Szlachnyi. Skew-monoidal categories and bialgebroids. *Advances in Mathematics*, 231(3):1694 – 1730, 2012.