# Internal language and classified theories of toposes in algebraic geometry

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September 2018

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# 1 Introduction

Topos theory originated from the desire to compute the cohomology of schemes in algebraic geometry. Only after that it was noticed that toposes carry a rich structure providing an internal language and that they "are logical theories" just as much as they "are spaces".

The theory of classifying toposes was started in [5], when the notion of a geometric theory was not developed yet. The initial insight being that, similarly to classifying spaces in algebraic topology, some structures in toposes correspond to geometric morphisms into a special classifying topos for that sort of structures. The first example, given in [5], was that of the big Zariski topos, which classifies local rings.

In [9], there is a remark that many toposes from algebraic geometry should be classifying toposes of reasonable geometric theories. However, not much of this vision seems to have been developed since. We give an answer to the question about the classified theory for the big infinitesimal topos. For this, we extensively utilize the concept of a theory of presheaf type, applying different theorems from [3].

# 2 Geometric theories

**Definition 1.** A *geometric theory* consists of a set of *sorts*, a set of *function symbols*, each equipped with a finite (possibly empty) list $A_1, \ldots, A_n$ of *input sorts* and one *output sort* $B$, denoted

$$f : A_1, \ldots, A_n \to B,$$

furthermore a set of *relation symbols*, each equipped with a finite (possibly empty) list of sorts $A_1, \ldots, A_n$, denoted

$$R \rightarrowtail A_1, \ldots, A_n,$$

and finally a set of *axioms*, each a sequent of the form

$$\phi \vdash_{x_1:A_1,\ldots,x_n:A_n} \psi,$$

where $\phi$ and $\psi$ are geometric formulas in the context $x_1 : A_1, \ldots, x_n : A_n$ over the *signature* given by the sorts, function symbols and relation symbols of the theory.

We don't go into the details of defining *terms* and *formulas* here (see [6, Section D.1.1] instead) but just want to mention that the *geometric formulas* are those (infinitary) first-order formulas which only use the logical connectives

$$\top, \bot, \wedge, \vee, \bigvee, \exists,$$

but not $\Rightarrow, \bigwedge, \forall$.

Also, we don't give a complete definition of a *model* of a theory $T$ in a topos $\mathcal{E}$. (And we will not need the notion of a model in more general categories.) But recall that if we are given a structure $M$ consisting of an object $A_M$ for every sort $A$, a morphism $f_M : A_1 \times \ldots \times A_n \to B_M$ for every function symbol and a subobject $R_M \hookrightarrow A_1 \times \ldots \times A_n$ for every relation symbol, then we can inductively (over the structure of the formula) define the *interpretation* of a formula $\phi$ in a context $x_1 : A_1, \ldots, x_n : A_n$, which is a subobject

$$[\![\phi]\!]_M \hookrightarrow M_{A_1} \times \ldots \times M_{A_n}.$$

Then a sequent $\phi \vdash_C \psi$ is *fulfilled* if $[\![\phi]\!]_M \leq [\![\psi]\!]_M$ as subobjects and $M$ is a *model* of $T$ if all axioms are fulfilled for $M$. Finally, a homomorphism of such structures $M$, $N$ for the same signature (this does not depend on the axioms of $T$) consists of morphisms $A_M \to A_N$ for each sort $A$, which are compatible with the morphisms assigned to function symbols and the subobjects assigned to relation symbols. For details see [6, Section D1.2].

We will of course need the following Soundness Theorem. For the notion of provability we refer to [6, Section D.1.3].

**Proposition 2** (Soundness Theorem)**.** *Let $M$ be a model of a geometric theory $T$ in a topos $\mathcal{E}$ (or in any geometric category). Then any geometric sequent which is provable in $T$ is also fulfilled in $M$.*

*Proof.* See [6, Proposition 1.3.2]. □

**Definition 3.**

- A geometric theory $T$ is *algebraic* if it has no relation symbols and all axioms of $T$ are of the form

  $$\top \vdash_C s = t$$

  where $s$ and $t$ are terms in the context $C$ and $C$ contains no other variables than those occuring in $s$ or in $t$.

- A geometric theory $T$ is *cartesian* if the only logical connectives occuring in its axioms (left and right of the turnstile $\vdash$) are $\top, \wedge, \exists$ and the axioms of $T$ can be given a well-founded partial ordering such that every occurence of existential quantification $(\exists x : A.\ \phi)$ in an axiom $\sigma$ can be shown to refer to unique existence ($\phi \wedge \phi[y/x] \vdash_{C,x,y} x = y$, where $C$ is the context in which $(\exists x : A.\ \phi)$ occured) relative to the axioms preceding $\sigma$.

Of course, any algebraic theory is cartesian.

**Definition 4.** A *quotient* of a geometric theory $T$ is another geometric theory $T'$ over the same signature such that all axioms of $T$ are provable in $T'$.

# 3 Classifying toposes and Morita equivalence

For any topos $\mathcal{E}$ and geometric theory $T$, the models of $T$ in $\mathcal{E}$ form a category which we simply denote by $T(\mathcal{E})$. Moreover, any geometric morphism $f : \mathcal{E}' \to \mathcal{E}$ yields a functor

$$f^* : T(\mathcal{E}) \to T(\mathcal{E}')$$

by applying the inverse image part $f^*$ of $f$ to the objects, morphisms and subobjects (for sorts, function symbols and relation symbols) of which a model consists.

Here it is crucial that we have restricted our attention to geometric theories instead of all first-order theories. For example, the interpretation of infinitary conjunction would have been intersection of infinitely many subobjects. This is an infinite limit, but $f^*$ is only required to preserve finite limits. However, $f^*$ preserves arbitrary colimits (since it is a left adjoint) and also the epi-mono factorization of morphisms (since this can be characterized using finite limits and colimits), which is all that is needed to construct infinite unions of subobjects, so $f^*$ does preserve the interpretation of infinite disjuctions.

We denote by $\mathrm{Geom}(\mathcal{E}', \mathcal{E})$ the category of geometric morphisms between toposes $\mathcal{E}'$ and $\mathcal{E}$.

**Definition 5.** A *classifying topos* for a geometric theory $T$ is a Grothendieck topos $\mathrm{Set}[T]$ such that for every Grothendieck topos $\mathcal{E}$ there is an equivalence of categories

$$\mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T]) \simeq T(\mathcal{E})$$

and this equivalence is natural in $\mathcal{E}$, meaning that for every geometric morphism $f : \mathcal{E}' \to \mathcal{E}$ the diagram of functors

$$\begin{array}{ccc} \mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T]) & \xrightarrow{\simeq} & T(\mathcal{E}) \\ \downarrow{\scriptstyle \cdot\circ f} & & \downarrow{\scriptstyle f^*} \\ \mathrm{Geom}(\mathcal{E}', \mathrm{Set}[T]) & \xrightarrow{\simeq} & T(\mathcal{E}') \end{array}$$

commutes up to isomorphism.

The model corresponding to the identity functor on Set$[T]$ under the equivalence

$$\mathrm{Geom}(\mathrm{Set}[T], \mathrm{Set}[T]) \simeq T(\mathrm{Set}[T])$$

is then called a *universal model* of $T$ and denoted $U_T$. And indeed, the equivalence $\mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T]) \simeq T(\mathcal{E})$ is then (up to isomorphism) given by $f \mapsto f^* U_T$, as can be seen by setting Set$[T]$ for $\mathcal{E}$ in the naturality square. So in particular, any model $M$ of $T$ in any topos is (up to isomorphism) the pullback of $U_T$ along a unique (up to isomorphism) geometric morphism to Set$[T]$.

We next ask the question when to consider two theories $T$ and $T'$ "essentially the same". Of course, if $T$ and $T'$ share the same signature, then we can simply ask if the axioms of each one can be deduced from the axioms of the other. (That is, if each is a quotient of the other.) Then (by Proposition 2) they also have the same models in any topos. But we also want to compare theories with different signatures. The appropriate general notion of "having the same models in all toposes" is given by Definition 6.

**Definition 6.** Two geometric theories $T$ and $T'$ are *Morita-equivalent* if for every Grothendieck topos $\mathcal{E}$ there is an equivalence of categories

$$T(\mathcal{E}) \simeq T'(\mathcal{E}),$$

and this equivalence is natural in $\mathcal{E}$, meaning that for every geometric morphism $f : \mathcal{E}' \to \mathcal{E}$ the diagram of functors

$$\begin{tikzcd} T(\mathcal{E}) \arrow[r, "\simeq"] \arrow[d, "f^*"] & T'(\mathcal{E}) \arrow[d, "f^*"] \\ T(\mathcal{E}') \arrow[r, "\simeq"] & T'(\mathcal{E}') \end{tikzcd}$$

commutes up to isomorphism.

**Lemma 7.** *Given two theories $T$, $T'$ and classifying toposes* Set$[T]$, Set$[T']$, *$T$ and $T'$ are Morita-equivalent if and only if* Set$[T]$ *and* Set$[T']$ *are equivalent.*

*Proof.* If $\mathrm{Set}[T] \simeq \mathrm{Set}[T']$ we have

$$T[\mathcal{E}] \simeq \mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T]) \simeq \mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T']) \simeq T'(\mathcal{E})$$

natural in $\mathcal{E}$. Conversely, if $\phi_{\mathcal{E}} : T(\mathcal{E}) \xrightarrow{\simeq} T'(\mathcal{E})$ is a Morita equivalence, we obtain geometric morphisms $f : \mathrm{Set}[T] \to \mathrm{Set}[T']$ and $g : \mathrm{Set}[T'] \to \mathrm{Set}[T]$

corresponding to the models $\phi_{\mathrm{Set}[T]}(U_T) \in T'(\mathrm{Set}[T])$ and $\phi^{-1}_{\mathrm{Set}[T']}(U_{T'}) \in T(\mathrm{Set}[T'])$. Then $f$ and $g$ are quasi-inverses by the various naturalities, one half provided by the following diagram.

$$\begin{tikzcd}
\mathrm{Geom}(\mathrm{Set}[T], \mathrm{Set}[T']) \arrow[r, "\simeq"] \arrow[d, "\cdot \circ g"] & T'(\mathrm{Set}[T]) \arrow[d, "g^*"] & T(\mathrm{Set}[T]) \arrow[l, "\simeq", "\phi"'] \arrow[d, "g^*"] \\
\mathrm{Geom}(\mathrm{Set}[T'], \mathrm{Set}[T']) \arrow[r, "\simeq"] & T'(\mathrm{Set}[T']) & T(\mathrm{Set}[T']) \arrow[l, "\simeq", "\phi"']
\end{tikzcd}$$

This is of course an application of a higher dimensional Yoneda lemma. □

# 4 Definition of the relevant theories

**Definition 8.** The *theory of rings*, denoted Ring, consists of a single sort $A$, two constant symbols $0, 1 : A$, two binary function symbols $+, \cdot : A, A \to A$, one unary function symbol $- : A \to A$ and axioms expressing that $(A, +, 0, -)$ is an abelian group

$$\top \vdash_{x:A} 0 + x = x, \qquad \top \vdash_{x,y:A} x + y = y + x,$$
$$\top \vdash_{x,y,z:A} (x + y) + z = x + (y + z), \qquad \top \vdash_{x:A} x + (-x) = 0,$$

$(A, \cdot, 1)$ is a commutative monoid

$$\top \vdash_{x:A} 1 \cdot x = x, \qquad \top \vdash_{x,y:A} x \cdot y = y \cdot x,$$
$$\top \vdash_{x,y,z:A} (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

and multiplication is distributive over addition

$$\top \vdash_{x,y,z:A} x \cdot (y + z) = (x \cdot y) + (x \cdot z).$$

*Remark* 9. We could leave out the function symbol for negation $- : A \to A$ and interpret "abelian group" to mean the axiom

$$\top \vdash_{x:A} \exists y : A.\, x + y = 0.$$

The existential quantifier here can easily be shown to refer to a unique existence, so this theory is not algebraic but it is still a cartesian theory. It is indeed Morita-equivalent to the theory given above: If we have a function symbol for negation, the axiom asserting the existence of negatives of course holds. So we only have to show from the existence axiom that there is a unique morphism $- : A \to A$ fulfilling $\top \vdash_{x:A} x + (-x) = 0$. For this, we observe that the formula $x + y = 0$ is provably functional (from the context

$x : A$ to the context $y : A$), meaning that in addition to the existence axiom we can also prove (using the other axioms of the theory of rings)

$$x + y = 0 \wedge x + y' = 0 \ \vdash_{x,y,y':A} \ y = y'.$$

From this we can conclude by [6, Proposition D1.3.12] that there indeed exists a unique morphism $- : A \to A$ as above. One can also check that negation is respected by any homomorphism of models of the theory without negation symbol, completing the Morita equivalence.

However, we prefer to think about the theory of rings as an algebraic theory and therefore included the function symbol for negation. In [7, Section X.3] this issue seems to have been overlooked, as it is stated that the theory of rings could be formulated with only $0, 1, +, \cdot$ and algebraic axioms.

**Definition 10.** Let $K$ be a ring. The *theory of $K$-algebras*, denoted $K$-Alg, consists of one sort $A$, all the function symbols (including constant symbols) and axioms of the theory of rings and additionally one constant symbol $c_\lambda : A$ for every element $\lambda \in K$, together with the axioms

$$\top \ \vdash_{[]} \ c_0 = 0, \qquad \top \ \vdash_{[]} \ c_1 = 1,$$
$$\top \ \vdash_{[]} \ c_\lambda + c_\mu = c_{\lambda+\mu}, \qquad \top \ \vdash_{[]} \ c_\lambda \cdot c_\mu = c_{\lambda\cdot\mu},$$

the latter two for all $\lambda, \mu \in K$, expressing the properties of a ring homomorphism from $K$ to $A$.

**Definition 11.** Let $K$ be a ring.

- The theory $K$-AlgQuot of *$K$-algebras with an ideal* consists of one sort $A$, all function symbols and axioms of $K$-Alg, one unary relation symbol $\mathfrak{a} \rightarrowtail A$ and axioms expressing that $\mathfrak{a}$ is an ideal (we denote the "application" of $\mathfrak{a}$ to a term $t$ as $t \in \mathfrak{a}$)

$$\top \ \vdash_{[]} \ 0 \in \mathfrak{a}$$
$$x \in \mathfrak{a} \wedge y \in \mathfrak{a} \ \vdash_{x,y:A} \ x + y \in \mathfrak{a}$$
$$x \in \mathfrak{a} \ \vdash_{x,y:A} \ x \cdot y \in \mathfrak{a}.$$

- The theory $K$-AlgNilQuot of *$K$-algebras with a nil ideal* is the quotient of $K$-AlgQuot with one additional axiom

$$x \in \mathfrak{a} \ \vdash_{x:A} \ \bigvee_{n \in \mathbb{N}_{\geq 0}} x^n = 0.$$

  Here, $x^n$ of course stands for the term $(\ldots((1 \cdot x) \cdot x) \ldots) \cdot x$ with $x$ occuring $n$ times.

An alternative way to think about a $K$-algebra $A$ together with an ideal is a surjective $K$-algebra homomorphism from $A$ to some other $K$-algebra (therefore the name $K$-AlgQuot). We will see in Proposition 13 below that this indeed leads to a Morita-equivalent theory. But first we give definitions of such theories involving a $K$-algebra homomorphism, generalized by additional structure on the codomain.

**Definition 12.** Let $K$ be a ring and $R$ be a $K$-algebra.

- The theory $K$-Alg-$R$-Alg of *$K$-algebra homomorphisms into an $R$-algebra* consists of two sorts $A$ and $B$, all function symbols and axioms of $K$-Alg for the sort $A$, all function symbols and axioms of $R$-Alg for the sort $B$ (we will denote the two distinct funtions symbols $+ : A, A \to A$ and $+ : B, B \to B$ of the theory in the same way, and similarly for all other funtion symbols mentioned so far), one additional function symbol $f : A \to B$ and axioms expressing that $f$ is a $K$-algebra homomorphism

$$\top \vdash_{x,y:A} f(x + y) = f(x) + f(y), \qquad \top \vdash_{x,y:A} f(x \cdot y) = f(x) \cdot f(y),$$
$$\top \vdash_{[]} f(c_\lambda) = c_\lambda,$$

  the last one for every $\lambda \in K$ (and reading $\lambda$ as an element of $R$ on the right side of the equation, such that $c_\lambda : B$).

- The theory $K$-Alg-$R$-Quot of *surjective $K$-algebra homomorphisms into an $R$-algebra* is the quotient of $K$-Alg-$R$-Alg with the additional axiom

$$\top \vdash_{y:B} \exists x : A.\ f(x) = y.$$

- The theory $K$-Alg-$R$-NilQuot of *surjective $K$-algebra homomorphisms into an $R$-algebra with nil kernel* is the quotient of $K$-Alg-$R$-Quot with the additional axiom

$$f(x) = 0 \vdash_{x:A} \bigvee_{n \in \mathbb{N}_{\geq 0}} x^n = 0.$$

**Proposition 13.** *The theories* $K$-Alg-$K$-Quot *(surjective $K$-algebra homomorphisms) and* $K$-AlgQuot *($K$-algebras with an ideal) are Morita-equivalent.*

*Proof.* Let $\mathcal{E}$ be a Grothendieck topos. We give explicit constructions to convert models of $K$-Alg-$K$-Quot and $K$-AlgQuot into each other, i. e. for the equivalence

$$K\text{-Alg-}K\text{-Quot}(\mathcal{E}) \simeq K\text{-AlgQuot}(\mathcal{E}).$$

First let $(f : A \twoheadrightarrow B) \in K\text{-Alg-}K\text{-Quot}(\mathcal{E})$. Then we take as subobject $\mathfrak{a}$ of $A$ the interpretation of the formula $f(x) = 0$ (in the context $x : A$), that is, the equalizer

$$\mathfrak{a} \hookrightarrow A \overset{f}{\underset{0}{\rightrightarrows}} B$$

(where the arrow annotated 0 is the composite $A \to 1 \xrightarrow{0} B$ over the terminal object 1). It is easy to show that the predicate $f(x) = 0$ fulfills the axioms for an ideal, so we have obtained a model $\mathfrak{a} \triangleleft A$ of $K$-AlgQuot.

If $\mathfrak{a} \triangleleft A$ is a given model of $K$-AlgQuot, we have to construct an epimorphism $f : A \twoheadrightarrow B$ (because the surjectivity axiom translates exactly to being an epimorphism) and also provide a $K$-algebra structure on $B$. For $f$ we take the coequalizer

$$\mathfrak{a} \times A \overset{+}{\underset{\pi_2}{\rightrightarrows}} A \overset{f}{\twoheadrightarrow} B.$$

The constants $c_\lambda$ for $\lambda \in K$ are easily defined for $B$, we just have to compose with $f$. For the other function symbols like $+ : B \times B \to B$, a bit more work is needed. One has to show that $f \circ + : A \times A \to B$ factors over $f \times f : A \times A \to B \times B$, mimicking the usual algebraic calculations needed to prove the ring operations on a quotient ring well-defined. The axioms for the $K$-algebra structure on $B$ follow then immediately from the same axioms for $A$ and the epimorphism $f$ commuting with the various operations. For example, the axiom $0 + x = x$ means that the horizontal composites in the diagram below are the identity on $A$ respectively $B$.

$$\begin{tikzcd}
A \arrow[r, "(0{,}\mathrm{id}_A)"] \arrow[d, two heads, "f"] & A \times A \arrow[r, "+"] \arrow[d, two heads, "f\times f"] & A \arrow[d, two heads, "f"] \\
B \arrow[r, "(0{,}\mathrm{id}_B)"] & B \times B \arrow[r, "+"] & B
\end{tikzcd}$$

Having defined the two parts of the equivalence on objects, one checks that for two such "extended models" $\mathfrak{a} \triangleleft A \twoheadrightarrow B$ and $\mathfrak{a}' \triangleleft A' \twoheadrightarrow B'$, a $K$-algebra homomorphism $A \to A'$ indeed allows a compatible map $B \to B'$ if and only if it sends $\mathfrak{a}$ into $\mathfrak{a}'$, that is, allows a compatible map $\mathfrak{a} \to \mathfrak{a}'$. (Here one can reason internally again, only giving a formula for the graph of the desired map, and then apply [6, Proposition D1.3.12] to obtain an actual morphism.) Finally, the above constructions are natural in $\mathcal{E}$, that is, they are preserved by the inverse image parts of geometric morphisms, as we have only used finite limits and colimits. $\square$

**Corollary 14.** *The theories* $K$-Alg-$K$-NilQuot *and* $K$-AlgNilQuot *are Morita-equivalent.*

*Proof.* These theories differ from $K$-Alg-$K$-Quot, respectively $K$-AlgQuot, by a nilpotence axiom, where the premise is $f(x) = 0$ in the first case and $x \in \mathfrak{a}$ in the second. But the subobjects defined by these formulas correspond to each other under the constructions in the proof of Proposition 13. So the two nilpotence axioms define "the same" full subcategory of $K$-Alg-$K$-Quot$(\mathcal{E}) \cong K$-AlgQuot$(\mathcal{E})$. $\square$

**Definition 15.**

- The *theory of local rings* is the quotient of the theory of rings with the additional axioms
$$x_1 + \ldots + x_n = 1 \ \vdash_{x_1,\ldots,x_n:A} \ \bigvee_{i=1,\ldots,n} \exists y : A.\ x_i \cdot y = 1$$
for all $n \in \mathbb{N}_{\geq 0}$.
- The above locality axioms can be added to any of the theories defined earlier as they all contain a sort $A$ with a ring structure. Thus we obtain in particular loc-$K$-Alg (the *theory of local $K$-algebras*), loc-$K$-AlgNilQuot and loc-$K$-Alg-$R$-NilQuot.

*Remark* 16. The (countably many) locality axioms given in Definition 15 can be replaced by two axioms

$$0 = 1 \ \vdash_{[]} \ \bot, \qquad x_1 + x_2 = 1 \ \vdash_{x_1,x_2:A} \ (\exists y : A.\ x_1 \cdot y = 1) \vee (\exists y : A.\ x_2 \cdot y = 1).$$

Indeed, these are special cases of the axioms from Definition 15 for $n = 0$ and $n = 2$. The case $n = 1$ is trivially fulfilled (i. e. provable from the theory of rings) and all other instances can be seen to follow from these by induction.

*Remark* 17. In a context of classical logic, local rings are usually defined as those with a unique maximal ideal. Using the Axiom of Choice this is equivalent to our definition: Our axioms say that the non-invertible elements form an ideal (as they are in any case closed under multiplication with arbitrary ring elements), which is then of course maximal. Conversely, any non-invertible element is contained in some maximal ideal by Zorn's lemma, so if there is only one maximal ideal it must contain precisely all non-invertibles.

For the theories with two ring structures we could also (or instead) have added locality axioms for $B$. However, this would have been redundant (or equivalent) in the case of $K$-Alg-$R$-NilQuot as we can conclude from the following lemma.

**Lemma 18.** *Let $f : A \twoheadrightarrow B$ be a surjective ring homomorphism with kernel a nil ideal $\mathfrak{a} \lhd A$. Then $A$ is a local ring if and only $B$ is a local ring. And this holds intuitionistically (with "local" understood as in Definition 15), so it holds in any geometric theory with an appropriate ring homomorphism.*

*Proof.* Let $A$ be local and $b_1 + \ldots + b_n = 1$ in $B$. Then there are preimages $a_1, \ldots a_n \in A$ and $a_1 + \ldots + a_n + \epsilon = 1$ for some $\epsilon \in \mathfrak{a}$. Since $A$ is local, one of the $a_i$ or $\epsilon$ must be invertible. But $\epsilon$ is nilpotent, so if it is invertible, $0 = 1$ holds in $A$, which by the locality axiom for $n = 0$ entails $\perp$.

Conversely, let $B$ be local and $a_1 + \ldots + a_n = 1$ in $A$. Then $f(a_1) + \ldots + f(a_n) = 1$ in $B$ and by locality of $B$, some $f(a_i)$ is invertible, $f(a_i) \cdot s = 1$. For $t \in A$ with $f(t) = s$ we obtain $a_i \cdot t + \epsilon = 1$ for some $\epsilon \in \mathfrak{a}$. Now let $k \in \mathbb{N}$ with $\epsilon^k = 0$, then calculate $1 = (\epsilon + a_i \cdot t)^k = \epsilon^k + a_i \cdot t \cdot (\ldots) = a_i \cdot t \cdot (\ldots)$, so $a_i$ is invertible.

This proof can clearly be carried out in any geometric theory with a surjective ring homomorphism $f : A \to B$ with nil kernel. □

*Remark* 19. The ring homomorphism $f : A \to B$ is then automatically local (if $f(x)$ is invertible, then $x$ is invertible), as we have seen in the second part of the proof (for $x = a_i$).

# 5 Theories of presheaf type

In this section we will introduce the abstract machinery which we intend to use later. The central notion is that of a geometric theory being *of presheaf type*. It turns out that theories of presheaf type allow a very convenient description of a classyfing topos and also provide access to quotients of the theory. For all proofs, we refer to the recent book [3].

**Definition 20.** A geometric theory $T$ is *of presheaf type* if there is a small category $C$ such that the presheaf topos $[C, \mathrm{Set}]$ is a classifying topos for $T$.

**Theorem 21.** *Every cartesian theory is of presheaf type.*

*Proof.* See [3, Theorem 2.1.8]. □

By Theorem 21 it is now clear that the theories Ring, $K$-Alg, $K$-AlgQuot and $K$-Alg-$R$-Alg are of presheaf type, since they are all cartesian. Ring, $K$-Alg and $K$-Alg-$R$-Alg are even algebraic, while $K$-AlgQuot is a *Horn theory*: its axioms only use finite conjunctions (including $\top$), no existential quantification at all. However, the exitential quantifier in the axioms of $K$-Alg-$R$-Quot can not be shown to refer to unique existence, so this is not a cartesian theory. Only in the special case $R = K$ we have already seen that the theory gets Morita-equivalent to $K$-AlgQuot and is therefore of presheaf type. $K$-AlgNilQuot and $K$-Alg-$R$-NilQuot even contain infinitary disjunctions and thus are not cartesian. We will answer the question whether they are still of presheaf type later.

The main reason why we are interested in theories of presheaf type is, that the classifying topos for any such theory has a very useful *canonical* site of definition, namely the dual of the category of finitely presentable models of the theory. We first recall the notions of compact object and filtered category.

**Definition 22.** A category $C$ is *filtered* if it has cocones on all finite diagrams. Equivalently, if the following three conditions are satisfied.

- There is at least one object $c \in C$.
- For any two objects $c_1, c_2 \in C$, there is an object $c' \in C$ and some arrows $c_1 \to c'$, $c_2 \to c'$.
- For any two parallel arrows $f, g : c \to c'$, there is an object $c'' \in C$ and an arrow $h : c' \to c''$ such that $h \circ f = h \circ g$.

A *filtered colimit* is one such that the index category is a filtered small category.

**Definition 23.** Let $C$ be a locally small category with all filtered colimits. An object $c$ of $C$ is *compact* if the corepresentable functor $\mathrm{Hom}(c, \cdot\,) : C \to \mathrm{Set}$ preserves filtered colimits. That is, if for every filtered small category $I$ and functor $F : I \to C$ the canonical map

$$\operatorname*{colim}_{i \in I} \mathrm{Hom}(c, F(i)) \to \mathrm{Hom}(c, \operatorname*{colim}_{i \in I} F(i))$$

is a bijection. We will denote the full subcategory on the compact objects of $C$ as $C_c$.

If $C$ happens to be the cagegory $T(\mathrm{Set})$ of Set-based models of a geometric theory $T$, then the compact objects of $C = T(\mathrm{Set})$ are called the *finitely presentable* models of the theory $T$.

**Theorem 24.** *Let $T$ be a theory of presheaf type. Let $T(\mathrm{Set})_c$ be the category of its finitely presentable models (that is, the full subcategory of $T(\mathrm{Set})$ on the compact objects). Then $[T(\mathrm{Set})_c, \mathrm{Set}]$ is a classifying topos for $T$.*

*Proof.* See [3, Section 6.1.1]. □

*Remark* 25. The category $T(\mathrm{Set})$ is usually not small, but $T(\mathrm{Set})_c$ can be shown to be essentially small. So what we meant by the functor category $[T(\mathrm{Set})_c, \mathrm{Set}]$ is to replace $T(\mathrm{Set})_c$ by a small skeleton first.

*Remark* 26. Theorem 24 implies that the classification of theories up to Morita equivalence can, for theories of presheaf type, be achieved by computing their compact Set-based models: If $T$ and $T'$ are Morita-equivalent, of course $T(\mathrm{Set})_c$ and $T'(\mathrm{Set})_c$ must be equivalent (because compactness is a categorical property). The theorem gives the converse implication in case $T$ and $T'$ are both of presheaf type.

In general, the Set-based models do not suffice to distinguish geometric theories. Indeed, there are geometric theories which have no models in Set at all, but are consistent and do have models in other topoi, so they are not Morita-equivalent to an inconsistent theory, which has no models in any topos.

The next question after knowing the classifying topos of a theory is of course to identify the universal model in that topos, so here we go.

**Theorem 27.** *In the situation of Theorem 24, a universal model of the theory $T$ is given by $N_T$ as follows. For a sort $A$, $A_{N_T}$ is the functor $T(\mathrm{Set})_c \to \mathrm{Set}$ given by $M \mapsto A_M$. For a function symbol $f$, $f_{N_T}$ is the natural transformation which assigns to an object $M \in T(\mathrm{Set})_c$ the map $f_M$. And for a relation Symbol $R \rightarrowtail A_1, \ldots, A_n$, $R_{N_T}$ evaluated at $M$ is the subset $R_M \subseteq A_{1M} \times \ldots \times A_{nM}$.*

*Proof.* See [3, Theorem 6.1.1]. □

*Remark* 28. There are weaker (and earlier) versions of Theorems 24 and 27, like [6, Corollary D3.1.2], which assumes $T$ to be a cartesian theory. While this is sufficient in many cases, we will need the theorems in the full generality stated here in the last section.

Note that we deviate from the notation in [3], where the category of finitely presentable models of $T$ is denoted f.p.$T$-mod(Set). This is because we like to emphasize that the notion of compactness, which distinguishes the finitely presentable models, is a categorical one and does not depend on the notion of a model. Also, this reduces the risk of confusion concerning another, quite similar term, namely that of a *finitely presented* model. It is a property of theories of presheaf type that for them the finitely presented models are exactly the finitely presentable, i. e. compact ones, as we will state in Theorem 30 below. For the same reasons we will also prefer the term "compact model" over the term "finitely presentable model", especially if it is not clear whether the theory in question is of presheaf type.

**Definition 29.** Let $T$ be a geometric theory. A Set-based model $M$ of $T$ is *(finitely) presented* by a geometric formula $\phi$ if it corepresents the functor

$$T(\mathrm{Set}) \to \mathrm{Set}, \quad N \mapsto [\![\phi]\!]_N$$

sending a model to the interpretation of $\phi$ in that model.

This can be spelled out as follows. Let $x_1 : A_1, \ldots, x_n : A_n$ be the context of the formula $\phi$. Then an element of $[\![\phi]\!]_N$ is given by elements $a_1 \in A_{1N}, \ldots, a_n \in A_{nN}$ such that $\phi$ holds for $a_1, \ldots, a_n$. So $M$ is presented by $\phi$ if and only if there are elements $a_i \in A_{iM}$ (the *generators* of M) such that $\phi$ holds for the $a_i$ and, for any model $N \in T(\mathrm{Set})$ and elements $b_i \in A_{iN}$ for which $\phi$ holds, there is a unique homomorphism $f : M \to N$ such that $f(a_i) = b_i$.

**Theorem 30.** *Let $T$ be a theory of presheaf type and $M$ a* Set*-based model of $T$. Then $M$ is compact as an object of $T(\mathrm{Set})$ if and only if $M$ is presented by some geometric formula.*

*Proof.* See [3, Corollary 6.1.15]. □

The final tool to be introduced here concernes quotients of a theory of presheaf type. If $T'$ is a quotient of a geometric theory $T$, the category $T'(\mathcal{E})$ of models of $T$ is a full subcategory of $T(\mathcal{E})$ for any Grothendieck topos $\mathcal{E}$. In terms of classifying toposes this means that $\mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T'])$ must be a full subcategory of $\mathrm{Geom}(\mathcal{E}, \mathrm{Set}[T])$. This is the case if $\mathrm{Set}[T']$ is a subtopos of $\mathrm{Set}[T]$, and indeed, there is a direct correspondence between quotients of a geometric theory $T$ and subtoposes of $\mathrm{Set}[T]$. For the details see [3, Theorem 3.2.5].

Now, if we have a Grothendieck topos $\mathrm{Sh}(C, J)$, the subtoposes of $\mathrm{Sh}(C, J)$ correspond to those Grothendieck topologies on $C$ which contain $J$ (and therefore "select" a smaller full subcategory of $[C^{\mathrm{op}}, \mathrm{Set}]$ than $\mathrm{Sh}(C, J)$). So we would like to know, for a theory $T$ of presheaf type, how to describe the Gothendieck topology on ${T(\mathrm{Set})_c}^{\mathrm{op}}$ corresponding to a quotient of the theory $T$.

For this purpose, we need to explain what it means for a formula to present a homomorphism of models (instead of a model). Let $\phi$ and $\psi$ be formulas presenting models $M_\phi, M_\psi \in T(\mathrm{Set})_c$. Let furthermore $\theta$ be a formula in the context $\vec{x}, \vec{y}$ where $\vec{x}$ (respectively $\vec{y}$) is the context of $\phi$ (respectively $\psi$). Assume that $\theta$ is *provably functional* from $\psi$ to $\phi$, meaning that the following sequents are provable in $T$.

$$\theta \vdash_{\vec{y},\vec{x}} \phi \wedge \psi$$
$$\psi \vdash_{\vec{y}} \exists \vec{x}.\, \theta$$
$$\theta \wedge \theta[\vec{\tilde{x}}/\vec{x}] \vdash_{\vec{y},\vec{x},\vec{\tilde{x}}} \vec{x} = \vec{\tilde{x}}$$

Then for any model $N \in T(\mathrm{Set})$, the interpretation $[\![\theta]\!]_N \subseteq [\![\psi]\!]_N \times [\![\phi]\!]_N$ is the graph of a map $[\![\psi]\!]_N \to [\![\phi]\!]_N$. In particular, if $\vec{a} \in [\![\psi]\!]_{M_\psi}$ is the tuple of generators of $M_\psi$, then there is a unique element $\vec{b} \in [\![\phi]\!]_{M_\psi}$ with $(\vec{a}, \vec{b}) \in [\![\theta]\!]_{M_\psi}$. And since $\phi$ presents $M_\phi$, there is a unique model homomorphism $s_\theta : M_\phi \to M_\psi$ sending the generators of $M_\phi$ to $\vec{b}$. This arrow $s_\theta$ is the arrow *presented* by the provably functional formula $\theta$.

**Theorem 31.** *Let $T$ be a theory of presheaf type. Let $\phi_i$, $i \in I$ be geometric formulas (in contexts $\vec{x}_i$) presenting models $M_i \in T(\mathrm{Set})_c$. For each $i \in I$, let $\psi_i^j, \theta_i^j$, $j \in J_i$ be geometric formulas, where each $\psi_i^j$ (in context $\vec{y}_i^j$) presents a model $M_i^j \in T(\mathrm{Set})_c$ and each $\theta_i^j$ is provably functional from $\psi_i^j$ to $\phi_i$. Finally, let $T'$ be the quotient of $T$ with the additional axioms*

$$\phi_i \vdash_{\vec{x}_i} \bigvee_{j \in J_i} \exists \vec{y}_i^j . \, \theta_i^j, \qquad i \in I.$$

*Then the Grothendieck topology on ${T(\mathrm{Set})_c}^{\mathrm{op}}$ induced by $T'$ is generated by the sieves $S_i$, $i \in I$, where $S_i$ is the dual of the cosieve on $M_i$ generated by the arrows $s_i^j : M_i \to M_i^j$ presented by $\theta_i^j$.*

*Proof.* See [3, Theorem 8.1.10]. □

# 6 Compact models of the relevant theories

Because compact models play such a central role for theories of presheaf type, we try to identify them for as many of the theories defined earlier as possible. Of course, we have not shown all of those theories to be of presheaf type; this problem will be handled later as needed. We start with a very useful lemma.

**Lemma 32.** *Finite colimits of compact objects are compact.*

*Proof.* Let $G : J \to C$ be a finite diagram in a locally small category $C$ with all filtered colimits such that all $G(j)$ for $j \in J$ are compact and assume that the colimit $\mathrm{colim}_{j \in J} G(j)$ exists. Then for any functor $F : I \to C$ with $I$ small and filtered we can calculate

$$\begin{aligned} \operatorname*{colim}_{i \in I} \mathrm{Hom}(\operatorname*{colim}_{j \in J} G(j), F(i)) &\cong \operatorname*{colim}_{i \in I} \lim_{j \in J} \mathrm{Hom}(G(j), F(i)) \\ &\cong \lim_{j \in J} \operatorname*{colim}_{i \in I} \mathrm{Hom}(G(j), F(i)) \\ &\cong \lim_{j \in J} \mathrm{Hom}(G(j), \operatorname*{colim}_{i \in I} F(i)) \\ &\cong \mathrm{Hom}(\operatorname*{colim}_{j \in J} G(j), \operatorname*{colim}_{i \in I} F(i)) \end{aligned}$$

because filtered colimits commute with finite limits in Set. And one can check that this composition of bijections is indeed the canonical map of Definition 23. □

To determine the compact objects of $K$-Alg(Set) we first have to describe filtered colimits in that category.

**Lemma 33.** *The forgetful functor* $U : K\text{-Alg(Set)} \to \text{Set}$ *creates filtered colimits. Explicitly, for any diagram* $(A_i)_{i\in I}$ *in* $K$-Alg(Set) *with* $I$ *small and filtered, there is a unique* $K$*-algebra structure on the set* $A := \operatorname{colim}_{i\in I} U(A_i)$ *making the canonical maps* $U(A_i) \to A$ *into* $K$*-algebra homomorphisms, and the resulting cocone in* $K$-Alg(Set) *is a colimit.*

*Proof.* Any two elements $a, a' \in A$ are the images of elements in some $A_i$ for a common $i \in I$ since $I$ is filtered. So we are forced to define their sum $a + a'$ as the sum in $A_i$ for $A_i \to A$ to have a chance to be a homomorphism. And this is well-defined as we can compare the results obtained in $A_i$ and in $A_j$ in some further $A_k$ with $i \to k \leftarrow j$ chosen carefully such that the two representatives of $a$ in $A_k$ are equal and likewise for $a'$. This works exactly the same for the product of two elements and also for functions of any other arity than two, including arity zero for constants (scalars from $K$) where we use that $I$ is inhabited to find a "common" $i \in I$ for the zero elements to be "combined".

The algebraic (equational) axioms for a $K$-algebra are fulfilled because each of them involves only finitely many elements and therefore every instance can be checked in some $A_i$.

For another $K$-algebra $A'$ and compatible homomorphisms $A_i \to A'$ we only have to check that the induced map of sets $A \to A'$ is again a homomorphism. But this again follows immediatly from the fact that sum, product and scalars form $K$ all involve only finitely many elements which can be assumed to lie in a commom $A_i$. □

*Remark* 34. The proof of Lemma 33 would clearly have worked for any single-sorted algebraic theory. For multi-sorted algebraic thories we would have to consider the forgetful functor to $\text{Set}^n$ instead where $n$ is the number of sorts.

**Definition 35.** Let $K$ be a ring.

- A $K$-algebra $A$ is *finitely generated* if there is a surjective $K$-algebra homomorphism
$$K[X_1, \ldots, X_n] \twoheadrightarrow A$$
for some $n \geq 0$.

- A $K$-algebra $A$ is *finitely presented* if it is isomorphic to one of the form
$$K[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$$
for some $n, m \geq 0$ and elements $f_1, \ldots, f_m \in K[X_1, \ldots, X_n]$.

**Proposition 36.** *Let $K$ be a ring. The compact objects of $K$-*Alg(Set) *are precisely the finitely presented $K$-algebras.*

The proof given here is inspired by the proof of [1, Theorem 3.12], which is a much more general statement.

*Proof.* By Lemma 33 the forgetful functor $K\text{-Alg(Set)} \to \text{Set}$ in particular preserves filtered colimits. But this functor is corepresented by the object $K[X]$, so $K[X]$ is compact. And if $A = K[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ is any finitely presented $K$-algebra, we can build $A$ from $K[X]$ using only finite colimits: The $n$-fold coproduct (tensor product) of $K[X]$ with itself is $K[X_1, \ldots, X_n]$ and then we obtain $A$ as the coequalizer

$$K[X_1, \ldots, X_m] \underset{X_i \mapsto 0}{\overset{X_i \mapsto f_i}{\rightrightarrows}} K[X_1, \ldots, X_n] \longrightarrow K[X_1, \ldots, X_n]/(f_1, \ldots, f_m).$$

Now let $A$ be a compact object of $K$-Alg(Set). We first show that $A$ is finitely generated as a $K$-algebra. Let $I$ be the set of all finitely generated sub-$K$-algebras of $A$, partially ordered by inclusion. Then $I$ is filtered as a category, i. e. a directed system. (The third condition from Definition 22 is trivially satisfied for partial orders.) Indeed, we have an object $\operatorname{im}(K \to A) \subseteq A$ (generated by 0 elements) and any two finitely generated subalgebras $A', A'' \subseteq A$ with $K[X_1, \ldots, X_n] \twoheadrightarrow A'$ and $K[Y_1, \ldots, Y_m] \twoheadrightarrow A''$ are contained in $\operatorname{im}(K[X_1, \ldots, X_n, Y_1, \ldots Y_m] \to A) \subseteq A$.

The $K$-algebras $A' \in I$ constitute a diagram in $K$-Alg(Set) and the inclusions $A' \hookrightarrow A$ form a cocone. We claim that the induced homomorphism

$$\operatorname*{colim}_{A' \in I} A' \to A$$

is an isomorphism: It is surjective since every $a \in A$ lies in the subalgebra $\operatorname{im}(K[X] \xrightarrow{X \mapsto a} A) \subseteq A$ finitely generated by the single element $a$. And it is injective since all the cocone arrows $A' \hookrightarrow A$ are injective and any two elements $a' \in A'$, $a'' \in A''$ can be mapped within $I$ into a common finitely generated subalgebra of $A$.

We can now apply the hypothesis that $A$ is compact to the map

$$\operatorname*{colim}_{A' \in I} \operatorname{Hom}(A, A') \to \operatorname{Hom}(A, \operatorname*{colim}_{A' \in I} A') \cong \operatorname{Hom}(A, A)$$

to obtain a preimage of $\mathrm{id}_A$. But this is a section $A \to A'$ of the inclusion $A' \hookrightarrow A$ of some $A' \in I$, showing that $A' = A$ and therefore that $A$ is finitely generated.

Now let $q : K[X_1, \ldots X_n] \twoheadrightarrow A$ be a fixed surjective $K$-algebra homomorphism. We show that the kernel of $q$ is a finitely generated ideal, completing the proof of the proposition. To this end, let $J$ be the partial order of all finitely generated ideals of $K[X_1, \ldots, X_n]$ contained in $\ker q$. This is again a directed system: The zero ideal always lies in $J$ and for two finitely generated ideals $(f_1, \ldots, f_m), (g_1, \ldots, g_k) \subseteq \ker q$ we also have $(f_1, \ldots, f_m, g_1, \ldots, g_k) \subseteq \ker q$. The assignment $\mathfrak{a} \mapsto K[X_1, \ldots, X_n]/\mathfrak{a}$ defines a $J$-shaped diagram in $K$-Alg(Set) and the quotient maps $K[X_1, \ldots, X_n]/\mathfrak{a} \to K[X_1, \ldots, X_n]/\ker q \cong A$ induce a homomorphism

$$\operatorname*{colim}_{\mathfrak{a} \in J} K[X_1, \ldots, X_n]/\mathfrak{a} \to A.$$

This is again an isomorphism: It is surjective since already $K[X_1, \ldots, X_n]/(0) \to A$ is surjective and injective since any $b \in K[X_1, \ldots, X_n]/\mathfrak{a}$ which becomes zero in $A$ already becomes zero in $K[X_1, \ldots, X_n]/(\mathfrak{a} + (b))$.

Using the compactness of $A$ a second time, we obtain a preimage of $\mathrm{id}_A$ under the map

$$\operatorname*{colim}_{\mathfrak{a} \in J} \mathrm{Hom}(A, K[X_1, \ldots, X_n]/\mathfrak{a}) \to \mathrm{Hom}(A, A).$$

This means for some $B := K[X_1, \ldots, X_n]/\mathfrak{a}$ with $\mathfrak{a} \in J$ we have a section $s : A \to B$ of the quotient map $q_B : B \to A$. We don't have $s \circ q_B = \mathrm{id}_B$ yet, but for any $B' = K[X_1, \ldots, X_n]/\mathfrak{b}$ with $\mathfrak{b} \in J$, $\mathfrak{a} \subseteq \mathfrak{b}$ we still have $q_{B'} \circ r \circ s = \mathrm{id}_A$ for the quotient maps $q_{B'} : B' \to A$, $r : B \to B'$ as in the diagram below.

$$\begin{tikzcd}
B \arrow[r, two heads, "q_B"'] \arrow[d, two heads, "r"'] & A \arrow[l, hook', bend right, "s"'] \\
B' \arrow[ur, "q_{B'}"'] &
\end{tikzcd}$$

So our aim is to find $B'$ such that $r \circ s \circ q_{B'} = \mathrm{id}_{B'}$. For this we remember that $B$ is a finitely presented $K$-algebra, so by the first part of this proof it is compact and the map

$$\operatorname*{colim}_{\mathfrak{b} \in J} \mathrm{Hom}(B, K[X_1, \ldots, X_n]/\mathfrak{b}) \to \mathrm{Hom}(B, A)$$

is bijective. This time we use injectivity rather than surjectivity: $s \circ q_B : B \to B$ and $\mathrm{id}_B$ considered as elements of the left side become equal on the right side: $q_B \circ s \circ q_B = q_B = q_B \circ \mathrm{id}_B$. Therefore they must be equal already on the left side, so there exists $B'$ as above with $r \circ s \circ q_B = r$. To see $r \circ s \circ q_{B'} = \mathrm{id}_{B'}$ we precompose with the surjection $r$: $r \circ s \circ q_{B'} \circ r = r \circ s \circ q_B = r$. Thus we have $B' \cong A$ and $A$ is finitely presented. $\square$

*Remark* 37. The last part of the proof showed that the kernel of any surjective homomorphism $K[X_1, \ldots, X_n] \twoheadrightarrow A$ is finitely generated if $A$ is compact. Combining this with the statement of the lemma, any surjection $K[X_1, \ldots, X_n] \twoheadrightarrow A$ with $A$ a finitely presented $K$-algebra has finitely generated kernel.

We now quickly illustrate Theorem 30, which said that every compact model is presentable by some geometric formula. For example, $K$ is the initial object of $K$-Alg(Set), so it is presented by the formula $\top$ in the empty context. If we consider $\top$ as a formula in the context $x_1, \ldots, x_n$, this presents $K[X_1, \ldots, X_n]$ instead: for elements $a_1, \ldots, a_n$ in any $K$-algebra A (fulfilling $\top$, i. e. no additional condition), there is a unique $K$-algebra homomorphism $K[X_1, \ldots, X_n] \to A$ sending $X_i$ to $a_i$. And for a general finitely generated $K$-algebra $K[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$ we find that it is presented by the formula $(f_1 = 0) \wedge \ldots \wedge (f_m = 0)$ in the context $x_1, \ldots, x_n : A$. (Here, $f_i$ has to be read as a term built from the variables $x_1, \ldots, x_n$ and the function symbols of $K$-Alg.) We could have tried to identify the compact models of $K$-Alg in this way, but observe that the formulas which occured here were of an especially simple form (finite conjunctions of equations). For a more complex geometric formula it might not be so easy to determine if there even exists a model presented by that formula. (In the case of a theory of presheaf type, the *irreducible* formulas suffice to present all compact models, see [3, Theorem 6.1.13] for more.)

**Lemma 38.** *Let $C$, $D$ be locally small categories with all filtered colimits. Let $F : C \to D$ be a functor that has a right adjoint $G : D \to C$ and assume that $G$ preserves filtered colimits. Then $F$ preserves compact objects.*

*Proof.* Let $c \in C$ be compact and let $(d_i)_{i \in I}$ be a diagram in $D$ with $I$ a filtered small category. Then we have

$$\begin{aligned} \mathrm{Hom}(F(c), \operatorname*{colim}_{i \in I} d_i) &\cong \mathrm{Hom}(c, G(\operatorname*{colim}_{i \in I} d_i)) \\ &\cong \mathrm{Hom}(c, \operatorname*{colim}_{i \in I} G(d_i)) \\ &\cong \operatorname*{colim}_{i \in I} \mathrm{Hom}(c, G(d_i)) \\ &\cong \operatorname*{colim}_{i \in I} \mathrm{Hom}(F(c), d_i). \end{aligned}$$

□

**Lemma 39.** *There are four adjoint functors between $K$-AlgQuot(Set) and $K$-Alg(Set) as follows.*

$K$-AlgQuot(Set)

$$
\begin{tikzcd}
\mathfrak{a} \lhd A \arrow[d, mapsto] & & (0) \lhd A' & & \mathfrak{a} \lhd A \arrow[d, mapsto] & & (1) \lhd A' \\
A/\mathfrak{a} \arrow[rr, phantom, "\dashv"] & & A' \arrow[u, mapsto] \arrow[rr, phantom, "\dashv"] & & A \arrow[rr, phantom, "\dashv"] & & A' \arrow[u, mapsto]
\end{tikzcd}
$$

$$K\text{-Alg(Set)}$$

*Proof.* This is easy to check. □

**Lemma 40.** *The forgetful functor $K$-AlgQuot(Set) $\to$ $K$-Alg(Set), $(\mathfrak{a} \lhd A) \mapsto A$ creates filtered colimits. Explicitly, for any diagram $(\mathfrak{a}_i \lhd A_i)_{i\in I}$ in $K$-AlgQuot(Set) with $I$ small and filtered the colimit $A := \mathrm{colim}_{i\in I} A_i$ computed in $K$-Alg(Set) becomes a colimit in $K$-AlgQuot(Set) by taking as ideal $\mathfrak{a} \lhd A$ the union of the images of all $\mathfrak{a}_i$.*

*Proof.* The union of the images of $\mathfrak{a}_i$ in A is indeed an ideal: The sum of elements from $\mathfrak{a}_i$ and $\mathfrak{a}_j$ can be computed in any $A_k$ with $i \to k \leftarrow j$, where they both lie in $\mathfrak{a}_k$. And for any object $(\mathfrak{a}' \lhd A') \in K$-AlgQuot(Set), and compatible morphisms $f_i : (\mathfrak{a}_i \lhd A_i) \to (\mathfrak{a}' \lhd A')$ inducing $f : A \to A'$ we have $f(\mathfrak{a}) \subseteq \mathfrak{a}'$ because any element of $\mathfrak{a}$ is represented by an element of some $\mathfrak{a}_i$ and $f_i(\mathfrak{a}_i) \subseteq \mathfrak{a}'$. □

**Lemma 41.** *An object $(\mathfrak{a} \lhd A) \in K$-AlgQuot is compact if and only if $A$ is a finitely presented $K$-algebra and $\mathfrak{a}$ is a finitely generated ideal.*

*Proof.* The functor $(\mathfrak{a} \lhd A) \mapsto A/\mathfrak{a}$ from Lemma 39 preserves compact objects by Lemma 38 because is right adjoint $A' \mapsto ((0) \lhd A')$ has a further right adjoint, so it even preserves all colimits. The functor $(\mathfrak{a} \lhd A) \mapsto A$ also preserves compact objects, because its right adjoint $A' \mapsto ((1) \lhd A')$ preserves filtered colimits by the description of filtered colimits in $K$-AlgQuot(Set) given in Lemma 40. (Note that this functor does not preserve all colimits as it does not preserve the initial object, which is $K$ in $K$-Alg(Set) and $(0) \lhd K$ in $K$-AlgQuot(Set).) So if $\mathfrak{a} \lhd A$ is compact, then both $A$ and $A/\mathfrak{a}$ are finitely presented $K$-algebras. To see that $\mathfrak{a}$ is finitely generated, observe that the kernel of a composite

$$K[X_1, \dots, X_n] \twoheadrightarrow A \twoheadrightarrow A/\mathfrak{a}$$

is finitely generated by Remark 37 and its image in $A$ is $\mathfrak{a}$. This completes the "only if" part.

For the "if" part we start by observing that the composite forgetful functor

$$K\text{-AlgQuot(Set)} \to K\text{-Alg}(Set) \to \text{Set}, \quad (\mathfrak{a} \lhd A) \mapsto A$$

preserves filtered colimits, so the object $(0) \lhd K[X]$ corepresenting it is compact. Just like in the proof of Lemma 33 we can build fom $(0) \lhd K[X]$ any $(0) \lhd A$ with $A$ finitely presented using only finite colimits. Next we see that also the functor

$$K\text{-AlgQuot(Set)} \to \text{Set}, \quad (\mathfrak{a} \lhd A) \mapsto A$$

preserves filtered colimits, by the description in Lemma 40. This functor is corepresented by $(X) \lhd K[X]$, which is therefore compact. And also, as a finite coproduct, $(X_1, \ldots, X_n) \lhd K[X_1, \ldots, X_n]$. Now for any finitely generated ideal $(f_1, \ldots, f_n) \lhd A$ we conclude that the pushout

$$\begin{tikzcd}
(0) \lhd K[X_1, \ldots, X_n] \arrow[r, "X_i \mapsto f_i"] \arrow[d] & (0) \lhd A \arrow[d] \\
(X_1, \ldots, X_n) \lhd K[X_1, \ldots, X_n] \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & (f_1, \ldots, f_n) \lhd A
\end{tikzcd}$$

is compact. □

**Proposition 42.** *An object* $(\mathfrak{a} \lhd A) \in K\text{-AlgNilQuot(Set)}$ *is compact if and only if it is compact as an object of* $K$-AlgQuot(Set)*, that is, if and only if* $A$ *is a finitely presented* $K$*-algebra and* $\mathfrak{a}$ *is finitely generated.*

*Proof.* $K$-AlgNilQuot(Set) is a full subcategory of $K$-AlgQuot(Set) closed under filtered colimits (again by the description in Lemma 40). Therefore, if $\mathfrak{a} \lhd A$ is compact in $K$-AlgQuot(Set), it is in particular compact in $K$-AlgNilQuot(Set).

For the converse, we consider adjoint functors similar to those from Lemma 39:

$$\begin{array}{c}
K\text{-AlgNilQuot(Set)} \\[1ex]
\begin{array}{ccccccc}
\mathfrak{a} \lhd A & & (0) \lhd A' & & \mathfrak{a} \lhd A & & \mathrm{Nil}(A') \lhd A' \\
\big\downarrow & \dashv & \big\uparrow & \dashv & \big\downarrow & \dashv & \big\uparrow \\
A/\mathfrak{a} & & A' & & A & & A'
\end{array} \\[1ex]
K\text{-Alg(Set)}
\end{array}$$

Only the rightmost functor had to be adjusted, because $(1) \lhd A'$ is not a nil ideal (unless $A'$ is the zero ring). One can easily check that $A' \mapsto (\mathrm{Nil}(A') \lhd A')$ is a functor, that it is indeed right adjoint to $(\mathfrak{a} \lhd A) \mapsto A$ and that it preserves filtered colimits. So as before we can conclude by Lemma 38 that for $\mathfrak{a} \lhd A$ compact in $K$-AlgNilQuot(Set), $A/\mathfrak{a}$ and $A$ are finitely presented $K$-algebras and $\mathfrak{a}$ is finitely generated. □

We now turn to the theories with two sorts.

**Lemma 43.** *The category* $K$-Alg-$R$-Alg(Set) *has all small colimits. Specifically, for* $(A_i \to B_i)_{i \in I}$ *a small diagram in* $K$-Alg-$R$-Alg(Set)*, the colimit is given by the canonical* $K$*-algebra homomorphism*

$$\operatorname*{colim}_{i \in I} A_i \to \operatorname*{colim}_{i \in I} B_i,$$

*where the first colimit is to be computed in* $K$-Alg(Set) *and the second in* $R$-Alg(Set).

*Proof.* By the canonical homomorphism we mean the one induced by the composites of $A_i \to B_i$ with $B_i \to \operatorname{colim}_{i \in I} B_i$, which indeed form a cocone of $K$-algebra homomorphisms. For a cocone $((A_i \to B_i) \to (A \to B))_{i \in I}$, we get an induced $K$-algebra homomorphism $\operatorname{colim}_i A_i \to A$ and an $R$-algebra homomorphism $\operatorname{colim}_i B_i \to B$. These form a morphism in $K$-Alg-$R$-Alg(Set), i. e. the right square in the diagram (in $K$-Alg(Set))

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d] \arrow[rr, bend left] & \operatorname{colim}_i A_i \arrow[r, dashed] \arrow[d] & A \arrow[d] \\
B_i \arrow[r] \arrow[rr, bend right] & \operatorname{colim}_i B_i \arrow[r, dashed] & B
\end{tikzcd}$$

kommutes, since this can be tested by precomposing with all $A_i \to \operatorname{colim}_i A_i$. This proof can in fact be carried out in the general setting of a comma category $(C \downarrow F)$ with $F : D \to C$ any functor such that both $C$ and $D$ have all small colimits. $\square$

**Proposition 44.** *An object* $(A \to B) \in K$-Alg-$R$-Alg(Set) *is compact if and only if* $A$ *is a finitely presented* $K$*-algebra and* $B$ *is a finitely presented* $R$*-algebra.*

*Proof.* We first want to show that $A$ must be finitely presented as a $K$-algebra, using again Lemma 38. So let

$$F_1 : K\text{-Alg-}R\text{-Alg(Set)} \to K\text{-Alg(Set)}, \quad (A \to B) \mapsto A$$

be the forgetful functor. It has a right adjoint, namely

$$G_1 : K\text{-Alg(Set)} \to K\text{-Alg-}R\text{-Alg(Set)}, \quad A \mapsto (A \to 0).$$

Now, $G_1$ does not (in general) preserve all small colimits, as it doesn't preserve (in general) the initial object, which would be $K \to R$ in $K$-Alg-$R$-Alg(Set). But it does preserve filtered colimits (and indeed, all inhabited colimits), as we can see from Lemma 43: if all $B_i$ are 0, and there is at least one $i \in I$, then $\operatorname{colim}_{i \in I} B_i = 0$.

Now consider the other forgetful functor

$$F_2 : K\text{-Alg-}R\text{-Alg(Set)} \to R\text{-Alg(Set)}, \quad (A \to B) \mapsto B.$$

We can again find a right adjoint,

$$G_2 : R\text{-Alg(Set)} \to K\text{-Alg-}R\text{-Alg(Set)}, \quad B \mapsto (B \xrightarrow{=} B).$$

To see that $G_2$ preserves filtered colimits, we remind ourselves of Lemma 33, which implies that filtered colimits are computed in $K$-Alg(Set) and in $R$-Alg(Set) in the same way (i. e., $R$-Alg(Set) $\to$ $K$-Alg(Set) preserves filtered colimits), so we are done by Lemma 43. We conclude that for compact $(A \to B) \in K$-Alg-$R$-Alg(Set), $A$ must be compact in $K$-Alg(Set) and $B$ must be compact in $R$-Alg(Set), which is one half of the claim.

For the other half, consider the composition of $F_1$ and $F_2$ with the respective underlying-set functors. They both preserve filtered colimits, and they are represented by $K[X] \xrightarrow{X \mapsto X} R[X]$ respectively $K \to R[Y]$. These are the building blocks which will suffice to obtain all of the objects from the statement using only finite colimits. So let $f : A \to B$ be an object with $A = K[X_1, \dots, X_n]/(f_1, \dots, f_m)$, $B = R[Y_1, \dots, Y_k]/(g_1, \dots, g_l)$. Start by taking the coproduct of $n$ copies of $K[X] \to R[X]$ and $m$ copies of $K \to R[Y]$, yielding

$$K[X_1, \dots, X_n] \to R[X_1, \dots, X_n, Y_1, \dots, Y_k].$$

Forming a coequalizer for each $f_i$ (using $K[X] \to R[X]$) and each $g_i$ (using $K \to R[Y]$), we have

$$A \to A \otimes_K B,$$

and it only remains to identify each $X_i$ with $f(X_i)$ on the right side by another $n$ coequalizers, again using $K \to R[Y]$. $\square$

**Lemma 45.** *The full subcategory* $K$-Alg-$R$-Quot(Set) *of* $K$-Alg-$R$-Alg(Set) *is closed under inhabited small colimits.*

*Proof.* Let $(A_i \twoheadrightarrow B_i)_{i \in I}$ be a small diagram in $K$-Alg-$R$-Quot(Set) with $I$ inhabited. Let $b \in \operatorname{colim}_i B_i \cong (\bigotimes_i B_i)/\sim$ be any element. Then $b$ can be written as a sum of products of elements coming from some $B_i$, i. e. we find a polynomial $p \in \mathbb{Z}[X_1, \dots, X_n]$ and elements $b_j \in B_{i_j}$ such that $p(b_1, \dots, b_k) =$

$b$ in $\operatorname{colim}_i B_i$. (Here we need $I$ to be inhabited to get rid of any scalars from $R$.) But for these $b_j$ we can find preimages $a_j \in A_{i_j}$ under the maps $A_{i_j} \to B_{i_j}$, and then $p(a_1, \ldots, a_n)$ is a preimage of $b$ under $\operatorname{colim}_i A_i \to \operatorname{colim}_i B_i$, which is therefore surjective. $\square$

*Remark* 46. The subcategory $K$-Alg-$R$-Quot(Set) is, in general, not closed under arbitrary small colimits. For instance, the initial object $K \to R$ of $K$-Alg-$R$-Alg(Set) doesn't have to lie in $K$-Alg-$R$-Quot(Set).

**Corollary 47.** *An object* $(A \twoheadrightarrow B) \in K$-Alg-$R$-Quot(Set) *is compact if and only if it is compact as an object of* $K$-Alg-$R$-Alg(Set)*, that is, if and only if $A$ is a finitely presented $K$-algebra and $B$ is a finitely presented $R$-algebra.*

*Proof.* The subcategory $K$-Alg-$R$-Quot(Set) is in particular closed under filtered colimits in $K$-Alg-$R$-Alg(Set) by Lemma 45. So if an object $(A \twoheadrightarrow B)$ of $K$-Alg-$R$-Quot(Set) is compact in $K$-Alg-$R$-Alg(Set), then it is also compact in $K$-Alg-$R$-Quot(Set). Furthermore, the functors $G_1$, $G_2$ in the proof of Proposition 44 factorize over $K$-Alg-$R$-Quot(Set), as $(A \to 0)$ and $B \xrightarrow{=} B$ are always surjective. So they are right adjoint to the restrictions of $F_1$, $F_2$ (and still preserve filtered colimits), which provides the other implication just like before. $\square$

*Remark* 48. Note that for a compact model $(f : A \twoheadrightarrow B) \in K$-Alg-$R$-Quot(Set), the kernel of $f$ doesn't have to be a finitely generated ideal, if $R$ is not a finitely presented $K$-algebra. For example, we can take $A = K$, $B = R$ with $K = \mathbb{Z}[X_1, X_2, \ldots] \xrightarrow{X_i \mapsto 0} R = \mathbb{Z}$.

**Lemma 49.** *The full subcategory* $K$-Alg-$R$-NilQuot(Set) *of* $K$-Alg-$R$-Quot(Set) *is closed under filtered colimits.*

*Proof.* Let $(f_i : A_i \to B_i)_{i \in I}$ be a filtered small diagram in $K$-Alg-$R$-NilQuot(Set) and let $a \in \operatorname{colim}_i A_i$ be an element which gets mapped to zero in $\operatorname{colim}_i B_i$. (One colimit being computed in $K$-Alg(Set), the other in $R$-Alg(Set).) We use the description of these filtered colimits given in Lemma 33 as follows. There is some $a' \in A_i$ representing $a$ for some $i \in I$, and $f_i(a') \in B_i$ represents zero in $\operatorname{colim}_i B_i$. But this means that there is an arrow $i \to j$ in $I$ such that $f_i(a')$ is zero in $B_j$. That is, $a'$ gets mapped to some $a'' \in A_j$ with $f_j(a'') = 0$. Since $f_j : A_j \to B_j$ is an object of $K$-Alg-$R$-NilQuot, this means that $a''$ is nilpotent, and since $a''$ also represents $a$, the latter was nilpotent. $\square$

**Corollary 50.** *Let $A \twoheadrightarrow B$ be an object of* $K$-Alg-$R$-NilQuot(Set) *with $A$ a finitely presented $K$-algebra and $B$ a finitely presented $R$-algebra. Then $A \twoheadrightarrow B$ is compact.*

*Proof.* This follows from Lemma 49 just like before. □

For the converse of Corollary 50, let's consider the pairs of adjoint functors $F_1 \dashv G_1$ and $F_2 \dashv G_2$ from the proof of Proposition 44 again. We have that $G_2$ factors over $K$-Alg-$R$-NilQuot(Set), since $B \xrightarrow{=} B$ has kernel $(0)$. But the kernel of $G_1(A) = (A \to 0)$ is certainly not a nil ideal in general. It doesn't seem obvious how to modify $G_1$ to get a new right adjoint. Instead, we will obtain the converse of Corollary 50 in a completely different manner in Corollary 80.

# 7 The big Zariski topos

**Definition 51.** A morphism of schemes $f : T \to S$ is *locally of finite presentation* if there exists an affine open cover $T = \bigcup_{i \in I} U_i$, $U_i = \operatorname{Spec} A_i$ and affine open subsets $V_i = \operatorname{Spec} K_i \subseteq S$ for $i \in I$ such that for all $i$, $f(U_i) \subseteq V_i$ and the corresponding ring homomorphism $K_i \to A_i$ makes $A_i$ into a finitely presented $K_i$-algebra.

**Definition 52.** Let $S$ be a scheme. The *big Zariski site* $\mathrm{ZAR}(S)$ of $S$ consists of the following.

- Objects: Schemes $T \to S$ locally of finite presentation over $S$.
- Morphisms: Scheme morphisms $T' \to T$ compatible with the structure morphisms to $S$.

$$\begin{tikzcd} T' \arrow[rr] \arrow[dr] & & T \arrow[dl] \\ & S & \end{tikzcd}$$

- Grothendieck topology: A sieve on $T$ is $J_{\mathrm{ZAR}(S)}$-covering if and only if it contains immersions of open subsets $U_i \hookrightarrow T$ such that $T = \bigcup_i U_i$.

The topos $S_{\mathrm{ZAR}} := \mathrm{Sh}(\mathrm{ZAR}(S))$ is the *big Zariski topos* of $S$.

For a discussion concerning the restriction to schemes locally of finite presentation over $S$, see [2, Section 15].

**Lemma 53.** *The topology from Definition 52 actually satisfies the axioms of a Grothendieck topology.*

*Proof.* The maximal sieve on $T$ is covering because it contains the open immersion $T \hookrightarrow T$. If $T = \bigcup_i U_i$ is an open cover and $f : T' \to T$ is a morphism, then the pullbacks $f^*U_i$ constitute an open cover of $T'$, so pullback sieves are again covering. Finally, for transitivity, let $(U_i)_{i\in I}$ be an open cover of $T$ and $(U_{i,j})_{j\in J_i}$ an open cover of $U_i$ for each $i \in I$. Then of course $(U_{i,j})_{i\in I, j\in J_i}$ is an open cover of $T$. □

We want to reduce the big Zariski topos as defined above to something more algebraic. This is possible using the Comparison Lemma (Theorem 56 below), which rests on the notion of a dense subcategory of a site.

**Definition 54.** Let $(C, J)$ be a site. A full subcategory $C'$ of $C$ is called *dense* (for the topology $J$) if for every object $c \in \operatorname{Ob} C$ the sieve generated by all morphisms $c' \to c$ with $c' \in \operatorname{Ob} C'$ is covering for $J$.

**Lemma 55.** *The full subcategory of* $\mathrm{ZAR}(S)$ *consisting of the affine schemes* $\operatorname{Spec} A = T \to S$ *is dense.*

*Proof.* Of course we can cover every $T \in \mathrm{ZAR}(S)$ by affine open subschemes. □

Let us denote this subcategory by $\mathrm{ZAR}^*(S)$.

**Theorem 56** (Comparison Lemma). *Let $(C, J)$ be a site and $C'$ a dense full subcategory of $C$. Then the restriction functor* $\mathrm{Set}^C \to \mathrm{Set}^{C'}$ *induces an equivalence of categories*

$$\mathrm{Sh}(C, J) \simeq \mathrm{Sh}(C', J|_{C'})$$

*where $J|_{C'}$ is the topology on $C'$ given as follows: A sieve $S$ on $c \in \operatorname{Ob} C'$ is $J'$-covering if and only if the sieve generated by $S$ in $C$ is $J$-covering.*

*Proof.* See [6, Theorem C2.2.3]. □

**Lemma 57.** *Let $J_{\mathrm{ZAR}^*(S)} = J_{\mathrm{ZAR}(S)}|_{\mathrm{ZAR}^*(S)}$ be the restriction of $J_{\mathrm{ZAR}(S)}$ to $\mathrm{ZAR}^*(S)$ as in the Comparison Lemma. Then a sieve $\tilde{S}$ on $T = \operatorname{Spec} A$ is $J_{\mathrm{ZAR}^*(S)}$-covering if and only if there are elements $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^k a_i = 1$ and the duals of all the localization homomorphisms*

$$A \to A[a_i^{-1}]$$

*for $i = 1, \ldots, k$ lie in $\tilde{S}$.*

*Proof.* Note that $\operatorname{Spec} A \leftarrow \operatorname{Spec} A[a_i^{-1}] = D(a_i)$ is locally of finite presentation, so $\operatorname{Spec} A[a_i^{-1}]$ is again an object of $\mathrm{ZAR}^*(S)$. And indeed, $\operatorname{Spec} A$ is covered (as a space) by some open subsets $U_j$ if and only if there are $a_1, \ldots, a_k \in A$ with $\sum_{i=1}^k a_i = 1$ and each $D(a_i)$ lies in some $U_j$ (see [8, Tag 01HS]). Now suppose $\tilde{S}$ is any sieve on $\operatorname{Spec} A$ in $\mathrm{ZAR}^*(S)$. The sieve generated by $\tilde{S}$ in $\mathrm{ZAR}(S)$ is covering for $J_{\mathrm{ZAR}(S)}$ if and only if it contains an open cover of $\operatorname{Spec} A$. But in this case it also contains a cover by affine opens, which were therefore already present in $\tilde{S}$, so we get exactly the asserted condition. $\square$

**Lemma 58.** *If $S = \operatorname{Spec} K$ is affine, the category $\mathrm{ZAR}^*(S)$ is (equivalent to) the dual of the category of finitely presented $K$-algebras.*

*Proof.* The category of affine schemes is equivalent to the dual of the category of rings, $\mathrm{AffSch} \simeq \mathrm{Ring}(\mathrm{Set})^{\mathrm{op}}$. So we also have a duality of comma categories $(\operatorname{Spec} K \downarrow \mathrm{AffSch}) \simeq (\mathrm{Ring}(\mathrm{Set}) \downarrow K)^{\mathrm{op}}$. The claim follows because an affine scheme $T = SpecA \to \operatorname{Spec} K$ is locally of finite presentation over $\operatorname{Spec} K$ if and only if $A$ is finitely presented as a $K$-Algebra (see e. g. [8, Tag 01TQ]). $\square$

We have seen in Proposition 36 that the category of finitely presented $K$-algebras is exactly the category $K\text{-Alg}(\mathrm{Set})_c$ of compact Set-based models of $K$-Alg. Since $K$-Alg is of presheaf type (as it is algebraic), we can immediately conclude by Theorem 24 that $[K\text{-Alg}(\mathrm{Set})_c, \mathrm{Set}] \simeq \mathrm{PSh}(\mathrm{ZAR}^*(\operatorname{Spec} K))$ is a classifying topos for $K$-Alg. What remains is to find the quotient of $K$-Alg corresponding to the subtopos $(\operatorname{Spec} K)_{\mathrm{ZAR}} \simeq \mathrm{Sh}(\mathrm{ZAR}^*(\operatorname{Spec} K), J_{\mathrm{ZAR}^*(\operatorname{Spec} K)})$ of $\mathrm{PSh}(\mathrm{ZAR}^*(\operatorname{Spec} K))$.

**Lemma 59.** *The Grothendieck topology $J_{\mathrm{ZAR}^*(\operatorname{Spec} K)}$ on $\mathrm{ZAR}^*(\operatorname{Spec} K) \simeq K\text{-Alg}(\mathrm{Set})_c{}^{\mathrm{op}}$ is generated by the duals of the following families for $n \in \mathbb{N}_{\geq 0}$.*

$$\begin{array}{c} K[X_1, \dots, X_n]/(X_1 + \dots + X_n - 1) \\ \big\downarrow \\ K[X_1, \dots, X_n, X_i^{-1}]/(X_1 + \dots + X_n - 1) \end{array} \qquad i = 1, \dots, n$$

(By the notation $X_i^{-1}$ we of course mean the localization away from the element $X_i$.)

*Proof.* The displayed families are special cases of the families from Lemma 57. And any such family $A \to A[a_i^{-1}]$ with $\sum_i a_i = 1$ can be obtained as a pushout (pullback in $\mathrm{ZAR}^*(\operatorname{Spec} K)$) of the displayed family (for the same $n$) along $K[X_1, \dots, X_n]/(X_1 + \dots + X_n - 1) \to A$, $X_i \mapsto a_i$. Thus the Grothendieck topology generated by the families in the statement indeed contains all the sieves from Lemma 57. $\square$

*Remark* 60. The given set of generators for the topology $J_{\mathrm{ZAR}^*(\operatorname{Spec} K)}$ is not the smallest possible: taking only the families for $n = 0$ and $n = 2$ suffices. The other families can be recovered by the transitivity property of a

Grothendieck topology. This corresponds to the locality axioms, where also the instances for $n = 0$ and $n = 2$ are sufficient.

**Theorem 61.** *The big Zariski topos* $(\operatorname{Spec} K)_{\mathrm{ZAR}}$ *classifies the theory* loc-$K$-Alg *of local $K$-algebras.*

*Proof.* We use Theorem 31 to find the Gothendieck topology on $K\text{-Alg(Set)}_c{}^{\mathrm{op}}$ corresponding to the axioms

$$x_1 + \ldots + x_n = 1 \ \vdash_{x_1,\ldots,x_n:A} \ \bigvee_{i=1,\ldots,n} \exists y : A.\ x_i \cdot y = 1.$$

They are of the required form

$$\phi_n \ \vdash_{\vec{x}_n} \ \bigvee_{j \in J_n} \exists \vec{y}^j_n.\ \theta^j_n$$

with $\phi_n :\equiv (x_1 + \ldots + x_n = 1)$ (in the context $x_1, \ldots, x_n : A$), $\psi^i_n :\equiv (\tilde{x}_1 + \ldots + \tilde{x}_n = 1) \wedge (\tilde{x}_i \cdot y = 1)$ (in the context $\tilde{x}_1, \ldots, \tilde{x}_n, y : A$) and $\theta^i_n :\equiv \psi^i_n \wedge (x_1 = \tilde{x}_1) \wedge \ldots \wedge (x_n = \tilde{x}_n)$. We actually obtain in this way

$$x_1 + \ldots + x_n = 1 \ \vdash_{\vec{x}:A^n} \ \bigvee_{i=1,\ldots,n} \exists \vec{\tilde{x}}, y.\ (\tilde{x}_1 + \ldots + \tilde{x}_n = 1) \wedge (\tilde{x}_i \cdot y = 1) \wedge (\vec{\tilde{x}} = \vec{x}),$$

but the right side is equivalent to $\bigvee_i \exists y : A.\ (x_1 + \ldots + x_n = 1) \wedge (\tilde{x}_i \cdot y = 1)$, which is equivalent to the right side of the locality axioms modulo the left side. Also, $\theta^i_n$ is provably functional from $\psi^i_n$ to $\phi_n$ by construction.

Furthermore, the model $M_{\phi_n} := K[X_1, \ldots, X_n]/(X_1 + \ldots + X_n - 1)$ is presented by the formula $\phi_n$, as explained after Proposition 36. Similarly, $M_{\psi^i_n} := K[X_1, \ldots, X_n, X_i^{-1}]/(X_1 + \ldots + X_n - 1) \cong K[X_1, \ldots, X_n, Y]/(X_1 + \ldots + X_n - 1, X_i Y - 1)$ is presented by $\psi^i_n$. So using Lemma 59, we only need to show that the formula $\theta^i_n$ presents the canonical localization homomorphism between these models. When we insert the generators $X_1, \ldots, X_n, Y$ of $M_{\psi^i_n}$ in $\theta^i_n$ for the variables $\tilde{x}_1, \ldots, \tilde{x}_n, y$, the unique elements of $M_{\psi^i_n}$ fulfilling the resulting formula when inserted for the variables $x_1, \ldots, x_1$ are of course $X_1, \ldots, X_n \in M_{\psi^i_n}$. And the unique homomorphism $M_{\phi_n} \to M_{\psi^i_n}$ sending the generators $X_1, \ldots, X_n$ of $M_{\phi_n}$ to the elements $X_1, \ldots, X_n \in M_{\psi^i_n}$ is indeed the localization homomorphism. $\square$

# 8 The simple big infinitesimal topos

We first give the general definition and reduce to an algebraic situation before explaining the simplifying assumption to be used in this section.

**Definition 62.** Let $X \to S$ be schemes. The *big infinitesimal topos* of $X$ over $S$ is the topos $X/S_{\mathrm{INF}} := \mathrm{Sh}(\mathrm{INF}(X/S))$, where $\mathrm{INF}(X/S)$ is the following site.

- Objects: Pairs of schemes $T \to S$, $U \to X$ locally of finite presentation over $S$ respectively $X$, together with a closed immersion $U \to T$ over $S$ such that the corresponding quasi-coherent ideal sheaf $I \subseteq \mathcal{O}_T$ is a sheaf of nil ideals (i. e. every section on every open subset is nilpotent).

$$\begin{tikzcd} T \arrow[d] & U \arrow[l] \arrow[d] \\ S & X \arrow[l] \end{tikzcd}$$

- Morphisms: Pairs of morphisms $T' \to T$, $U' \to U$ compatible with the other data.

$$\begin{tikzcd} T' \arrow[d] \arrow[dr] & & U' \arrow[ll] \arrow[d] \arrow[dr] & \\ & S & & X \arrow[ll] \\ T \arrow[ur] & & U \arrow[ll] \arrow[ur] & \end{tikzcd}$$

- Grothendieck topology: A sieve on $T \leftarrow U$ is $J_{\mathrm{INF}(X/S)}$-covering if and only if it contains the arrows

$$\begin{tikzcd} T_i \arrow[d, hook] & U \times_T T_i \arrow[l] \arrow[d, hook] \\ T & U \arrow[l] \end{tikzcd}$$

  for some open covering $T_i \hookrightarrow T$.

This definition is in analogy to the little infinitesimal topos as defined in [4, Section 4.1.]. Note that $T_i \leftarrow U \times_T T_i$ is indeed again a closed immersion corresponding to a sheaf of nil ideals (namely the restriction of the sheaf corresponding to $U$ to the open set $T_i$).

**Lemma 63.** *The full subcategory of* $\mathrm{INF}(X/S)$ *on the objects* $T \leftarrow U$ *with both* $T$ *and* $U$ *affine is dense.*

*Proof.* For any object $T \leftarrow U$, we can cover $T$ by affine opens $T_i \hookrightarrow T$. Then the pullbacks $U \times_T T_i$ are also affine, since $U \to T$ is a closed immersion. $\square$

We denote this dense full subcategory by $\mathrm{INF}^*(X/S)$.

**Lemma 64.** *Let $J_{\mathrm{INF}^*(X/S)} = J_{\mathrm{INF}(X/S)}|_{\mathrm{INF}^*(X/S)}$ be the restriction of $J_{\mathrm{INF}(X/S)}$ to $\mathrm{INF}^*(X/S)$ as in the Comparison Lemma (Theorem 56). Then a sieve $\tilde{S}$ on $T = \operatorname{Spec} A \leftarrow U = \operatorname{Spec} B$ (given by $f : A \to B$) is $J_{\mathrm{INF}^*(X/S)}$-covering if and only if there are elements $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^k a_i = 1$ and all the morphisms given by*

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[d] & B \arrow[d] \\ A[a_i^{-1}] \arrow[r, two heads] & B[f(a_i)^{-1}] \end{tikzcd}$$

*for $i = 1, \ldots, k$ lie in $\tilde{S}$.*

*Proof.* The dual of the displayed family generates a $J_{\mathrm{INF}^*(X/S)}$-covering sieve, because the $\operatorname{Spec} A[a_i^{-1}] \to \operatorname{Spec} A$ are an open covering and $\operatorname{Spec} B[f(a_i)^{-1}]$ is the pushout as in Definition 62. Now let $\tilde{S}$ be a sieve in $\mathrm{ZAR}^*(X/S)$ which generates a covering sieve in $\mathrm{ZAR}(X/S)$. Then the generated sieve contains arrows $(T_i \leftarrow U_i) \to (\operatorname{Spec} A \leftarrow \operatorname{Spec} B)$, where $U_i = U \times_{\operatorname{Spec} A} T_i$, such that $T_i \to \operatorname{Spec} A$ is an open covering. This can be chosen so that the $T_i$ are affine, i. e. of the above form $\operatorname{Spec} A[a_i^{-1}] \to \operatorname{Spec} A$. Then the $U_i$ are affine too, namely $U_i \cong \operatorname{Spec} B[f(a_i)^{-1}]$, so the dual of a family as displayed in the statement was contained in $\tilde{S}$. $\square$

**Lemma 65.** *If $S = \operatorname{Spec} K$, $X = \operatorname{Spec} R$ are both affine, then the category* $\mathrm{INF}^*(\operatorname{Spec} R/\operatorname{Spec} K)$ *is equivalent to the full subcategory of* $K$-Alg-$R$-NilQuot(Set)$^{\mathrm{op}}$ *on those objects $A \twoheadrightarrow B$ such that $A$ is a finitely presented $K$-algebra and $B$ is a finitely presented $R$-algebra.*

*Proof.* Of course we want to map such an $f : A \twoheadrightarrow B$ to $\operatorname{Spec} A \leftarrow \operatorname{Spec} B$. That $A$ (respectively $B$) is a finitely presented $K$-algebra (respectively $R$-algebra) is the same as to say that $\operatorname{Spec} A$ (respectively $\operatorname{Spec} B$) is locally of finite presentation over $\operatorname{Spec} K$ (respectively $\operatorname{Spec} R$). The homomorphism $f$ being surjective corresponds to $\operatorname{Spec} A \leftarrow \operatorname{Spec} B$ being a closed immersion, and the kernel of $f$ is a nil ideal if and only if the corresponding ideal sheaf on $\operatorname{Spec} A$ is a sheaf of nil ideals. $\square$

To make use of Lemma 65, we need to prove that $K$-Alg-$R$-NilQuot is of presheaf type and show that said subcategory is $K$-Alg-$R$-NilQuot(Set)$_c{}^{\mathrm{op}}$. This is the point where we temporarily restrict our attention to a special case, namely that of $R = K$.

**Definition 66.** Let $S$ be a scheme. We call $S_{\mathrm{INF}} := (S/S)_{\mathrm{INF}} = \mathrm{Sh}(\mathrm{INF}(S))$ with $\mathrm{INF}(S) := \mathrm{INF}(S/S)$ the *simple big infinitesimal topos.*

We have seen in Corollary 14 that $K$-Alg-$K$-NilQuot is Morita-equivalent to $K$-AlgNilQuot, and identified the compact models in Proposition 42 to be those described in Lemma 65. It is still not obvious whether $K$-AlgNilQuot is of presheaf type, since it contains the following nilpotence axiom, which prevents it from being a cartesian theory.

$$x \in \mathfrak{a} \ \vdash_{x:A} \ \bigvee_{n \in \mathbb{N}_{\geq 0}} x^n = 0$$

We will now further reformulate this theory to a Morita-equivalent theory which is cartesian.

If we had an upper bound $N$ on the nilpotence index of (generalized) elements of any model, we could simply replace the infinitary disjunction by $x^N = 0$. This is of course not possible, as there are models (already in Set) with elements of arbitrarily high nilpotence index. Instead of deciding for one of the formulas $x^n = 0$ for a fixed $n$, we will impose them all, but under different premises.

Let $A$ be a ring (in Set) and $N \subseteq A$ an arbitrary subset containing only nilpotent elements. Then we can "stratify" $N$ into the subsets $N_n := \{a \in N \mid a^n = 0\}$, for each of which there is a bound $n$ on the nilpotence index of its elements. To make this stratification unique, we must additionally require $N_n = \{a \in N_{n+1} \mid a^n = 0\}$. Finally, can we formulate the requirement that $N = \bigcup N_n$ should be an ideal, only refering to the $N_n$ but not to their union? This is possible, since the nilpotence index of elements can not increase arbitrarily under the operations needed to define an ideal.

**Proposition 67.** *The theory $K$-AlgNilQuot is Morita-equivalent to the theory obtained from $K$-Alg by adding unary relation symbols $\mathfrak{a}_n \rightarrowtail A$ for $n \in \mathbb{N}_{\geq 0}$ and the following axioms (for each $n, m \in \mathbb{N}_{\geq 0}$ wherever occurring).*

$$\begin{aligned} x \in \mathfrak{a}_n \ &\dashv\vdash_{x:A} \ (x^n = 0) \wedge (x \in \mathfrak{a}_{n+1}) \\ \top \ &\vdash_{[]} \ 0 \in \mathfrak{a}_1 \\ x \in \mathfrak{a}_n \ &\vdash_{x,y:A} \ x \cdot y \in \mathfrak{a}_n \\ (x \in \mathfrak{a}_n) \wedge (y \in \mathfrak{a}_m) \ &\vdash_{x,y:A} \ x + y \in \mathfrak{a}_{n+m-1} \end{aligned}$$

*Proof.* The conversion between models of the two theories is relatively simple here, because the signatures only differ in relation symbols ($\mathfrak{a}$ for one, $\mathfrak{a}_n$, $n \geq 0$ for the other). Given a model $M$ of $K$-AlgNilQuot in any Grothendieck topos, we define subobjects $\mathfrak{a}_{nM} := [\![(x \in \mathfrak{a}) \wedge (x^n = 0)]\!]_M \hookrightarrow A_M$. We have obtained a model of the second theory, since all of the above axioms are provable in $K$-AlgNilQuot when every application $x \in \mathfrak{a}_n$ of one of the relation symbols not present in $K$-AlgNilQuot is replaced by the formula $(x \in \mathfrak{a}) \wedge (x^n = 0)$. (Here one calculates, within the theory, that $(xy)^n = 0$ for $x^n = 0$, and $(x+y)^{n+m-1} = 0$ for $x^n = 0$, $y^m = 0$.) Next, let instead $\mathfrak{a}_{nM} \hookrightarrow A_M$ be subobjects such that the above axioms are valid. Then we set $\mathfrak{a}_M := [\![\bigvee_{n \geq 0} x \in \mathfrak{a}_n]\!] \hookrightarrow A_M$, that is, $\mathfrak{a}_M$ is the union of the subobjects $\mathfrak{a}_{nM}$. And this time, we have to check that $\mathfrak{a}_M$ fulfills the axioms for an ideal, which is easily done.

To see that these constructions are inverses of each other (up to isomorphism, i. e. different representations of the same subobjects), we only have to prove the following sequents, which follow readily from the respective axioms.

$$x \in \mathfrak{a} \;\dashv\vdash_{x:A}\; \bigvee_{n \geq 0} ((x \in \mathfrak{a}) \wedge (x^n = 0))$$

$$x \in \mathfrak{a}_n \;\dashv\vdash_{x:A}\; (x^n = 0) \wedge \bigvee_{n \geq 0} x \in \mathfrak{a}_n$$

There is no additional restriction imposed on homomorphisms of models by the relation symbols $\mathfrak{a}_n$, since $(f(x))^n = 0$ follows from $x^n = 0$ if $f$ satisfies the ring homomorphism axioms. (Or alternatively, since model homomorphisms respect the interpretation of geometric formulas.) So the established correspondence between models is indeed an equivalence of categories. Finally, naturality in the topos argument is given because geometric morphisms preserve the interpretation of geometric formulas. □

**Corollary 68.** *The theory* $K$-AlgNilQuot *is of presheaf type, and therefore classified by* $\mathrm{PSh}(\mathrm{INF}^*(\operatorname{Spec} K))$.

*Proof.* The theory given in Proposition 67 is a cartesian theory (in fact, it is a Horn theory), so it is of presheaf type (by Theorem 21), and so is $K$-AlgNilQuot. Theorem 24 and Lemma 65 provide the second part of the statement. □

**Lemma 69.** *The Grothendieck topology* $J_{\mathrm{INF}^*(\operatorname{Spec} K)}$ *on* $\mathrm{INF}^*(\operatorname{Spec} K) \simeq K\text{-AlgNilQuot}(\mathrm{Set})_c{}^{\mathrm{op}}$ *is generated by the duals of the following families for* $n \in \mathbb{N}_{\geq 0}$.

$$\begin{array}{c} (0) \lhd K[X_1, \ldots, X_n]/(X_1 + \ldots + X_n - 1) \\ \downarrow \\ (0) \lhd K[X_1, \ldots, X_n, X_i^{-1}]/(X_1 + \ldots + X_n - 1) \end{array} \qquad i = 1, \ldots, n$$

*Proof.* For $X = S$, the families displayed in Lemma 64 can be written as

$$\begin{array}{c} \mathfrak{a} \lhd A \\ \downarrow \\ \mathfrak{a}[a_i^{-1}] \lhd A[a_i^{-1}] \end{array} \qquad i = 1, \ldots, n$$

with $\mathfrak{a} := \ker(A \twoheadrightarrow B)$, because $B$ carries no extra structure than being a quotient of $A$ by a nil ideal. The families in the statement are clearly special cases of this, so they do generate $J_{\mathrm{INF}^*(\operatorname{Spec} K)}$-covering sieves. And by pushout along $((0) \lhd K[X_1, \ldots, X_n]/(X_1 + \ldots + X_n - 1)) \to (\mathfrak{a} \lhd A), X_j \mapsto a_j$ we can again recover all covering sieves from these special cases. □

**Theorem 70.** *The simple big infinitesimal topos* $(\operatorname{Spec} K)_{\mathrm{INF}}$ *classifies the theory* loc-$K$-AlgNilQuot*.*

*Proof.* Just as for the big Zariski topos, we use Theorem 31 to describe the topology on $K\text{-AlgNilQuot}(\mathrm{Set})_c{}^{\mathrm{op}}$ corresponding to the locality axioms. Indeed, we can take the same formulas as before, and only have to find new models presented by them, this time in $K$-AlgNilQuot(Set) instead of $K$-Alg(Set): $\phi_n :\equiv (x_1 + \ldots + x_n = 1)$, $\psi_n^i :\equiv (\tilde{x}_1 + \ldots + \tilde{x}_n = 1) \wedge (\tilde{x}_i \cdot y = 1)$ and $\theta_n^i :\equiv \psi_n^i \wedge (x_1 = \tilde{x}_1) \wedge \ldots \wedge (x_n = \tilde{x}_n)$.

One can check that $\phi_n$ and $\psi_n^i$ present precisely the models they presented in $K$-Alg(Set), with $\mathfrak{a}$ set to the zero ideal.

$$\begin{aligned} M_{\phi_n} &= ((0) \lhd K[X_1, \ldots, X_n]/(X_1 + \ldots + X_n - 1)) \\ M_{\psi_n^i} &= ((0) \lhd K[X_1, \ldots, X_n, X_i^{-1}]/(X_1 + \ldots + X_n - 1)) \end{aligned}$$

This is because of the adjunction

$$\operatorname{Hom}_{K\text{-AlgNilQuot(Set)}}((0) \lhd A, \mathfrak{a}' \lhd A') \cong \operatorname{Hom}_{K\text{-Alg(Set)}}(A, A')$$

we have already met in Proposition 42: The interpretation of a formula of $K$-AlgNilQuot, which does not contain the relation symbol $\mathfrak{a}$, is, as a functor $K\text{-AlgNilQuot(Set)} \to \mathrm{Set}$, the composition of the interpretation of the same formula read as a formula of $K$-Alg and the forgetful functor $K\text{-AlgNilQuot(Set)} \to K\text{-Alg(Set)}, (\mathfrak{a} \lhd A) \mapsto A$. One can also check that $\theta_n^i$ still presents the canonical localization homomorphism, so we have obtained, as generating families for the topology corresponding to the locality axioms, exactly the families from Lemma 69. □

# 9 The big infinitesimal topos

We now return to the general case of the big infinitesimal topos $(\operatorname{Spec} R/\operatorname{Spec} K)_{\mathrm{INF}}$ for affine base schemes. As noted before, it would be very helpful to know that $K$-Alg-$R$-NilQuot is of presheaf type. We do not even know this for $K$-Alg-$R$-Quot, but here it seems very plausible in the following sense: $K$-Alg-$R$-Quot differs from $K$-Alg-$K$-Quot, which is a theory of presheaf type, only in additional "algebraic data". We will start by showing that $K$-Alg-$R$-Quot is of presheaf type, using the concept of a rigid Grothendieck topology.

**Definition 71.** Let $(C, J)$ be a site.

- An object $c \in C$ is *irreducible* for the Grothendieck topology $J$ if every $J$-covering sieve on $c$ is the maximal sieve (contains $\mathrm{id}_c$).
- The Grothendieck topology $J$ is *rigid* if for every object $c \in C$, the sieve on $c$ generated by all arrows $c' \to c$ with $c'$ irreducible is $J$-covering.

**Lemma 72.** *Let $(C, J)$ be a site with $J$ a rigid Grothendieck topology. Let $C'$ be the full subcategory of $C$ on the irreducible objects. Then we have $\mathrm{Sh}(C, J) \simeq \mathrm{PSh}(C')$ via the restriction of functors defined on $C^{\mathrm{op}}$ to $C'^{\mathrm{op}}$.*

*Proof.* We use the Comparison Lemma (Theorem 56). The only thing to show is that the topology induced on $C'$ by $J$ is the trivial topology. So let $c \in C'$, $S'$ a sieve on $c$ in $C'$, such that the sieve $S$ generated by $S'$ in $C$ is $J$-covering. Then $S$ is the maximal sieve, $\mathrm{id}_c \in S$, since $c$ is irreducible. This means that there is an arrow $f : d \to c$ in $S'$ such that $\mathrm{id}_c$ factorizes over $f$, but then we already had $\mathrm{id}_c \in S'$, since $S'$ is a sieve. □

Let us denote by $J_{\mathrm{Quot}}$ the topology on $K\text{-Alg-}R\text{-Alg}(\mathrm{Set})_c{}^{\mathrm{op}}$ induced by the quotient $K$-Alg-$R$-Quot of the theory $K$-Alg-$R$-Alg.

**Lemma 73.** *The topology $J_{\mathrm{Quot}}$ is generated by a single covering family consisting of a single arrow, namely the dual of $(K \to R[X]) \to (K[X] \to R[X])$.*

*Proof.* We use Theorem 31. The axiom

$$\top \ \vdash_{y:B} \ \exists x : A.\ f(x) = y$$

is of the required form with $\phi :\equiv \top$ in the context $y : B$, $\psi :\equiv \top$ in the context $x : A$ and $\theta :\equiv (f(x) = y)$. (The potentially infinitary disjunction was not needed here.) Of course, $\theta$ is provably functional from $\psi$ to $\phi$. And $\phi$, $\psi$ respectively present the models $K \to R[X]$ and $K[X] \to R[X]$, as the interpretation functors are the forgetful functors $(A \to B) \mapsto B$ and $(A \to B) \mapsto A$. Finally, $\theta$ presents the obvious arrow $(K \to R[X]) \to (K[X] \to R[X])$. □

**Lemma 74.** *A cosieve on* $(A \to B) \in K\text{-Alg-}R\text{-Alg(Set)}_c$ *is* $J_{\text{Quot}}$*-covering if and only if it contains an arrow of the form*

$$(A \to B) \to (A[X_1, \dots, X_n] \xrightarrow{X_i \mapsto b_i} B)$$

*for some elements* $b_1, \dots, b_n \in B$*,* $n \in \mathbb{N}_{\geq 0}$*.*

*Proof.* Via pushout along $(K \to R[X]) \xrightarrow{X \mapsto b} (A \to B)$, the sieve generated by the single arrow $(A \to B) \to (A[X] \xrightarrow{X \mapsto b} B)$ is $J_{\text{Quot}}$-covering for any $b \in B$. The arrows in the statement are $n$-fold compositions of such arrows, so they generate $J_{\text{Quot}}$-covering sieves by transitivity $J_{\text{Quot}}$.

It is also clear that the generator from Lemma 73 is a special case of the arrows in the statement. But we have to show that the indicated collection of (co)sieves $J$ is a Grothendieck topology. Maximal sieves lie in $J$ by, e. g., $n = 0$. The pushout of $(A \to B) \to (A[X_1, \dots, X_n] \to B)$ along $(A \to B) \to (A' \to B')$ is simply $A'[X_1, \dots, X_n] \to B'$, so pullback of sieves (in $K\text{-Alg-}R\text{-Alg(Set)}_c{}^{\text{op}}$) remains in $J$. Finally, transitivity is satisfied by $J$, since the composition of two arrows as in the statement is again of that form. □

**Proposition 75.** *If* $R$ *is a finitely generated* $K$*-algebra, the Grothendieck topology* $J_{\text{Quot}}$ *is rigid and* $K\text{-Alg-}R\text{-Quot}$ *is of presheaf type.*

*Proof.* Those $(f : A \to B) \in K\text{-Alg-}R\text{-Alg(Set)}_c$ with $f$ surjective (or rather their duals) are irreducible for $J_{\text{Quot}}$: any covering cosieve contains an arrow $(A \twoheadrightarrow B) \to (A[X_1, \dots, X_n] \xrightarrow{X_i \mapsto b_i} B)$, and since $f$ is surjective, there are $a_i \in A$ with $f(a_i) = b_i$. Then, $(A[X_1, \dots, X_n] \to B) \xrightarrow{X_i \mapsto a_i} (A \twoheadrightarrow B)$ is a retract of the previous arrow, so the cosieve was maximal.

Secondly, any object $A \to B$ can be covered by such irreducibles. Since $B$ is a finitely presented $R$-algebra and $R$ is finitely generated as a $K$-algebra, $B$ is also finitely generated as a $K$-algebra. Then $B$ is in particular finitely generated as an $A$-algebra and the sieve generated by the dual of the following arrow is covering.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A[X_1, \dots, X_n] \arrow[r, two heads] & B \end{tikzcd}$$

It follows by Lemma 72 that $K\text{-Alg-}R\text{-Quot}$ is of presheaf type. □

We have identified the compact models of $K\text{-Alg-}R\text{-Quot}$ in Corollary 47. The Grothendieck topology on $K\text{-Alg-}R\text{-Quot(Set)}_c{}^{\text{op}}$ corresponding to the further quotient $K$-Alg-$R$-NilQuot will be denoted $J_{\mathrm{Nil}}$. This time, applying Theorem 31 will mean a bit more effort, since the formulas we use must present models fulfilling the surjectivity axiom.

**Lemma 76.** *Let $R$ be finitely generated as a $K$-algebra by $r_1, \dots, r_m \in R$. Then the Grothendieck topology $J_{\mathrm{Nil}}$ on $K$-Alg-$R$-Quot(Set)$_c{}^{\mathrm{op}}$ is generated by the following family indexed by $n \in \mathbb{N}_{\geq 0}$.*

$$(K[R_1, \dots, R_m, X] \xrightarrow[X \mapsto 0]{R_i \mapsto r_i} R) \to (K[\vec{R}, X]/(X^n) \xrightarrow[X \mapsto 0]{R_i \mapsto r_i} R)$$

(We allow ourselves to write $\vec{R}$ for $R_1, \dots, R_m$ and similar extensively now.)

*Proof.* The additional axiom is

$$f(x) = 0 \vdash_{x:A} \bigvee_{n \geq 0} x^n = 0.$$

Since we do not know models of $K$-Alg-$R$-Quot which are presented by these simple formulas, we introduce $m$ new variables $\vec{x}_R : A^m$ of which we will always require $f(\vec{x}_R) = \vec{r}$ to make the models presented by our formulas in $K$-Alg-$R$-Alg(Set) into such ones lying in $K$-Alg-$R$-Quot(Set). The axiom is equivalent modulo $K$-Alg-$R$-Quot to

$$\phi \vdash_{x:A, \vec{x}_R:A^m} \bigvee_{n \geq 0} \exists \tilde{x} : A.\ \exists \vec{\tilde{x}}_R : A^m : \theta._n,$$

$$\phi :\equiv (f(x) = 0) \wedge (f(\vec{x}_R) = \vec{r}),$$

$$\theta :\equiv \phi \wedge (\tilde{x} = x) \wedge (\vec{\tilde{x}}_R = \vec{x}_R) \wedge (\tilde{x}^n = 0).$$

Note, that the existence of $\vec{x}_R$ with $f(\vec{x}_R) = \vec{r}$ is indeed derivable in $K$-Alg-$R$-Quot. The formula $\theta_n$ is provably functional from $\psi_n$ to $\phi$ with

$$\psi_n :\equiv (f(\tilde{x}) = 0) \wedge (\tilde{x}^n = 0) \wedge (f(\vec{\tilde{x}}_R) = \vec{r}).$$

The models presented by $\phi$ and $\psi_n$ are indeed the ones from the statement and $\theta$ of course presents the obvious homomorphism. $\square$

To give a very explicit description of $J_{\mathrm{Nil}}$, we strengthen the requirement of $R$ being a finitely generated $K$-algebra to even being finitely presented as a $K$-algebra.

**Lemma 77.** *Let $R$ be a finitely presented $K$-algebra. A cosieve on $(f : A \twoheadrightarrow B) \in K$-Alg-$R$-Quot(Set)$_c$ is $J_{\mathrm{Nil}}$-covering if and only if it contains all the arrows*

$$(A \twoheadrightarrow B) \to (A/\mathfrak{a}^n \twoheadrightarrow B)$$

*for $n \geq 1$, where $\mathfrak{a} := \ker f$.*

*Proof.* Let us give the collection of sieves from the statement the name $J$ again. We first show that $J$ is a Grothendieck topology. While maximal sieves are trivially covering for $J$ again, the pushout of a sieve containing $(A \twoheadrightarrow B) \to (A/\mathfrak{a}^n \twoheadrightarrow B)$ along $(A \twoheadrightarrow B) \xrightarrow{g} (A' \twoheadrightarrow B')$ will contain the arrows $(A' \twoheadrightarrow B') \to (A'/\mathfrak{a}' \twoheadrightarrow B')$, since $g(\mathfrak{a}) \subseteq \mathfrak{a}'$, so $g(\mathfrak{a}^n) \subseteq \mathfrak{a}'^n$. For transitivity, a sieve of composites will contain the arrows $(A \twoheadrightarrow B) \to (A/\mathfrak{a}^n \twoheadrightarrow B) \to ((A/\mathfrak{a}^n)/(\mathfrak{a}/\mathfrak{a}^n)^n \twoheadrightarrow B) = (A/\mathfrak{a}^n \twoheadrightarrow B)$.

Next, $J$ contains the generator of $J_{\mathrm{Nil}}$: in $K[\vec{R}, X] \twoheadrightarrow R$, we have $X \in \ker$, so $(K[\vec{R}, X] \twoheadrightarrow R) \to (K[\vec{R}, X]/(X^n) \twoheadrightarrow R)$ factorizes over $(K[\vec{R}, X]/\ker^n \twoheadrightarrow R)$. This shows $J_{\mathrm{Nil}} \subseteq J$.

For $J \subseteq J_{\mathrm{Nil}}$, let $(f : A \twoheadrightarrow B)$ be any object, then $\ker f = \mathfrak{a} = (a_1, \dots, a_k)$ is finitely generated, because both $A$ and $B$ are finitely presented $K$-algebras, using that $R$ is a finitely presented $K$-algebra. Let furthermore $s_1, \dots, s_m \in A$ be such that $f(s_i) = r_i$ by surjectivity of $f$. Then form the following pushout.

$$\begin{tikzcd}
(K[\vec{R}, X] \twoheadrightarrow R) \arrow[r, "R_i \mapsto s_i", "X \mapsto a_1"'] \arrow[d] & (A \twoheadrightarrow B) \arrow[d] \\
(K[\vec{R}, X]/(X^n) \twoheadrightarrow R) \arrow[r] & (A/a_1^n \twoheadrightarrow B)
\end{tikzcd}$$

By transitivity, the sieve generated by $(A \twoheadrightarrow B) \to (A/(a_1^n, a_2^{(}n')) \twoheadrightarrow B)$ for $n, n' \geq 1$ lies in $J_{\mathrm{Nil}}$. But this is already generated by $(A \twoheadrightarrow B) \to (A/(a_1^n, a_2^{(}n)) \twoheadrightarrow B)$ for $n \geq 1$. By induction, we get the same with the ideals $(a_1^n, \dots, a_k^n)$, which are contained in $\mathfrak{a}^n$. $\square$

**Proposition 78.** *If $R$ is a finitely presented $K$-algebra, the Grothendieck topology $J_{\mathrm{Nil}}$ on $K$-Alg-$R$-Quot(Set)$_c{}^{\mathrm{op}}$ is rigid and the theory $K$-Alg-$R$-NilQuot is of presheaf type.*

*Proof.* The objects with nilpotent kernel $\mathfrak{a}^n = 0$ are irreducible, as this means $(A/\mathfrak{a}^n \twoheadrightarrow B) = (A \twoheadrightarrow B)$. Also, the kernel of $A/\mathfrak{a}^n \twoheadrightarrow B$ is nilpotent, so any object can be covered by irreducibles. $\square$

The reasoning from rigid Grothendieck topologies to quotient theories being of presheaf type, that we have used twice now, can be strengthend to the following theorem.

**Theorem 79.** *Let $T$ be a geometric theory of presheaf type. Let $T'$ be a quotient of $T$ corresponding to a Grothendieck topology $J$ on $T$(Set)$_c{}^{\mathrm{op}}$. Then $J$ is rigid if and only if $T'$ is of presheaf type and every compact object of $T'$(Set) is also compact as an object of $T$(Set).*

*Proof.* See [3, Theorem 8.2.6]. □

This completes our identification of the compact models of $K$-Alg-$R$-NilQuot.

**Corollary 80.** *Let $R$ be finitely presented as a $K$-algebra. Then an object $A \twoheadrightarrow B$ of $K$-Alg-$R$-NilQuot(Set) is compact if and only if $A$ is a finitely presented $K$-algebra and $B$ is a finitely presented $R$-algebra.*

*Proof.* Corollary 50 was the "if" part. The "only if" part is provided by Proposition 78 and Thoerem 79. □

Let $J_{\mathrm{Loc}}$ be the Grothendieck topology on $K$-Alg-$R$-NilQuot$(\mathrm{Set})_c{}^{\mathrm{op}}$ corresponding to the quotient theory loc-$K$-Alg-$R$-NilQuot.

**Lemma 81.** *Let $R$ be a finitely presented $K$-algebra. Specifically, let $r_1, \ldots, r_m \in R$ be generators for $R$ as a $K$-algebra and let $p_1, \ldots, p_l \in K[R_1, \ldots, R_m]$ be generators of the kernel. Then the Grothendieck topology $J_{\mathrm{Loc}}$ on $K$-Alg-$R$-NilQuot$(\mathrm{Set})_c{}^{\mathrm{op}}$ is generated by the duals of the following families, one for each $k \geq 0, n \geq 1$.*

$$\begin{tikzcd}
K[\vec{R}, \vec{X}]/(\vec{p}^n, X_1 + \ldots + X_k - 1) \arrow[r, two heads] \arrow[d] & R[\vec{X}]/(X_1 + \ldots + X_k - 1) \arrow[d] \\
K[\vec{R}, \vec{X}, X_j^{-1}]/(\vec{p}^n, X_1 + \ldots + X_k - 1) \arrow[r, two heads] & R[\vec{X}, X_j^{-1}]/(X_1 + \ldots + X_k - 1)
\end{tikzcd}$$

*Proof.* The additional axioms are the locality axioms.

$$x_1 + \ldots + x_k = 1 \ \vdash_{y:A} \bigvee_{j=1,\ldots,k} x_j \cdot y = 1$$

We have to express this with formulas presenting models in $K$-Alg-$R$-NilQuot(Set). First, $\alpha_0 :\equiv (f(\vec{x}_R) = \vec{r})$ presents in $K$-Alg-$R$-Alg(Set) the model $K[\vec{R}] \xrightarrow{R_i \mapsto r_i} R$. This is surjective, but the kernel is not a nil ideal yet. The formulas $\alpha^n :\equiv (f(\vec{x}_R) = \vec{r}) \wedge \bigwedge_{i=1,\ldots,l}(p_i(\vec{x}_R)^n = 0)$ present $K[\vec{R}]/(p_1^n, \ldots, p_l^n) \twoheadrightarrow R$, which does lie in $K$-Alg-$R$-NilQuot(Set).

We formulate the axioms

$$\phi_k^n \ \vdash_{\vec{x}_R, \vec{x}} \bigvee_{j=1,\ldots,k} \exists \vec{\tilde{x}}_R.\ \exists \vec{\tilde{x}}.\ \exists y : A.\ \theta_{k,j}^n,$$

$$\phi_k^n :\equiv \alpha^n \wedge (x_1 + \ldots + x_k = 1),$$

$$\theta_{k,j}^n :\equiv (\vec{\tilde{x}}_R = \vec{x}_R) \wedge (\vec{\tilde{x}} = \vec{x}) \wedge \psi_{k,j}^n,$$

$$\psi_{k,j}^n :\equiv \alpha^n[\vec{\tilde{x}}_R / \vec{x}_R] \wedge (\tilde{x}_1 + \ldots + \tilde{x}_k = 1) \wedge (y \cdot \tilde{x}_j = 1).$$

One can see that $\theta^n_{k,j}$ is provably functional from $\psi^n_{k,j}$ to $\phi^n_k$. The models presented by $\phi^n_k$ and $\psi^n_{k,j}$ in $K$-Alg-$R$-Alg(Set), and therefore in $K$-Alg-$R$-NilQuot(Set), are the ones in the statement. And $\theta^n_{k,j}$ presents the canonical localization homomorphism.

One can also see that these axioms, for fixed $k$ and varying $n$, are equivalent modulo $K$-Alg-$R$-NilQuot to the locality axiom for the same $k$. Indeed, we need both the surjectivity axiom and the nilpotence axiom to eliminate the additional condition $\bigvee_{n\geq 1} \alpha^n$ which can be separated out when combining the axioms for varying $n$. $\square$

**Theorem 82.** *Let $K$ be a ring and $R$ be a finitely presented $K$-algebra. Then the big infinitesimal topos* $(\operatorname{Spec} R/\operatorname{Spec} K)_{\mathrm{INF}}$ *classifies the theory* loc-$K$-Alg-$R$-NilQuot.

*Proof.* We only have to show that the families of Lemma 81 generate the Grothendieck topology $J_{\mathrm{INF}^*} := J_{\mathrm{INF}^*(\operatorname{Spec} R/\operatorname{Spec} K)}$ described in Lemma 64. Our generators are clearly instances of the families for $J_{\mathrm{INF}^*}$. Conversely, let $(f : A \twoheadrightarrow B) \in K\text{-Alg-}R\text{-NilQuot(Set)}_c$ be any object and $a_1 + \ldots + a_k = 1$ in $A$. Then there are elements $a_{R,i} \in A$ with $f(a_{R,i}) = r_i$, because $f$ is surjective. And since $f(p_i(\vec{a}_R)) = 0$ and the kernel of $f$ is a nil ideal, we find an $n$ such that $p_i(\vec{a}_R)^n = 0$ for all $i$. Then we can use our generator for that $k$ and $n$ to obtain the required family from Lemma 64 as a pushout along $X_i \mapsto a_i, R_i \mapsto a_{R,i}$. $\square$

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