# INTERNAL LANGUAGES OF HIGHER TOPOSES

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The internal language of toposes, a form of higher-order logic, is well-established as a powerful tool for internalizing mathematics in many geometric contexts. The corresponding internal language for *higher* toposes has long been conjectured to be a form of dependent type theory satisfying Voevodsky's univalence principle. There are now many such "homotopy type theories", but the original and simplest is Martin-Löf's original intensional type theory, with univalence and "higher inductive types" added using axioms; this is known as "Book HoTT" after the 2013 book "Homotopy Type Theory".

About a decade ago, Voevodsky constructed an interpretation of Martin-Löf type theory with univalence in the basic $(\infty, 1)$-topos of $\infty$-groupoids, using simplicial sets. Extending this to an interpretation of all of Book HoTT in all $(\infty, 1)$-toposes involves a number of coherence issues, many of which have been resolved piecemeal since then. Earlier this year I announced a solution to the largest remaining gap: every (Grothendieck–Lurie) $(\infty, 1)$-toposes can be presented by a model category that contains strict univalent universes. Thus, Book HoTT can now be used confidently as an internal language for all $(\infty, 1)$-toposes.

In this talk I will motivate the problem of internal languages for higher toposes, sketch the construction of strict universes, and describe some applications. For simplicity and wider familiarity I will focus on (2,1)-toposes (categories of stacks of ordinary 1-groupoids), where many of the important issues and ideas already arise.

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