Non-commutative disintegrations and regular conditional probabilities

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   - Existence and uniqueness
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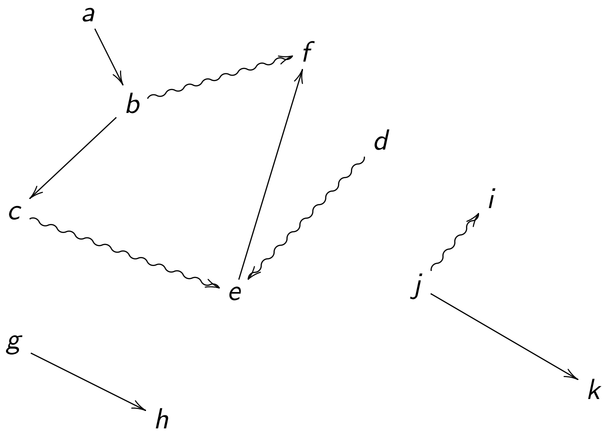
Category theory as a theory of processes

Processes can be deterministic or non-deterministic



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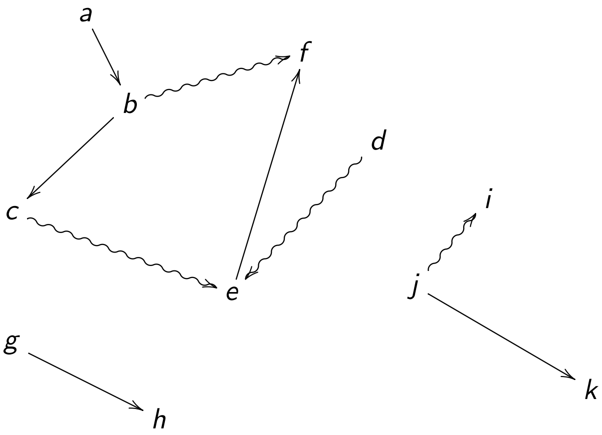
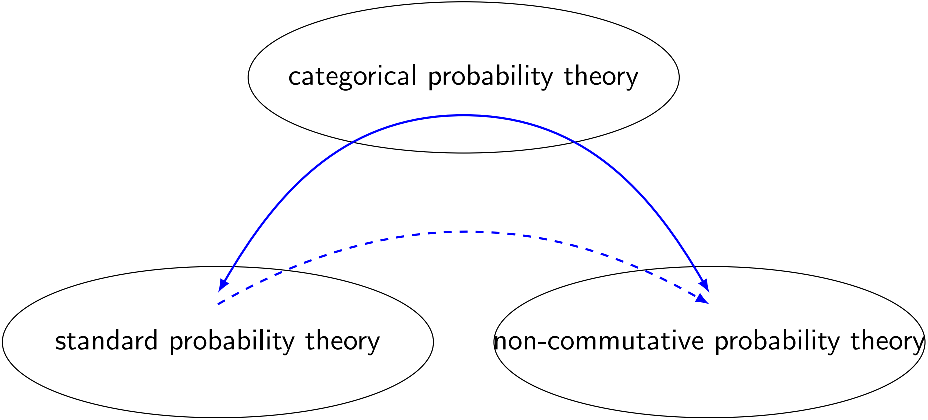


The Kleisli category associated to a monad is one way to distinguish between two such kinds of morphisms.

Goal for non-commutative regular conditional probabilities

Our goal will be to formulate concepts in probability theory categorically. This will enable us to abstract these concepts to contexts beyond their initial domain. We will focus our attention on quantum probability.



Stochastic maps: "if y then x" probabilistic statements

Let X and Y be finite sets. A *stochastic map* $r : Y \rightsquigarrow X$ assigns a probability measure on X to every point in Y. It is a function whose value at a point "spreads out" over the codomain.

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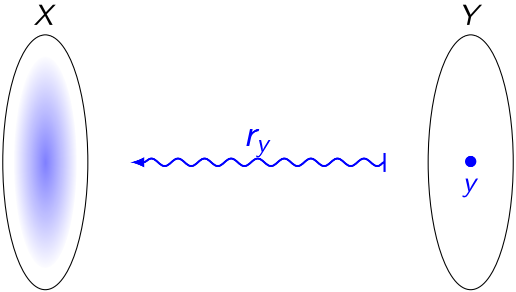

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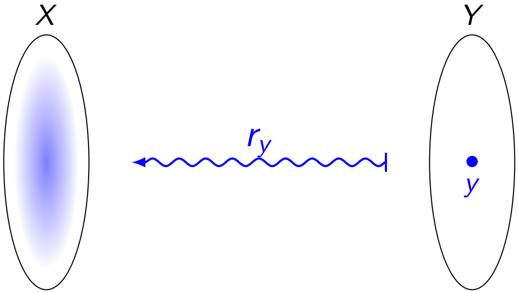
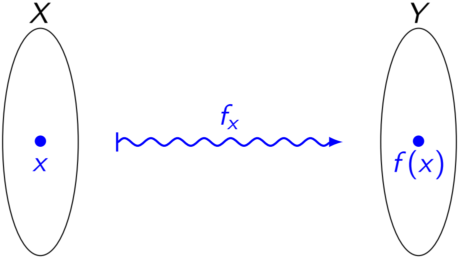


The value $r_y(x)$ of r_y at x is denoted by r_{xy}. Since r_y is a probability measure, $r_{xy} \geq 0$ for all x and y. Also, $\sum_{x\in X} r_{xy} = 1$ for all y.

Stochastic maps from functions: "if x then y" statements

A function $f : X \to Y$ induces a stochastic map $f : X \rightsquigarrow Y$ via

$$f_{yx} := \delta_{yf(x)}$$



where $\delta_{yy'}$ is the Kronecker delta and equals 1 if and only if $y = y'$ and is zero otherwise.

Composing stochastic maps

The *composition* $\nu \circ \mu : X \rightsquigarrow Z$ of $\mu : X \rightsquigarrow Y$ followed by $\nu : Y \rightsquigarrow Z$ is defined by matrix multiplication

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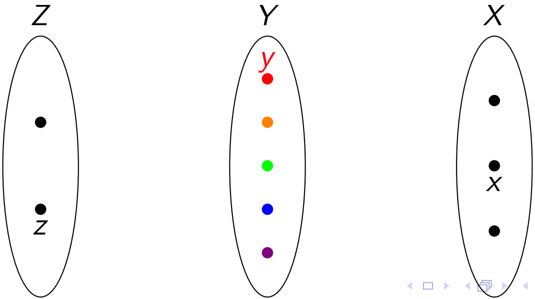

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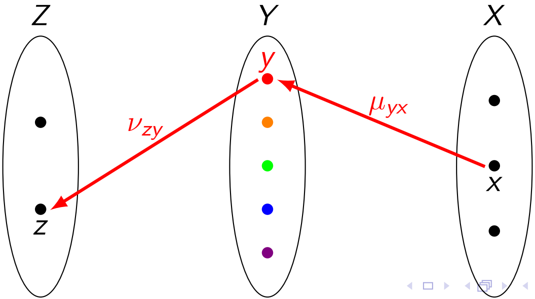

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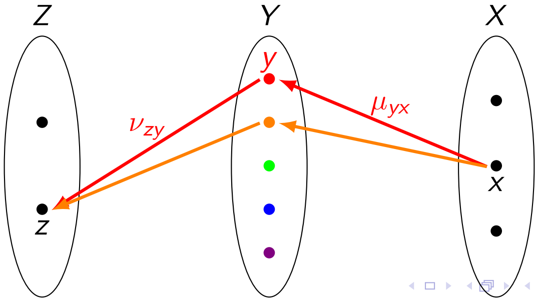

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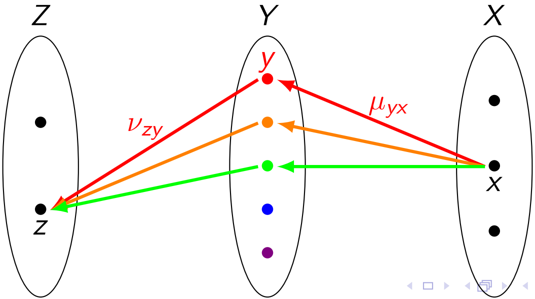

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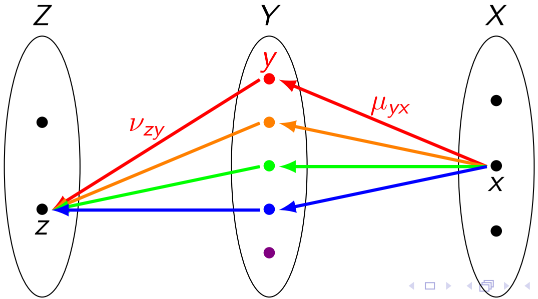

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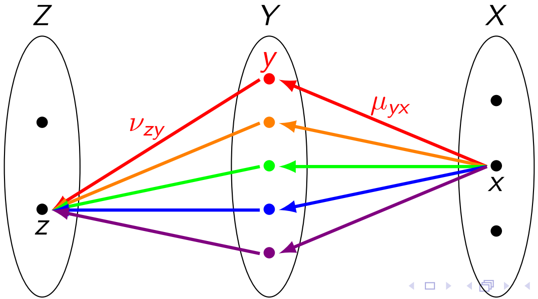

Special case: probability measures

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- A probability measure μ on X can be viewed as a stochastic map $\mu : \{\bullet\} \rightsquigarrow X$ from a single element set. Compare this to $\{\bullet\} \to X$, which picks out a single element of X.
- If $f : X \to Y$ is a function, the composition $f \circ \mu : \{\bullet\} \rightsquigarrow Y$ is the pushforward of μ along f.
- If $f : X \rightsquigarrow Y$ is a stochastic map, the composition $f \circ \mu : \{\bullet\} \rightsquigarrow Y$ is a generalization of the pushforward of a measure. The measure $f \circ \mu$ on Y is given by $(f \circ \mu)(y) = \sum_{x \in X} f_{yx}\mu(x)$ for each $y \in Y$.

Stochastic maps and their compositions form a category

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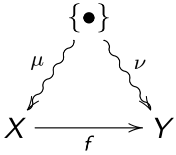
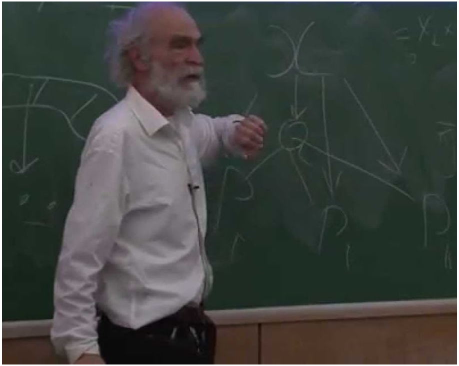
Thus, a commutative diagram of the form

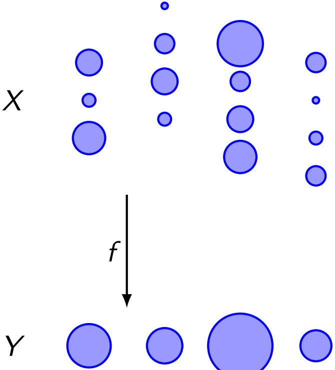


says that μ is a probability measure on X and its pushforward to Y along f is the probability measure ν.

A disintegration is a stochastic section

Let X and Y be finite sets equipped with probability measures. Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. f combines the water droplets and their volume (probabilities) add when they combine under f.





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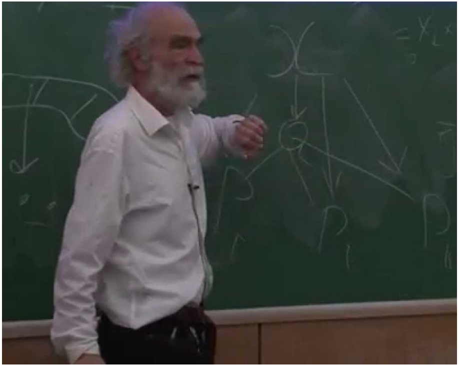
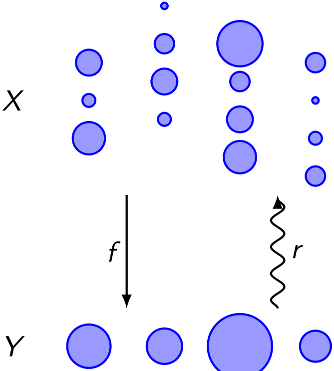
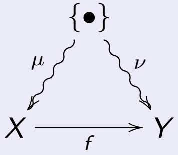




Disintegrations: diagrammatic definition

Definition

Let (X, μ) and (Y, ν) be probability spaces and let $f : X \to Y$ be a function such that the diagram on the right commutes.



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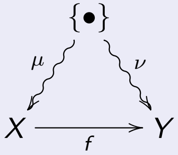
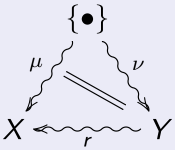


A *disintegration* of (f,μ,ν) is a stochastic map $Y \overset{r}{\rightsquigarrow} X$ such that



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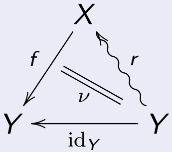

the latter diagram signifying commutativity ν-a.e.

A disintegration is also called a *regular conditional probability* and an *optimal hypothesis*.

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Classical disintegrations exist and are unique a.e. (almost everywhere). That's really all you need to know!

Application: Bayes' theorem

Question: Where do disintegrations show up?

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Corollary (Bayes' theorem)

Given $\{\bullet\} \overset{p}{\rightsquigarrow} X \overset{f}{\rightsquigarrow} Y$, *there exists a* $Y \overset{g}{\rightsquigarrow} X$ *such that*

$$
\begin{array}{ccc}
 & \{\bullet\} & \\
{\scriptstyle p}\swarrow & = & \searrow{\scriptstyle f\circ p} \\
X & \overset{g}{\leftsquigarrow} & Y
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccccc}
Y & \overset{f\circ p}{\leftsquigarrow} & \{\bullet\} & \overset{p}{\rightsquigarrow} & X \\
\downarrow{\scriptstyle \Delta_Y} & & = & & \downarrow{\scriptstyle \Delta_X} \\
Y \times Y & \overset{g\times \mathrm{id}_Y}{\rightsquigarrow} & X \times Y & \overset{\mathrm{id}_X \times f}{\leftsquigarrow} & X \times X
\end{array}
.
$$

Furthermore, for any other g' *satisfying these two conditions,* $g \underset{f\circ p}{=} g'$.

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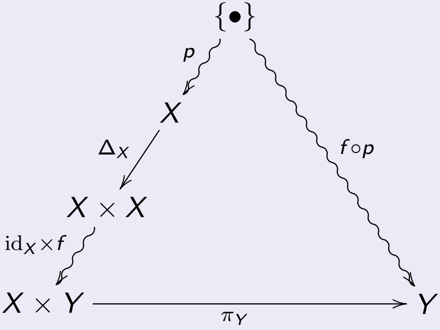

□

Objects: Finite-dimensional C^*-algebras

- Let $\mathcal{M}_n(\mathbb{C})$ denote the set of complex $n \times n$ matrices. It is an example of a C^*-algebra: we can add and multiply $n \times n$ matrices, the operator norm gives a norm, and A^* is the conjugate transpose of A.

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- Every finite-dimensional C^*-algebra is (C^*-algebraically isomorphic to) a direct sum of matrix algebras.
- In particular, $\mathbb{C}^X$, functions from a finite set X to $\mathbb{C}$, is a *commutative* C^*-algebra (it is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$). A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$.

Morphisms: *-homomorphisms and CPU maps

- Every *completely positive unital (CPU) map* $\varphi : \mathcal{M}_m(\mathbb{C}) \rightsquigarrow \mathcal{M}_n(\mathbb{C})$ preserves positivity of matrices and their tensor products with finite-dimensional identities.

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- Every (unital) *$*$-homomorphism* $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is of the form

$$F(A) = U \begin{bmatrix} A & & 0 \\ & \ddots & \\ 0 & & A \end{bmatrix} U^*,$$

where U is unitary. In particular $m = np$ for some $p \in \mathbb{N}$.

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- For every CPU map $\omega : \mathcal{M}_n(\mathbb{C}) \rightsquigarrow \mathbb{C}$ (called a *state*), there exists a unique $n \times n$ positive matrix ρ such that $\mathrm{tr}(\rho) = 1$ and $\mathrm{tr}(\rho A) = \omega(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$. ρ is called a *density matrix*.

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- For every CPU map $\omega : \mathbb{C}^X \rightsquigarrow \mathbb{C}$ (also called a *state*), there exists a unique probability measure $p : \{\bullet\} \rightsquigarrow X$ such that $\omega(\varphi) = \sum_{x \in X} p_x \varphi(x)$ for all $\varphi \in \mathbb{C}^X$. We write this state as $\langle p, \cdot \rangle$.

From finite sets to finite-dimensional C^*-algebras

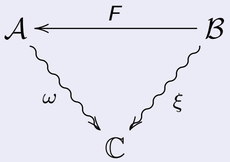
There is a (contravariant) fully faithful functor from finite sets and stochastic maps to finite-dimensional C^*-algebras and CPU maps.

category theory	classical/ commutative	quantum/ noncommutative	physics/ interpretation
object	set	C^*-algebra	phase space observables
$\rightarrow$ morphism	function	$*$-homomorphism	deterministic process
$\rightsquigarrow$ morphism	stochastic map	CPU map	non-deterministic process
monoidal product	cartesian product $\times$	tensor product $\otimes$	combining systems
$\rightsquigarrow$ to/from monoidal unit	probability measure	C^*-algebra state/ density matrix	physical state

Non-commutative disintegrations

Definition (P–Russo)

Let $(\mathcal{A},\omega)$ and $(\mathcal{B},\xi)$ be C^*-algebras equipped with states. Let $F:\mathcal{B}\to\mathcal{A}$ be a $*$-homomorphism such that the diagram on the right commutes.



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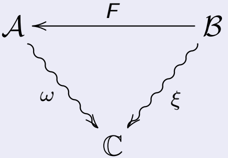
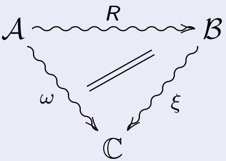


A *disintegration* of ω over ξ consistent with F is a CPU map $R:\mathcal{A}\rightsquigarrow\mathcal{B}$ such that



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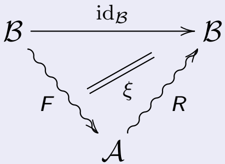

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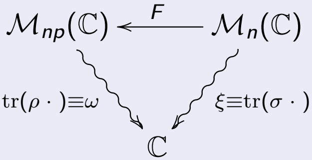

be a commutative diagram with F *the* $*$*-homomorphism given by the block diagonal inclusion* $F(A) = \mathrm{diag}(A, \ldots, A)$.

Existence and uniqueness of disintegrations

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$$\begin{array}{ccc} \mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{\;F\;} & \mathcal{M}_n(\mathbb{C}) \\ & \mathrm{tr}(\rho\,\cdot\,)\equiv\omega \searrow \quad \swarrow \xi\equiv\mathrm{tr}(\sigma\,\cdot\,) & \\ & \mathbb{C} & \end{array}$$

be a commutative diagram with F *the* $*$*-homomorphism given by the block diagonal inclusion* $F(A) = \mathrm{diag}(A, \ldots, A)$. *A disintegration of* ω *over* ξ *consistent with* F *exists* **if and only if** *there exists a density matrix* $\tau \in \mathcal{M}_p(\mathbb{C})$ *such that* $\rho = \tau \otimes \sigma$.

Example 1: Einstein–Podolsky–Rosen

Theorem (P–Russo)

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map.*

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and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ *be the diagonal map. Then* $\mathrm{tr}(\sigma A) = \mathrm{tr}(\rho F(A))$ *for all* A *but there does not exist a disintegration of* ρ *over* σ *consistent with* F*.*

Example 1: Einstein–Podolsky–Rosen

Theorem (P–Russo)

Let

$$\rho := \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \& \qquad \sigma := \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

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Proof.

ρ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two 2×2 density matrices. □

Example 2: Diagonal density matrices

Theorem (P–Russo)

Fix $p_1, p_2, p_3, p_4 \geq 0$ *with* $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, *and* $p_2 + p_4 > 0$. *Let*

$$\rho = \begin{bmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & p_3 & 0 \\ 0 & 0 & 0 & p_4 \end{bmatrix} \qquad \& \qquad \sigma = \begin{bmatrix} p_1 + p_3 & 0 \\ 0 & p_2 + p_4 \end{bmatrix}$$

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$$\rho = \sum_{\lambda \in \sigma(A)} P_\lambda \rho P_\lambda,$$

where the right-hand-side is called the Lüders projection of ρ with respect to the measurement of A.

Example 4: A “no-go” theorem for pure to mixed states

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Theorem (P–Russo)

Given a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{\;F\;} & \mathcal{M}_{n}(\mathbb{C}) \\ & {\scriptstyle \operatorname{tr}(\rho\,\cdot\,)} \searrow \quad \swarrow {\scriptstyle \operatorname{tr}(\sigma\,\cdot\,)} & \\ & \mathbb{C} & \end{array}$$

of CPU maps with ρ pure, if a disintegration exists, then σ must necessarily be pure as well.

Thank you!

Thank you for your attention!