Lax Gray tensor product for 2-quasi-categories

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Lax Gray tensor product of 2-categories

In lax Gray tensor product $\mathscr{A} \boxtimes \mathscr{B}$,

$$\left.\begin{array}{l} x \xrightarrow{f} x' \text{ in } \mathscr{A} \\[2em] y \xrightarrow{g} y' \text{ in } \mathscr{B} \end{array}\right\} \quad \rightsquigarrow \quad \begin{tikzcd} (x,y) \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\ (x,y') \arrow[r, "{(f,y')}"'] & (x',y') \end{tikzcd}$$

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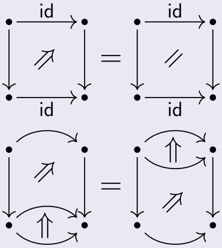
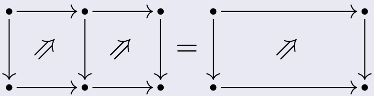
Coherence conditions





$+$ "vertical" counterparts

Δ

Δ consists of free categories $[n]$:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n$$

Δ and Θ_2

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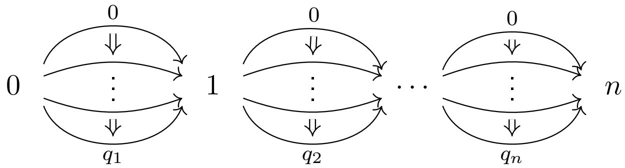
Θ_2 consists of free 2-categories $[n; q_1, \ldots, q_n]$:



Definition

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2-quasi-categories

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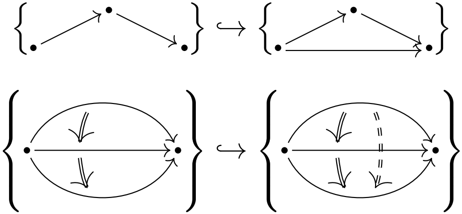
Theorem

2-quasi-categories and fibrations into them can be characterised by RLP wrt inner horn inclusions and equivalence extensions (introduced by Oury).

Some pictures

Inner horns look like:



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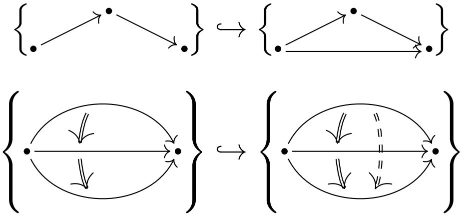
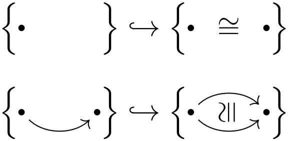


Equivalence extensions look like:



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$$\left\{\begin{array}{c}
\begin{tikzcd} \bullet \arrow[r, bend left] & \bullet \arrow[r] & \bullet \end{tikzcd} \\[2em]
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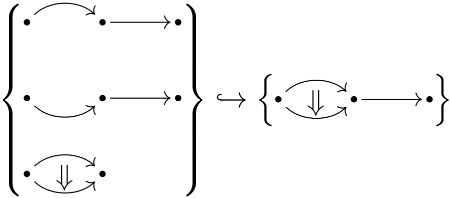

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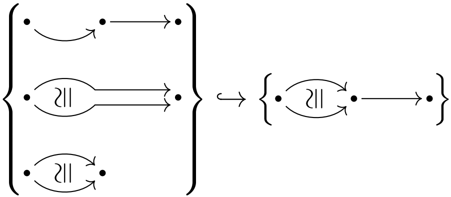

Lax Gray tensor product of Θ_2-sets

Definition

We define the lax Gray tensor product of Θ_2-sets by extending

$$\Theta_2 \times \Theta_2 \hookrightarrow 2\text{-}\underline{\mathrm{Cat}} \times 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\boxtimes} 2\text{-}\underline{\mathrm{Cat}} \xrightarrow{\text{nerve}} \widehat{\Theta_2}$$

cocontinuously in each variable.

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Solution: prove $X \otimes \{\cdot \cong \cdot\} \simeq X \times \{\cdot \cong \cdot\}$.

Non-asssociativity

$\Theta_2[1;0] \otimes \Theta_2[1;0]$

$\longrightarrow \otimes \big\downarrow$ consists of

$$\underbrace{\begin{tikzcd} \bullet \arrow[d] & \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[d] \\ & \bullet \end{tikzcd}}_{(2;0,0)\text{-cells}} \qquad \underbrace{\begin{tikzcd} \bullet \arrow[r, dash] \arrow[d, dash] & {} \arrow[d] \\ {} \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \bullet \end{tikzcd}}_{(1;1)\text{-cell}}$$

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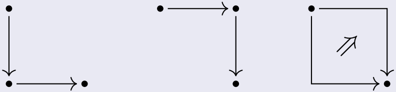
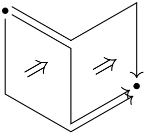
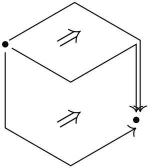


$\left(\nearrow \otimes \searrow \right) \otimes \downarrow$ contains



and



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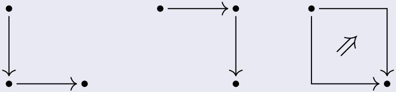
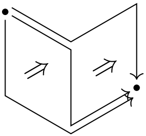
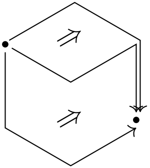
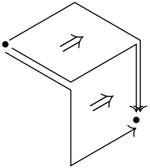
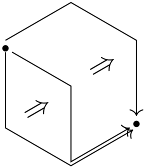


$\left(\nearrow \otimes \searrow \right) \otimes \downarrow$ contains  and 

$\nearrow \otimes \left(\searrow \otimes \downarrow \right)$ contains  and 

n-ary tensor

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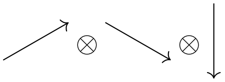
So that:



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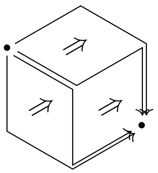

Associativity up to homotopy

Proposition

These form a lax monoidal structure on $\widehat{\Theta_2}$.

e.g. We have comparison maps

$$\otimes_2(\otimes_2(X, Y), \otimes_1(Z)) \to \otimes_3(X, Y, Z).$$

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Theorem

(The relative version of) these comparison maps are *trivial cofibrations.*

That's it!

Thank you!