

Picard Cohomology

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Picard Categories

- Groupoid
- Symmetric monoidal
- Group-like:
 - For all X , there is a Y such that $X \otimes Y \cong I \cong Y \otimes X$
- Essential data:
 - $\pi_0(\mathcal{C}) = \text{Obj}(\mathcal{C}) / \cong$
 - $\pi_1(\mathcal{C}) = \mathcal{C}(I, I)$
 - $K : \pi_0(\mathcal{C}) \rightarrow \pi_1(\mathcal{C})$,
 $X \mapsto \beta_{X, X} \in \mathcal{C}(X \otimes X, X \otimes X) \cong \pi_1(\mathcal{C})$

Examples

- $Pic(R) := R\text{-Mod}_{inv}^{\cong}$, for $R \in \mathbf{CRing}$
 - Note: $\pi_0(Pic(R)) = pic(R)$
- $\Pi_1 X$ for $X \in \Omega^3 \mathbf{Top}$
- \mathcal{Z} , “Super Integers”
 - $Obj(\mathcal{Z}) = \mathbb{Z}$, $\mathcal{Z}(n, m) \cong \begin{cases} \mathbb{Z}/2, & \text{if } n = m \\ 0, & \text{else} \end{cases}$
 - Call $\mathcal{Z}(n, n) = \{\pm 1_n\}$
 - $(\beta : n + m \rightarrow m + n) = (-1_{n+m})^{nm}$

Free Picard Categories

Theorem (H)

The forgetful functor $U : \mathbf{Pic} \rightarrow \mathbf{Grpd}$ has a left adjoint given by

$$\mathcal{Z}[_] : \mathbf{Grpd} \rightarrow \mathbf{Pic}.$$

Specifically: For $\mathcal{G} \in \mathbf{Grpd}$ and $\mathcal{A} \in \mathbf{Pic}$,

$$\mathbf{Pic}(\mathcal{Z}[\mathcal{G}], \mathcal{A}) \simeq \mathbf{Grpd}(\mathcal{G}, \mathcal{A})$$

as Picard categories, pseudonatural in \mathcal{G} and \mathcal{A} .

Tensoring over **Grpd**

Corollary (H)

Pic is tensored over **Grpd**

Specifically: For $\mathcal{G} \in \mathbf{Grpd}$ and $\mathcal{A} \in \mathbf{Pic}$, there exists $\mathcal{A}[\mathcal{G}] \in \mathbf{Pic}$ so that for all $\mathcal{B} \in \mathbf{Pic}$,

$$\mathbf{Pic}(\mathcal{A}[\mathcal{G}], \mathcal{B}) \simeq \mathbf{Grpd}(\mathcal{G}, \mathbf{Pic}(\mathcal{A}, \mathcal{B}))$$

pseudonaturally.

Picard Cohomology

Definition: Category of Modules over $\mathcal{G} \in \mathbf{Pic}$

For $\mathcal{G} \in \mathbf{Pic}$, $\mathcal{G}\text{-Mod} := \text{PsFunk}(\Sigma\mathcal{G}, \mathbf{Pic})$

Definition: Picard Cohomology

For $\mathcal{G} \in \mathbf{Pic}$ and $\mathcal{M} \in \mathcal{G}\text{-Mod}$, define

$$\mathcal{H}^n(\mathcal{G}; \mathcal{M}) := \mathbb{R}^n(\mathcal{G}\text{-Mod}(\mathcal{Z}_{\text{triv}}, _))(\mathcal{M})$$

i.e.

$$\mathcal{H}^n(\mathcal{G}; \mathcal{M}) = \text{Ext}_{\mathcal{G}\text{-Mod}}^n(\mathcal{Z}_{\text{triv}}, \mathcal{M})$$

i.e.

$$\mathcal{H}^n(\mathcal{G}; \mathcal{M}) = \mathbb{L}^n(\mathcal{G}\text{-Mod}(_, \mathcal{M}))(\mathcal{Z}_{\text{triv}})$$

A first computation: $\mathcal{H}^m(\mathbb{Z}/n; \mathcal{Z})$

Proposition (H)

The chain complex

$$\dots \xrightarrow{N} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{N} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{x-1} \mathcal{Z}[\mathbb{Z}/n] \xrightarrow{0} 0 \xrightarrow{0} \dots$$

provides a projective resolution of

$$\dots \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z}_{\text{triv}} \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Lemma (H)

Applying $\mathbb{Z}/n\text{-Mod}(_, \mathcal{Z}_{\text{triv}})$ to the above yields

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \mathcal{Z} \xrightarrow{0} \mathcal{Z} \xrightarrow{n} \dots$$

A first computation: $\mathcal{H}^m(\mathbb{Z}/n; \mathbb{Z})$

Theorem (H)

For n even,

$$\mathcal{H}^m(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2 & m = 0 \\ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z} \times \mathbb{Z}/2) & m = 1 \\ \mathbb{Z}/n \oplus \mathbb{Z}/2 & m > 1 \text{ is even} \\ \mathbb{Z}/2 \oplus \Sigma(\mathbb{Z}/n \times \mathbb{Z}/2) & m > 1 \text{ is odd} \end{cases}$$

For n odd,

$$\mathcal{H}^m(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0 \\ \Sigma(\mathbb{Z}) & m = 1 \\ \mathbb{Z}/n & m > 1 \text{ is even} \\ \Sigma(\mathbb{Z}/n) & m > 1 \text{ is odd} \end{cases}$$

A second computation: $\mathcal{H}^m(\mathbb{Z}/n; \mathbb{Z})$

Lemma (H)

Applying $\mathbb{Z}/n\text{-Mod}(_, \mathbb{Z}_{\text{triv}})$ to the previous resolution yields

$$0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \dots$$

Theorem (H)

For all n ,

$$\mathcal{H}^m(\mathbb{Z}/n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & m = 0 \\ \Sigma(\mathbb{Z}) & m = 1 \\ \mathbb{Z}/n & m > 1 \text{ is even} \\ \Sigma(\mathbb{Z}/n) & m > 1 \text{ is odd} \end{cases}$$

Chain complexes of Picard categories

- Chain complex of Picard categories

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & \curvearrowright & \uparrow \partial_{n-2} & \curvearrowright & \uparrow \partial_n & \curvearrowright & \\
 \dots & \rightarrow & \mathcal{A}_{n-2} & \xrightarrow{d_{n-2}} & \mathcal{A}_{n-1} & \xrightarrow{d_{n-1}} & \mathcal{A}_n & \xrightarrow{d_n} & \mathcal{A}_{n+1} & \xrightarrow{d_{n+1}} & \mathcal{A}_{n+2} & \rightarrow & \dots \\
 & & & & \downarrow \partial_{n-1} & & & & & & & & \\
 & & & & 0 & & & & & & & &
 \end{array}$$

- Compute cohomology using *relative* kernel and cokernel

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow \varphi & & \\
 \text{Ker}(F, \varphi) & \xrightarrow{k} & \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\
 & & \downarrow \kappa & & & & \\
 & & 0 & & & &
 \end{array}$$

- If $\mathcal{H}^n(\mathcal{A}_\bullet) \simeq 0$, call \mathcal{A}_\bullet *relative exact* at \mathcal{A}_n

Projective Picard categories

Definition

$\mathcal{P} \in \mathbf{Pic}$ is *projective* if for all

$$\begin{array}{ccc} & & \mathcal{P} \\ & \swarrow \text{---} & \downarrow H \\ \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array}$$

\cong

with G essentially surjective, such a lift exists.

- Problems:
 - No homological rephrasing
 - \mathcal{P} is projective if and only if $\mathcal{P} = \mathcal{Z}^{\oplus \kappa} = \mathcal{Z}[\mathcal{G}]$ for discrete \mathcal{G}

Relative projective Picard categories

Definition (H)

$\mathcal{P} \in \mathbf{Pic}$ is *relative projective* if for all

$$\begin{array}{ccccc} & & & & \mathcal{P} \\ & & & \swarrow & \downarrow H \\ & & & \cong & \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\ & \searrow & \downarrow \varphi & \nearrow & \\ & & 0 & & \end{array}$$

with G essentially surjective and φ -full, such a lift exists.

- “ φ -full” is a generalization of full

Relative projective Picard categories

Definition, rephrased

$\mathcal{P} \in \mathbf{Pic}$ is *relative projective* if for all

$$\begin{array}{ccccccc} & & & & \mathcal{P} & & \\ & & & & \swarrow \text{---} & \downarrow H & \\ & & & & & \cong & \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} & \xrightarrow{0} & 0 \xrightarrow{0} 0 \\ & & \downarrow \varphi & \nearrow & & & \\ & & 0 & & & & \end{array}$$

with row relative exact at \mathcal{C} , such a lift exists.

Relative projective Picard categories

Theorem (H)

$\mathcal{P} \in \mathbf{Pic}$ is relative projective if and only if $\mathbf{Pic}(\mathcal{P}, _)$ is exact.

- But not all free Picard categories are relative projective!

Proposition (H)

$\mathcal{Z}[\mathcal{G}]$ is relative projective if and only if for all $G \in \mathcal{G}$, $\text{End}(G)$ is free.

Proposition (H)

For free $A \in \mathbf{Ab}$, $\Sigma A \in \mathbf{Pic}$ is relative projective but not free.

Enough relative projectives

Note: If G is φ -full, then it is κ_G -full

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \varphi & \nearrow & \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array}$$

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \kappa_G & \nearrow & \\ \text{Ker}(G) & \xrightarrow{k_G} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array}$$

Definition

Say **Pic** has *enough relative projectives* if for all $\mathcal{C} \in \mathbf{Pic}$, there is a relative projective \mathcal{P} and $G : \mathcal{P} \rightarrow \mathcal{C}$ that is essentially surjective and κ_G -full.

Corollary (H)

Pic has *enough relative projectives*.

Thank you~!

References



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