Graphical regular logic

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University of Edinburgh
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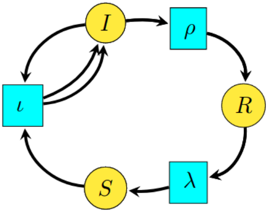
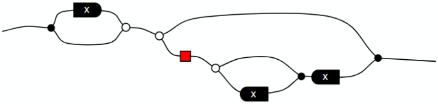
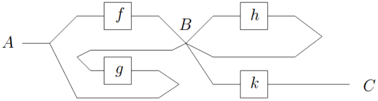


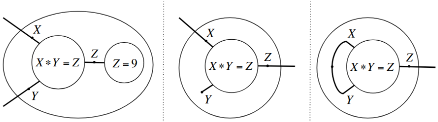
Baez, Pollard: *A compositional framework for reaction networks*



Bonchi, Sobocinski, Zanasi: *A categorical semantics of signal flow graphs*



Rosebrugh, Sabadini, Walters: *Calculating colimits compositionally*

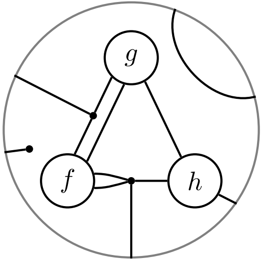


"all pairs of integers whose product is 9"

"all pairs of integers in which one is divisible by the other."

"all perfect squares"

Spivak: *The operad of wiring diagrams*



g
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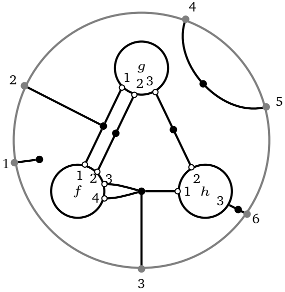
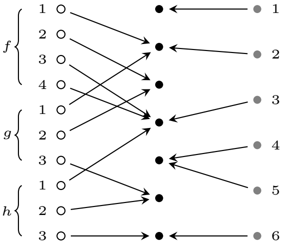




$$A \longrightarrow N \longleftarrow B$$

$$\left\{\mathsf{Cospan}_T \xrightarrow{\text{lax monoidal}} \mathsf{Set}\right\}$$

$$\uparrow\downarrow$$

$$\left\{\text{hypergraph} \quad \text{categories}\right\}$$

$$\left\{ \mathsf{Cospan}_T \xrightarrow{\text{lax monoidal}} \mathsf{Set} \right\}$$

$$\mathcal{H}(I,-)\Big\uparrow \quad \Big\downarrow \text{decorated corelations}$$

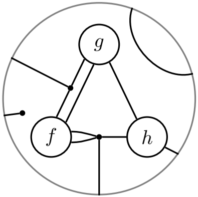
$$\left\{ \text{hypergraph} \quad \text{categories} \right\}$$

The category of cospan algebras is equivalent to the category of objectwise-free hypergraph categories

$$\left\{ \mathsf{Cospan}_T \xrightarrow{\text{lax monoidal}} \mathsf{Set} \right\}$$

$$\uparrow\downarrow$$

$$\left\{ \text{hypergraph} \quad \text{categories} \right\}$$



$$\left\{ \mathsf{Cospan}_T \xrightarrow{\text{lax monoidal}} \mathsf{Set} \right\}$$

$$\uparrow\downarrow$$

$$\left\{ \qquad\qquad\qquad \text{categories} \right\}$$

$$\left\{ \mathsf{Cospan}_T \xrightarrow{\text{lax monoidal}} \mathbb{P}\mathsf{oset} \right\}$$

$$\uparrow\downarrow$$

$$\left\{ \qquad\qquad\qquad \text{categories} \right\}$$

$$\left\{ \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \xrightarrow{\text{lax monoidal}} \mathbb{P}\mathsf{oset} \right\}$$

$$\uparrow\downarrow$$

$$\left\{ \qquad\qquad\qquad \text{categories} \right\}$$

$$\left\{ \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \xrightarrow{\text{right ajax monoidal}} \mathbb{P}\mathsf{oset} \right\}$$

$$\uparrow\downarrow$$

$$\left\{ \qquad\qquad\qquad \text{categories} \right\}$$

$$\left\{ \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \xrightarrow{\text{right ajax monoidal}} \mathbb{P}\mathsf{oset} \right\}$$

$$\text{subobject lattices} \uparrow \quad \downarrow \text{syntactic category}$$

$$\left\{ \textit{regular} \ \text{ categories} \right\}$$

$$
\left\{ \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \xrightarrow{\text{right ajax monoidal}} \mathbb{P}\mathsf{oset} \right\}
$$

$$
\text{subobject lattices}\Big\uparrow \quad \Big\downarrow\text{syntactic category}
$$

$$
\left\{ \ \textit{regular} \ \ \text{categories} \right\}
$$

Key idea: Regular calculi present regular categories.

Outline

II. The theorem

RgCat

A **regular category** is a category with finite limits and pullback stable image factorisations.

A **regular functor** is a functor between regular categories that preserves finite limits and image factorisations.

Examples: FinSet, $\mathsf{FinSet}^{\mathrm{op}}$, Set, $\mathsf{Set}^{\mathrm{op}}$, FDVect, Vect, abelian categories, toposes, any category monadic over Set, ...

Given a regular category $\mathcal{R}$, we may construct its **relations bicategory** $\mathbb{R}\mathsf{el}_{\mathcal{R}}$ with the same objects, but where 1-morphisms are jointly-monic spans.

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

Ajax functors

A **right ajax (monoidal) functor** is a lax monoidal functor $P\colon \mathbb{C} \to \mathbb{D}$ in which the laxators are right adjoints.

$$I_{\mathbb{D}} \underset{\lambda_I}{\overset{\rho_I}{\underset{\Leftarrow}{\rightleftarrows}}} P(I_{\mathbb{C}}) \qquad\qquad P(c_1) \otimes P(c_2) \underset{\lambda_{c_1,c_2}}{\overset{\rho_{c_1,c_2}}{\underset{\Leftarrow}{\rightleftarrows}}} P(c_1 \otimes c_2)\ .$$

Example: a right ajax functor $P\colon 1 \to \mathbb{P}\mathrm{oset}$ is a meet semilattice.

$$1 \underset{!}{\overset{\top}{\underset{\Leftarrow}{\rightleftarrows}}} P(\bullet) \qquad\qquad P(\bullet) \times P(\bullet) \underset{\delta}{\overset{\wedge}{\underset{\Leftarrow}{\rightleftarrows}}} P(\bullet)\ .$$

RgCalc

A **regular calculus** (T, P) is a set T and a right ajax functor

$$P \colon \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \longrightarrow \mathbb{P}\mathsf{oset}.$$

A **morphism** $(F, F^\sharp)\colon (T, P) \to (T', P')$ **of regular calculi** is a function F and a monoidal natural transformation $F^\sharp$:

$$\begin{tikzcd} T \arrow[d, "F"'] \\ T' \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \arrow[d, "\mathbb{C}\mathsf{ospan}_F^{\mathrm{co}}"'] \arrow[dr, bend left, "P"] \arrow[d, phantom, shift left=6ex, "F^\sharp\Downarrow"] & \\ \mathbb{C}\mathsf{ospan}_{T'}^{\mathrm{co}} \arrow[r, bend right, "P'"'] & \mathbb{P}\mathsf{oset} \end{tikzcd}$$

Theorem
We have an adjunction

$$\mathsf{RgCalc} \underset{\mathbf{prd}}{\overset{\mathbf{syn}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\Rightarrow}}}} \mathsf{RgCat}.$$

where $\mathbf{prd}$ *is fully faithful, and for any regular category* $\mathcal{R}$*, the counit map* $\mathbf{syn}(\mathbf{prd}(\mathcal{R})) \to \mathcal{R}$ *is an equivalence.*

$$\left\{\mathbb{C}\mathsf{ospan}_T^{\mathrm{co}} \xrightarrow{\text{right ajax monoidal}} \mathbb{P}\mathsf{oset}\right\}$$

$$\text{subobject lattices}\uparrow \quad \downarrow\text{syntactic category}$$

$$\left\{\ \textit{regular}\ \ \text{categories}\right\}$$

Key idea: Regular calculi present regular categories.

$$\left\{\mathbb{R}\mathsf{el}_{\mathsf{FreeReg}_T} \xrightarrow{\text{right ajax monoidal}} \mathbb{P}\mathsf{oset}\right\}$$

$$\text{subobject lattices}\Big\uparrow \quad \Big\downarrow\text{syntactic category}$$

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Key idea: Regular calculi present regular categories.

III. Proof sketch

$$\mathsf{RgCalc} \underset{\mathbf{prd}}{\overset{\mathbf{syn}}{\rightleftarrows}} \mathsf{RgCat}. \quad (\Rightarrow)$$

Given a regular category $\mathcal{R}$, we construct the regular calculus

$$\mathbf{prd}(\mathcal{R}): \quad \mathbb{C}\mathsf{ospan}^{\mathrm{co}}_{\mathrm{Ob}\mathcal{R}} \xrightarrow{\mathsf{Frob}} \mathbb{R}\mathsf{el}_{\mathcal{R}} \xrightarrow{\mathbb{R}\mathsf{el}_{\mathcal{R}}(1,-)} \mathbb{P}\mathsf{oset}$$

where Frob is given by the hypergraph structure on $\mathbb{R}\mathsf{el}_{\mathcal{R}}$.

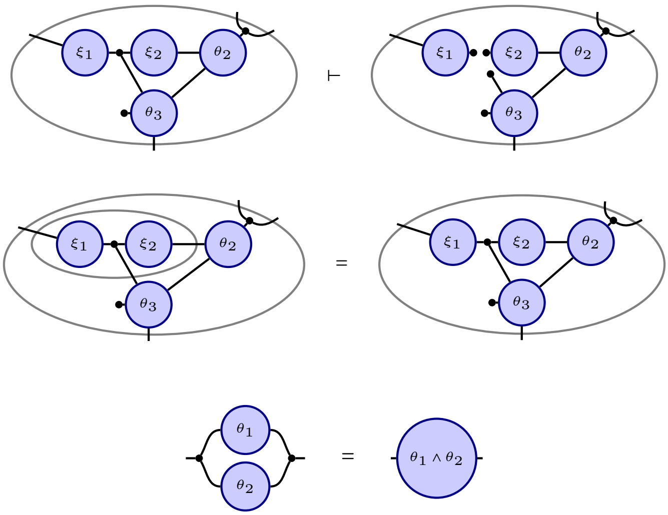
Given a regular calculus $P\colon \mathbb{C}\mathsf{ospan}^{\mathrm{co}}_{T} \to \mathbb{P}\mathsf{oset}$, we may construct the bicategory $\mathbb{R}\mathsf{el}_{\mathbf{syn}(P)}$:

objects: $\{(\Gamma, s) \mid \Gamma \in \mathbb{C}\mathsf{ospan}^{\mathrm{co}}_{T}, s \in P(\Gamma)\}$

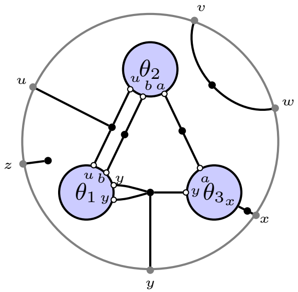
hom-posets: $\mathrm{Hom}((\Gamma, s), (\Gamma', s')) = P(\Gamma \oplus \Gamma')_{- \leq \rho(s, s')}$

The *syntactic category* $\mathbf{syn}(P)$ is the category of left adjoints in $\mathbb{R}\mathsf{el}_{\mathbf{syn}(P)}$.

Given any regular calculus, we may draw and interpret diagrams such as those below. The properties of regular calculi give 'deduction rules':



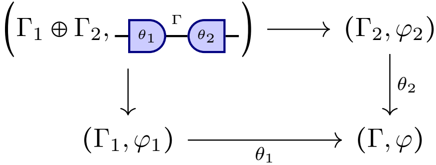
One might view this as a graphical regular logic, where **regular logic** is the fragment of first order logic given by $=$, $\top$, $\wedge$, $\exists$.



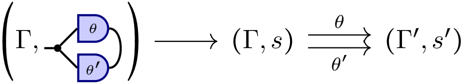
$$\psi(u, v, w, x, y, z) =$$

$$\exists a, b.\theta_1(u, b, y, y) \wedge \theta_2(a, b, u) \wedge \theta_3(b, x, y) \wedge (v = w) \wedge (z = z).$$

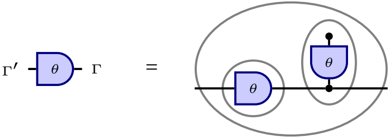
Pullback:



Equaliser:



Epi-mono factorisation:



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