HOMOTOPY-COHERENT ALGEBRAS AND POLYNOMIAL MONADS

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homotopy-coherent algebras

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Let $\mathbb{F}_*$ = category of pointed finite sets and let $\mathcal{S}$ = ∞-category of spaces. A special Γ-space is a functor

$$F\colon \mathbb{F}_* \to \mathcal{S}$$

satisfying the *Segal condition* $(F(\langle n\rangle) \xrightarrow{\sim} \prod_{i=1}^{n} F(\langle 1\rangle))$.

Examples

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- ∞-operads: the dendroidal category Ω^{op} of Moerdijk and Weiss
- ∞-properads: Γ^{op} of Hackney, Robertson and Yau

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But in which generality can we talk about “Segal conditions”?

We observe that all the examples have certain key features in common. This leads us to the following definition.

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An **algebraic pattern** $\mathcal{P}$ is an ∞-category equipped with

- an inert-active factorization system $(\mathcal{P}^{\mathrm{int}}, \mathcal{P}^{\mathrm{act}})$,
- a full subcategory $\mathcal{P}^{\mathrm{el}} \subseteq \mathcal{P}^{\mathrm{int}}$ (its objects are called *elementary*).

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Let $\mathcal{P}$ be an algebraic pattern and let $\mathcal{P}^{\mathrm{el}}_{X/} := \mathcal{P}^{\mathrm{el}} \times_{\mathcal{P}^{\mathrm{int}}} \mathcal{P}^{\mathrm{int}}_{X/}$.

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Then a *Segal $\mathcal{P}$-space* or *object* is a functor $F \colon \mathcal{P} \to \mathcal{S}$ (or $\mathcal{C} = \mathrm{Cat}_\infty$, ∞-topos, …) which satisfies the "Segal condition", i.e.

$$F(X) \xrightarrow{\sim} \lim_{E \in \mathcal{P}^{\mathrm{el}}_{X/}} F(E).$$

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- elementary objects $\xrightarrow{F}$ building blocks
- inert maps $\xrightarrow{F}$ geometric operations = “restriction maps” which are used for the Segal condition
- active maps $\xrightarrow{F}$ algebraic operations (multiplications, compositions,...)

Notation

We write $\mathrm{Seg}_{\mathcal{P}}(\mathcal{S})$ for the full subcategory of $\mathsf{Fun}(\mathcal{P}, \mathcal{S})$ spanned by Segal $\mathcal{P}$-spaces.

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- Ξ = category of unrooted trees defined by HRY. $\Rightarrow$ Cyclic ∞-operads = Segal Ξ^{op}-spaces.

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- Ξ = category of unrooted trees defined by HRY. $\Rightarrow$ Cyclic ∞-operads = Segal Ξ^{op}-spaces.
- $\mathcal{O}$ = Lurie's (generalized) ∞-operad $\Rightarrow$ $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ = Segal $\mathcal{O}$-objects in $\mathcal{C}$.

Remark

A functor $f\colon \mathcal{P} \to \mathcal{Q}$ induces adjunctions

$f_! : \mathsf{Fun}(\mathcal{P}, \mathcal{S}) \rightleftarrows \mathsf{Fun}(\mathcal{Q}, \mathcal{S}) : f^*$ and $f^* : \mathsf{Fun}(\mathcal{Q}, \mathcal{S}) \rightleftarrows \mathsf{Fun}(\mathcal{P}, \mathcal{S}) : f_*$.

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Under certain *checkable* conditions f^*, $f_!$ and f_* restrict to functors

$$f^*\colon\ \mathsf{Seg}_{\mathcal{Q}}(\mathcal{S}) \to \mathsf{Seg}_{\mathcal{P}}(\mathcal{S}) \text{ and } f_!, f_*\colon\ \mathsf{Seg}_{\mathcal{P}}(\mathcal{S}) \to \mathsf{Seg}_{\mathcal{Q}}(\mathcal{S}).$$

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$\mathcal{P}$ is *extendable* if $i_!\colon \mathsf{Fun}(\mathcal{P}^{\mathrm{el}}, \mathcal{S}) \simeq \mathsf{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S}) \to \mathsf{Seg}_{\mathcal{P}}(\mathcal{S})$ is given by restriction, where $i\colon \mathcal{P}^{\mathrm{int}} \hookrightarrow \mathcal{P}$ is the inclusion.

Applications

By checking conditions

- for right Kan extensions f_*: The forgetful functor $\mathrm{CycOpd}_\infty \to \mathrm{Opd}_\infty$ has a right adjoint.

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By checking conditions

- for right Kan extensions f_*: The forgetful functor $\mathrm{CycOpd}_\infty \to \mathrm{Opd}_\infty$ has a right adjoint.
- for left Kan extensions $f_!$: We recover the formula for operadic left Kan extensions in the sense of Lurie.
- for extendability: Θ_n^{op}, $\Delta^{n,\mathrm{op}}$ and Ω^{op} are extendable. $\Rightarrow$ Formula for free (∞, n)-categories, free n-fold ∞-categories and free ∞-operads.

Proposition

For every $\mathcal{P}$ the adjunction $\mathsf{Fun}(\mathcal{P}^{\mathrm{el}}, \mathcal{S}) \rightleftarrows \mathsf{Seg}_{\mathcal{P}}(\mathcal{S})$ is monadic.

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Questions

Which monads on presheaf ∞-categories can be described as the free Segal $\mathcal{P}$-space monad for an extendable pattern $\mathcal{P}$? How far is this correspondence from being an equivalence?

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  - where a morphism is a commutative square whose mate transformation is cartesian.

Theorem

If $\mathcal{P}$ is extendable, then the associated monad $T_{\mathcal{P}}$ on $\mathsf{Fun}(\mathcal{P}^{\mathrm{el}}, \mathcal{S})$ is polynomial. Hence, we have a functor

$$\mathfrak{M} \colon \mathsf{ExtPatt} \to \mathsf{PolyMnd}.$$

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We now want to prove that this functor is essentially surjective. For a given polynomial monad we want to construct the associated algebraic pattern.

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The main input for the proof for the essential surjectivity is an ∞-categorical version of work of Berger, Melliés and Weber and it is closely related to the “nerve theorem” studied by Leinster, Kock and Weber.

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The difference between cartesian monads (induced by Σ-free operads) and *weakly* cartesian monads in the 1-categorical world vanishes by going to that of ∞-categories.

patterns from polynomial monads: existence

Construction

Given a polynomial monad T on the presheaf ∞-category $\mathrm{Fun}(\mathcal{I}, \mathcal{S})$, define $\mathcal{P}_T$ by

$$
\begin{array}{ccccc}
 & & \mathcal{P}_T^{\mathrm{op}} & \hookrightarrow & \mathrm{Alg}_T(\mathrm{Fun}(\mathcal{I}, \mathcal{S})) \\
 & & {\scriptstyle F_T}\uparrow & & {\scriptstyle F_T}\uparrow \downarrow{\scriptstyle U_T} \\
\mathcal{I}^{\mathrm{op}} & \hookrightarrow & \mathcal{P}_T^{\mathrm{int,op}} & \hookrightarrow & \mathrm{Fun}(\mathcal{I}, \mathcal{S})
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The construction defines an algebraic pattern $\mathcal{P}_T$ with a factorization system $(\mathcal{P}_T^{\mathrm{int}}, \mathcal{P}_T^{\mathrm{act}})$ and $\mathcal{P}_T^{\mathrm{el}} = \mathcal{I}$.

Theorem

The assignment $T \mapsto \mathcal{P}_T$ gives a functor

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The assignment $T \mapsto \mathcal{P}_T$ gives a functor

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Moreover, this functor is fully faithful and $\mathfrak{M}\mathfrak{P} \simeq \mathrm{id}$.

Examples

- Free operad monad on $\mathsf{Fun}(\mathcal{I}, \mathcal{S})$ ($\mathcal{I}$ = the category of trees with at most one vertex) $\overset{\mathfrak{P}}{\mapsto} \Omega^{\mathrm{op}}$.

patterns from polynomial monads: uniqueness

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What is the essential image of the fully faithful functor $\mathfrak{P}\colon \mathrm{PolyMnd} \to \mathrm{ExtPatt}$?

patterns from polynomial monads: uniqueness

Theorem

An extendable algebraic patterns lies in the essential image of $\mathfrak{P}$ iff it is *nice*, i.e. every object $X \in \mathcal{P}$ admits an active map $X \to E$, $E \in \mathcal{P}^{\mathrm{el}}$ and $\mathsf{Map}_{\mathcal{P}^{\mathrm{int}}}(X, -) \in \mathsf{Seg}_{\mathcal{P}^{\mathrm{int}}}(\mathcal{S})$.

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Many algebraic patterns are nice: $\Delta^{n,\mathrm{op}}, \Theta_n^{\mathrm{op}}, \Omega^{\mathrm{op}}, \dots$

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Many algebraic patterns are nice: $\Delta^{n,\mathrm{op}}, \Theta_n^{\mathrm{op}}, \Omega^{\mathrm{op}}, \ldots$

$\mathbb{F}_*$ is not nice, $\mathfrak{PM}(\mathbb{F}_*) \simeq \mathsf{Span}^{\mathrm{inj}}(\mathbb{F})$.

patterns from polynomial monads: uniqueness

$\mathrm{ExtPatt}_n$ = the full subcategory of ExtPatt spanned by nice algebraic patterns.

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Corollary

The adjunction $\mathfrak{M} : \mathrm{ExtPatt} \rightleftarrows \mathrm{PolyMnd} : \mathfrak{P}$ restricts to an equivalence

$$\mathrm{ExtPatt}_n \simeq \mathrm{PolyMnd}.$$

In particular, PolyMnd is a localization of ExtPatt.