A Quillen model structure for bigroupoids and pseudofunctors

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Some similar results

Model structures exist on:

- Groupoids (Anderson)
- 2-Groupoids (Moerdijk-Svensson)
- 2-Categories (Lack)
- Bicategories (Lack)
- Pseudogroupoids (Lack)

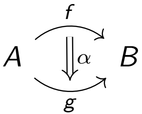
Bigroupoids

A bigroupoid consists of:

- 0-cells $A, B, C, \ldots$
- 1-cells (between 0-cells)

$$A \xrightarrow{f} B$$

- 2-cells (between parallel 1-cells)



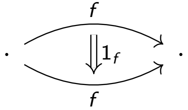
Bigroupoids

Bigroupoids have (on two levels):

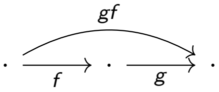
- Identity

$$A \xrightarrow{1_A} A$$

and



- Composition



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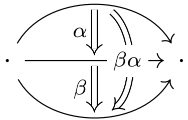
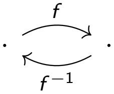


- Inversion



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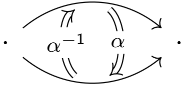

Bigroupoids

- The 2-cells follow the familiar laws:

$$(\gamma\beta)\alpha = \gamma(\beta\alpha), \qquad \alpha\alpha^{-1} = 1, \qquad \alpha 1 = \alpha, \qquad \text{etc.}$$

- The 1-cells follow these laws only up to a 2-cell. Example:

$$\text{In general:} \qquad (hg)f \neq h(gf)$$

$$\text{Instead:} \qquad (hg)f \stackrel{\alpha}{\Longrightarrow} h(gf)$$

- Plus coherence laws . . .

Morphisms

- A pseudofunctor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is structure preserving on 2-cells:

$$F\beta\alpha = F\beta F\alpha, \qquad F\alpha^{-1} = (F\alpha)^{-1}, \qquad F1 = 1$$

- On 1-cells this holds only up to a 2-cell. Example:

$$\text{In general:} \qquad Fgf \neq FgFf$$

$$\text{Instead:} \qquad Fgf \overset{\alpha}{\Longrightarrow} FgFf$$

- Plus coherence laws . . .

Bigroupoids + pseudofunctors form a category.

Model structures

A model structure on a category C consists of:

$$\mathcal{C}, \mathcal{F}, \mathcal{W} \subset \mathrm{Mor}(\mathsf{C}),$$

such that:

- $\mathsf{Iso} \subset \mathcal{W}$.
- $\mathcal{W}$ satifies 2-out-of-3.
- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

We do not require that C is (co)complete.

Model structure on bigroupoids

The classes of maps are given by:

- $F \in \mathcal{C} \iff F$ is injective on 0-cells and locally injective on 1-cells.
- $F \in \mathcal{F} \iff F$ lifts 1-cells and 2-cells.
- $F \in \mathcal{W} \iff F$ is a biequivalence.

Model structure on bigroupoids

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Lifting property for $F : \mathcal{A} \longrightarrow \mathcal{B}$ (on 1-cells):

$$\exists A \overset{\exists a}{\dashrightarrow} A'$$

$$B \overset{b}{\longrightarrow} FA'$$

Model structure on bigroupoids

Theorem

The category of bigroupoids and pseudofunctors carries a model structure, with $\mathcal{C}$, $\mathcal{F}$ and $\mathcal{W}$ as defined on the previous slide.

Theorem

The inclusion $I : 2-\mathrm{Grpd} \longrightarrow \mathrm{Bigrpd}$, of the category of 2-groupoids and 2-functors into the category of bigroupoids and pseudofunctors is the right adjoint part of a Quillen equivalence.

Coherence laws

For every propery (associativity, functoriality, ...) that needs to hold up to a 2-cell, we have a favourite witness. Example:

$$(hg)f \xRightarrow{\mathbf{a}_{h,g,f}} h(gf)$$

These witnesses need to interact in a coherent way. Example:

$$\begin{array}{ccc} (g1)f & \xRightarrow{\quad\mathbf{a}_{g,1,f}\quad} & g(1f) \\ & {\scriptstyle \mathbf{r}_g * 1_f} \searrow \qquad \swarrow {\scriptstyle 1_g * \mathbf{l}_f} & \\ & gf & \end{array}$$

Coherence theorem

Theorem

Every formal diagram ($\approx$ a diagram consisting of favourite witnesses) commutes.

Equivalently:

Theorem

For every morphism $F : \mathcal{G} \longrightarrow \mathcal{H}$ of (groupoid enriched) graphs, the induced 2-functor $\Delta : \mathrm{Free}_F(\mathcal{H}) \longrightarrow \mathrm{Free}_{2-\mathsf{Grpd}}(\mathcal{H})$, from the codomain of the free pseudofunctor on F to the free 2-groupoid on $\mathcal{H}$ is a biequivalence.

Remarks on the proof

- The coherence theorem is used to construct both the model structure and the Quillen equivalence.
- The small object argument is not used. Instead, more explicit constructions are given.
- Pullbacks of fibrations exist.
- Some constructions are made by mimicking constructions familiar from groupoids.
- The model structure on groupoids is used locally.

Future research

- Does there exist a model structure on bicategories + pseudofunctors?
- Or other similar weak higher order structures?
- Does it give rise to an (interesting) model of (some) type theory?