Exponentiability in Double Categories and the Glueing Construction

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Idea

What are the "exponentiable" objects Y in a double category

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{D}_0 \quad ?$$

For $\mathbb{C}\mathrm{at}$, $\mathbb{P}\mathrm{os}$, $\mathbb{T}\mathrm{op}$, $\mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{opos}$, can show directly:

$$Y \text{ is exponentiable in } \mathbb{D} \iff Y \text{ is exponentiable } \mathbb{D}_0$$

Showed they satisfy $\mathbb{D}_1 \simeq \mathbb{D}_0/\mathbb{2}$, generalizing Artin-Wraith glueing. [N 2012; JPAA]

Goal

To prove:

$$Y \text{ is exponentiable in } \mathbb{D} \iff Y \text{ is exponentiable in } \mathbb{D}_0$$

in a general theorem assuming $\mathbb{D}_1 \simeq \mathbb{D}_0/2$ plus ...

Plan

1. Double categories and the examples
2. Glueing categories
3. Lax Functors and Adjoints
4. Exponentiability in double categories

Double Categories

A double category $\mathbb{D}$ is a (pseudo) category object in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbb{D}_0 \qquad \mathbb{D}_1 \xleftarrow{\mathrm{id}^\bullet} \mathbb{D}_0$$

Objects: objects of $\mathbb{D}_0$

Horizontal morphisms: morphisms $f \colon X \longrightarrow Y$ of $\mathbb{D}_0$

Vertical morphism: objects of $\mathbb{D}_1$, denoted by $v \colon X_s \xrightarrow{\bullet} X_t$

Cells: morphisms of $\mathbb{D}_1$, denoted by

$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\varphi"] & Y_s \arrow[d, "w", "\bullet" marking] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$

Double Categories: Examples [N 2012; JPAA]

$\mathbb{T}\mathrm{op}$: top spaces X, $\underset{\text{cont maps}}{X \longrightarrow Y}$, $\underset{\text{lex}}{\dfrac{X_s \xrightarrow{\bullet} X_t}{\mathcal{O}(X_s) \longrightarrow \mathcal{O}(X_t)}}$,

$$\begin{tikzcd} \mathcal{O}(X_s) \arrow[r, "\mathcal{O}(f_s)"] \arrow[d, "v"'] \arrow[dr, phantom, "\supseteq"] & \mathcal{O}(Y_s) \arrow[d, "w"] \\ \mathcal{O}(X_t) \arrow[r, "\mathcal{O}(f_t)"'] & \mathcal{O}(Y_t) \end{tikzcd}$$

$\mathbb{L}\mathrm{oc}$: locales X, $\underset{\text{locale maps}}{X \longrightarrow Y}$, $\underset{\text{lex}}{X_s \xrightarrow{\bullet} X_t}$,

$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"'] \arrow[dr, phantom, "\geq"] & Y_s \arrow[d, "w"] \\ X_t \arrow[r, "f_t"'] & Y_t \end{tikzcd}$$

$\mathbb{T}\mathrm{opos}$: $\mathcal{S}$-toposes $\mathcal{X}$, $\underset{\text{geom. morph.}}{\mathcal{X} \longrightarrow \mathcal{Y}}$, $\underset{\text{lex}}{\mathcal{X}_s \xrightarrow{\bullet} \mathcal{X}_t}$,

$$\begin{tikzcd} \mathcal{X}_s \arrow[r, "f_s"] \arrow[d, "v"'] \arrow[dr, phantom, "\rightarrow"] & \mathcal{Y}_s \arrow[d, "w"] \\ \mathcal{X}_t \arrow[r, "f_t"'] & \mathcal{Y}_t \end{tikzcd}$$

Double Categories: Examples (cont.)

$\mathbb{C}\mathrm{at}$: categories X, $\underset{\text{functors}}{X \longrightarrow Y}$, $\underset{\text{profunctors}}{X_s \xrightarrow{\bullet} X_t}$,
$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\rightarrow"] & Y_s \arrow[d, "w", "\bullet"' marking] \\ X_t \arrow[r, "f_t"'] & Y_1 \end{tikzcd}$$

$\mathbb{P}\mathrm{os}$: posets X, $\underset{\text{monotone}}{X \longrightarrow Y}$, $\underset{\text{order ideals}}{X_s \xrightarrow{\bullet} X_t}$,
$$\begin{tikzcd} X_s \arrow[r, "f_s"] \arrow[d, "v"', "\bullet" marking] \arrow[dr, phantom, "\leq"] & Y_s \arrow[d, "w", "\bullet"' marking] \\ X_t \arrow[r, "f_t"'] & Y_1 \end{tikzcd}$$

Glueing Categories

(G1) $\mathbb{D}_0$ has finite limits

(G2) $\mathrm{id}^\bullet \colon \mathbb{D}_0 \longrightarrow \mathbb{D}_1$ has a left adjoint Γ with unit

$$\begin{tikzcd} X_s \arrow[r, "i_s"] \arrow[d, "v"'] \arrow[dr, phantom, "\gamma_v"] & \Gamma v \arrow[d, "\mathrm{id}^\bullet_{\Gamma v}"] \\ X_t \arrow[r, "i_t"'] & \Gamma v \end{tikzcd}$$

"*cotabulator*"

(G3) $\Gamma_{\mathbb{2}} \colon \mathbb{D}_1 \longrightarrow \mathbb{D}_0/\mathbb{2}$ is an equivalence, where $\mathbb{2} = \Gamma(\mathrm{id}^\bullet_1)$, and the following are pullbacks in $\mathbb{D}_0$

$$\begin{tikzcd} X_s \arrow[r, "i_s"] \arrow[d] & \Gamma v \arrow[d, "\Gamma_{\mathbb{2}} v"] \\ 1 \arrow[r, "i_s"'] & \mathbb{2} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_t \arrow[r, "i_t"] \arrow[d] & \Gamma v \arrow[d, "\Gamma_{\mathbb{2}} v"] \\ 1 \arrow[r, "i_t"'] & \mathbb{2} \end{tikzcd}$$

(G4) $\mathbb{D}$ is "horizontally invariant"

Glueing Categories: Examples

$\mathbb{T}\mathrm{op}$: Given $v\colon \mathcal{O}(X_s) \to \mathcal{O}(X_t)$, define $\Gamma v = X_s \sqcup X_t$ with $U = U_s \sqcup U_t$ open, if U_s, U_t are open and $U_t \subseteq v(U_s)$

$\mathbb{2}$ is the Sierpinski space

$\mathbb{L}\mathrm{oc}$: Γv defined by "Artin-Wraith glueing" along v

$\mathbb{2}$ is the Sierpinski locale $\mathcal{O}(\mathbb{2})$

$\mathbb{T}\mathrm{opos}$: Γv defined by "Artin-Wraith glueing" along v

$\mathbb{2}$ is the Sierpinski topos $\mathcal{S}^{\mathbb{2}}$

Glueing Categories: Examples, cont.

$\mathbb{C}\mathrm{at}$: Γv is the "collage" of the profunctor v

$|\Gamma v| = |X_s| \sqcup |X_t|$, morphisms in X_s, X_t, and via v

$\mathbb{2}$ is the arrow category

$\mathbb{P}\mathrm{os}$: Γv is the "collage" of the ideal v

$\mathbb{2}$ is the non-discrete 2-point poset

Note

Companions and conjoints are used for $\Gamma_{\mathbb{2}}^{-1}$ in the examples, but not in general, so they are not part of glueing categories.

Lax Functors

Definition

A lax functor $F \colon \mathbb{D} \longrightarrow \mathbb{E}$ consists of functors $F_0 \colon \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ and $F_1 \colon \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ compatible with s and t, and cells

$$\mathrm{id}^{\bullet}_{F_0 X} \longrightarrow F_1(\mathrm{id}^{\bullet}_X) \quad \text{and} \quad F_1 w \odot F_1 v \longrightarrow F_1(w \odot v)$$

satisfying naturality and coherence conditions.

Oplax and pseudo functors are defined with the cells in the opposite direction and invertible, respectively.

Get a 2-category **LxDbl** of double categories and lax functors.

Note

Why **LxDbl**?

Adjoints in **LxDbl**

Lemma (Grandis/Paré 2004)

The following are equivalent for a lax functor $F \colon \mathbb{D} \longrightarrow \mathbb{E}$, and functors $G_0 \colon \mathbb{E}_0 \longrightarrow \mathbb{D}_0$ and $G_1 \colon \mathbb{E}_1 \longrightarrow \mathbb{D}_1$ compatible with s, t.

(a) *G is lax and $F \dashv G$ in* **LxDbl**.

(b) *$F_0 \dashv G_0$, $F_1 \dashv G_1$, and G is lax.*

(c) *$F_0 \dashv G_0$, $F_1 \dashv G_1$, and F is oplax.*

Definition (Aleiferi 2018)

$\mathbb{D}$ is pre-cartesian (cartesian) if $\mathbb{D} \xrightarrow{\Delta} \mathbb{D} \times \mathbb{D}$ and $\mathbb{D} \xrightarrow{!} \mathbb{1}$ have (pseudo) right adoints $\times$ and 1.

Proposition

Every glueing category is pre-cartesian.

Proof.

Δ, $!$ are pseudo, and $\mathbb{D}_1 \simeq \mathbb{D}_0/2$ has finite limits since $\mathbb{D}_0$ does. $\square$

Exponentiability in Pre-cartesian Double Categories

Definition

An object Y is pre-exponentiable in $\mathbb{D}$ if the lax functor $- \times Y \colon \mathbb{D} \longrightarrow \mathbb{D}$ has a right adjoint in **LxDbl**, and $\mathbb{D}$ is pre-cartesian closed if every object is pre-exponentiable.

Theorem

If Y is pre-exponentiable in $\mathbb{D}$, then $- \times Y$ is oplax and Y is exponentiable in $\mathbb{D}_0$. The converse holds, if $\mathbb{D}$ is a glueing category.

Proof.

By the Lemma, Y is pre-exp iff $- \times Y$ is oplax and Y, $\mathrm{id}_Y^\bullet$ are exp in $\mathbb{D}_0, \mathbb{D}_1$, resp. But, $\mathrm{id}_Y^\bullet \mapsto (Y \times 2 \to 2)$ via $\mathbb{D}_1 \simeq \mathbb{D}_0/2$, which is exp in $\mathbb{D}_1$ when Y is exp in in $\mathbb{D}_0$, and so the result follows. $\square$

Note

For Proposition and Theorem, horizontal invariance of $\mathbb{D}$ is used to show compatibility with s, t required in the Lemma.

Exponentiability: Examples

From [N, 2012; TAC]: $- \times Y$ is pseudo, if Y is exponentiable in $\mathbb{D}_0$, for $\mathbb{D} = \mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}, \mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}, \mathbb{T}\mathrm{opos}$, and so for these $\mathbb{D}$:

Corollary

Y is pre-exponentiable in $\mathbb{D} \iff Y$ is exponentiable in $\mathbb{D}_0$.
In particular, $\mathbb{C}\mathrm{at}$ and $\mathbb{P}\mathrm{os}$ are pre-cartesian closed.

Note

In [N 2012; TAC], we assumed more, i.e., $\mathbb{D}$ is fibrant.

What can we add to (G1) - (G4) so that $- \times Y$ will be oplax for all glueing categories? How can we deal with $\odot$?

Exponentiability: Examples, cont.

Suppose $\mathbb{D}_0$ has pushouts and consider the pushout $\mathbb{3}$

$$\begin{tikzcd}
1 \arrow[r, "i_s"] \arrow[d, "i_t"'] & \mathbb{2} \arrow[d, "i_{12}"] \\
\mathbb{2} \arrow[r, "i_{01}"'] & \mathbb{3}
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[dd, "\mathrm{id}_1^\bullet"'] \arrow[dr, "i_s"] \arrow[drr, "i_0"] \arrow[dd, phantom, shift left=4ex, "\gamma"] & & \\
 & \mathbb{2} \arrow[r, "i_{01}"] & \mathbb{3} \\
1 \arrow[ur, "i_t"] \arrow[urr, "i_1"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[dd, "\mathrm{id}_1^\bullet"'] \arrow[dr, "i_s"] \arrow[drr, "i_1"] \arrow[dd, phantom, shift left=4ex, "\gamma"] & & \\
 & \mathbb{2} \arrow[r, "i_{12}"] & \mathbb{3} \\
1 \arrow[ur, "i_t"] \arrow[urr, "i_2"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
1 \arrow[dd, "\mathrm{id}_1^\bullet"'] \arrow[dr, "i_s"] \arrow[drr, "i_0"] \arrow[dd, phantom, shift left=4ex, "\gamma"] & & \\
 & \mathbb{2} \arrow[r, "i_{02}"] & \mathbb{3} \\
1 \arrow[ur, "i_t"] \arrow[urr, "i_2"'] & &
\end{tikzcd}$$

where i_{02} is induced by vertically pasting along $i_1 = i_{12} i_s = i_{01} i_t$.

Exponentiability: Examples, cont.

The diagram below induces a morphism j s.t. $(\star)$ is commutative.

$$
\begin{tikzcd}
X_s \arrow[dr] \arrow[dd, "v"' , "\bullet" marking] & & \\
 & \Gamma v \arrow[dr] & \\
X_t \arrow[ur] \arrow[dr] \arrow[dd, "w"', "\bullet" marking] \arrow[u, phantom, "\gamma_v" near start, shift left=3] & & \Gamma w \sqcup_{X_t} \Gamma v \\
 & \Gamma w \arrow[ur] & \\
X_u \arrow[ur] \arrow[u, phantom, "\gamma_w" near start, shift left=3] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(w \odot v) \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "(\star)"] & \Gamma w \sqcup_{X_t} \Gamma v \arrow[d] \\
\mathbb{2} \arrow[r, "i_{02}"'] & \mathbb{3}
\end{tikzcd}
$$

Definition

We say $\mathbb{D}$ has the 02-pullback condition if $\mathbb{D}_0$ has pushouts and $(\star)$ is a pullback, for all $X_s \overset{v}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} X_t \overset{w}{\longrightarrow\!\!\!\!\!\!\bullet\;\;} X_u$.

Note

$\mathbb{C}\mathrm{at}, \mathbb{P}\mathrm{os}, \mathbb{T}\mathrm{op}, \mathbb{L}\mathrm{oc}$, and $\mathbb{T}\mathrm{opos}$ satisfy the 02-pullback condition.

Exponentiability: Examples, cont.

Corollary

Suppose $\mathbb{D}$ is a glueing category with the 02-pullback condition.
Y is pre-exponentiable in $\mathbb{D}$ $\iff$ Y is exponentiable in $\mathbb{D}_0$

Proof. (Sketch)

It suffice to show $\Gamma\varphi$ is iso, for $(w \times Y) \odot (v \times Y) \xrightarrow{\varphi} (w \odot v) \times Y$.

$$\begin{tikzcd}
\Gamma((w \times Y) \odot (v \times Y)) \arrow[r] \arrow[d, "\Gamma\varphi"'] & \Gamma(w \times Y) \sqcup_{X_t \times Y} \Gamma(v \times Y) \arrow[dd, "\cong \ Y\mathrm{exp\ in\ } \mathbb{D}_0"] \\
\Gamma((w \odot v) \times Y) \arrow[d, "\cong"'] \arrow[r, phantom, "\mathrm{pb}"] & \\
\Gamma(w \odot v) \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & (\Gamma w \sqcup_{X_t} \Gamma v) \times Y \arrow[d] \\
2 \arrow[r, "i_{02}"'] & 3
\end{tikzcd}$$

□

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