

# Exponentiability in Double Categories and the Glueing Construction

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## Idea

What are the “exponentiable” objects  $Y$  in a double category

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\text{id} \bullet} \\ \xrightarrow{t} \end{array} \mathbb{D}_0 \quad ?$$

For  $\mathbb{C}at$ ,  $\mathbb{P}os$ ,  $\mathbb{T}op$ ,  $\mathbb{L}oc$ , and  $\mathbb{T}opos$ , can show directly:

$$Y \text{ is exponentiable in } \mathbb{D} \iff Y \text{ is exponentiable } \mathbb{D}_0$$

Showed they satisfy  $\mathbb{D}_1 \simeq \mathbb{D}_0/2$ , generalizing Artin-Wraith glueing.  
[N 2012; JPAA]

# Goal

To prove:

$Y$  is exponentiable in  $\mathbb{D} \iff Y$  is exponentiable in  $\mathbb{D}_0$

in a general theorem assuming  $\mathbb{D}_1 \simeq \mathbb{D}_0/2$  plus ...

## Plan

1. Double categories and the examples
2. Glueing categories
3. Lax Functors and Adjoints
4. Exponentiability in double categories

# Double Categories

A **double category**  $\mathbb{D}$  is a (pseudo) category object in CAT

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\odot} \mathbb{D}_1 \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{\text{id} \bullet} \\ \xrightarrow{t} \end{array} \mathbb{D}_0$$

Objects: objects of  $\mathbb{D}_0$

Horizontal morphisms: morphisms  $f : X \rightarrow Y$  of  $\mathbb{D}_0$

Vertical morphism: objects of  $\mathbb{D}_1$ , denoted by  $v : X_s \rightarrow X_t$

Cells: morphisms of  $\mathbb{D}_1$ , denoted by

$$\begin{array}{ccc} X_s & \xrightarrow{f_s} & Y_s \\ v \downarrow & \varphi & \downarrow w \\ X_t & \xrightarrow{f_t} & Y_t \end{array}$$

# Double Categories: Examples [N 2012; JPAA]

**Top:** top spaces  $X$ ,  $X \rightarrow Y$ ,  $\frac{X_s \xrightarrow{\bullet} X_t}{\mathcal{O}(X_s) \xrightarrow{\text{lex}} \mathcal{O}(X_t)}$ ,  $\frac{\mathcal{O}(X_s) \xrightarrow{\mathcal{O}(f_s)} \mathcal{O}(Y_s)}{v \downarrow \supseteq \downarrow w} \frac{\mathcal{O}(X_t) \xrightarrow{\mathcal{O}(f_t)} \mathcal{O}(Y_t)}$

cont maps lex

**Loc:** locales  $X$ ,  $X \rightarrow Y$ ,  $\frac{X_s \xrightarrow{\bullet} X_t}{\text{locale maps} \quad \text{lex}}$ ,  $\frac{X_s \xrightarrow{f_s} Y_s}{v \downarrow \supseteq \downarrow w} \frac{X_t \xrightarrow{f_t} Y_t}$

**Topos:**  $\mathcal{S}$ -toposes  $\mathcal{X}$ ,  $\mathcal{X} \rightarrow \mathcal{Y}$ ,  $\frac{\mathcal{X}_s \xrightarrow{\bullet} \mathcal{X}_t}{\text{geom. morph.} \quad \text{lex}}$ ,  $\frac{\mathcal{X}_s \xrightarrow{f_s} \mathcal{Y}_s}{v \downarrow \supseteq \downarrow w} \frac{\mathcal{X}_t \xrightarrow{f_t} \mathcal{Y}_t}$

# Double Categories: Examples (cont.)

**Cat:** categories  $X$ ,  $X \xrightarrow{\text{functors}} Y$ ,  $X_s \xrightarrow{\text{profunctors}} X_t$ ,

$$\begin{array}{ccc}
 X_s & \xrightarrow{f_s} & Y_s \\
 v \downarrow & \rightarrow & \downarrow w \\
 X_t & \xrightarrow{f_t} & Y_t
 \end{array}$$

**Pos:** posets  $X$ ,  $X \xrightarrow{\text{monotone}} Y$ ,  $X_s \xrightarrow{\text{order ideals}} X_t$ ,

$$\begin{array}{ccc}
 X_s & \xrightarrow{f_s} & Y_s \\
 v \downarrow & \leq & \downarrow w \\
 X_t & \xrightarrow{f_t} & Y_t
 \end{array}$$

# Glueing Categories

(G1)  $\mathbb{D}_0$  has finite limits

(G2)  $\text{id}^\bullet : \mathbb{D}_0 \rightarrow \mathbb{D}_1$  has a left adjoint  $\Gamma$  with unit

$$\begin{array}{ccc}
 X_s & \xrightarrow{i_s} & \Gamma v \\
 \downarrow & \gamma_v & \downarrow \text{id}_{\Gamma v} \\
 X_t & \xrightarrow{i_t} & \Gamma v
 \end{array} \quad \text{“cotabulator”}$$

(G3)  $\Gamma_2 : \mathbb{D}_1 \rightarrow \mathbb{D}_0/2$  is an equivalence, where  $2 = \Gamma(\text{id}_1^\bullet)$ , and the following are pullbacks in  $\mathbb{D}_0$

$$\begin{array}{ccc}
 X_s & \xrightarrow{i_s} & \Gamma v \\
 \downarrow & & \downarrow \Gamma_{2v} \\
 1 & \xrightarrow{i_s} & 2
 \end{array} \quad \text{and} \quad \begin{array}{ccc}
 X_t & \xrightarrow{i_t} & \Gamma v \\
 \downarrow & & \downarrow \Gamma_{2v} \\
 1 & \xrightarrow{i_t} & 2
 \end{array}$$

(G4)  $\mathbb{D}$  is “horizontally invariant”

## Glueing Categories: Examples

**Top:** Given  $v: \mathcal{O}(X_s) \dashrightarrow \mathcal{O}(X_t)$ , define  $\Gamma v = X_s \sqcup X_t$  with  $U = U_s \sqcup U_t$  open, if  $U_s, U_t$  are open and  $U_t \subseteq v(U_s)$   
 $\mathbb{2}$  is the Sierpinski space

**Loc:**  $\Gamma v$  defined by “Artin-Wraith glueing” along  $v$   
 $\mathbb{2}$  is the Sierpinski locale  $\mathcal{O}(\mathbb{2})$

**Topos:**  $\Gamma v$  defined by “Artin-Wraith glueing” along  $v$   
 $\mathbb{2}$  is the Sierpinski topos  $\mathcal{S}^2$



## Glueing Categories: Examples, cont.

**Cat:**  $\Gamma v$  is the “collage” of the profunctor  $v$

$|\Gamma v| = |X_s| \sqcup |X_t|$ , morphisms in  $X_s$ ,  $X_t$ , and via  $v$

$\mathbb{2}$  is the arrow category

**Pos:**  $\Gamma v$  is the “collage” of the ideal  $v$

$\mathbb{2}$  is the non-discrete 2-point poset

### Note

Companions and conjoints are used for  $\Gamma_2^{-1}$  in the examples, but not in general, so they are not part of glueing categories.

# Lax Functors

## Definition

A **lax functor**  $F: \mathbb{D} \rightarrow \mathbb{E}$  consists of functors  $F_0: \mathbb{D}_0 \rightarrow \mathbb{E}_0$  and  $F_1: \mathbb{D}_1 \rightarrow \mathbb{E}_1$  compatible with  $s$  and  $t$ , and cells

$$\text{id}_{F_0 X} \rightarrow F_1(\text{id}_X) \quad \text{and} \quad F_1 w \odot F_1 v \rightarrow F_1(w \odot v)$$

satisfying naturality and coherence conditions.

Oplax and pseudo functors are defined with the cells in the opposite direction and invertible, respectively.

Get a 2-category **LxDbl** of double categories and lax functors.

## Note

Why **LxDbl**?

# Adjoints in **LxDbl**

Lemma (Grandis/Paré 2004)

The following are equivalent for a lax functor  $F: \mathbb{D} \rightarrow \mathbb{E}$ , and functors  $G_0: \mathbb{E}_0 \rightarrow \mathbb{D}_0$  and  $G_1: \mathbb{E}_1 \rightarrow \mathbb{D}_1$  compatible with  $s, t$ .

- (a)  $G$  is lax and  $F \dashv G$  in **LxDbl**.
- (b)  $F_0 \dashv G_0$ ,  $F_1 \dashv G_1$ , and  $G$  is lax.
- (c)  $F_0 \dashv G_0$ ,  $F_1 \dashv G_1$ , and  $F$  is oplax.

Definition (Aleiferi 2018)

$\mathbb{D}$  is **pre-cartesian (cartesian)** if  $\mathbb{D} \xrightarrow{\Delta} \mathbb{D} \times \mathbb{D}$  and  $\mathbb{D} \xrightarrow{!} \mathbb{1}$  have (pseudo) right adjoints  $\times$  and  $1$ .

Proposition

Every glueing category is pre-cartesian.

Proof.

$\Delta, !$  are pseudo, and  $\mathbb{D}_1 \simeq \mathbb{D}_0/2$  has finite limits since  $\mathbb{D}_0$  does.

# Exponentiability in Pre-cartesian Double Categories

## Definition

An object  $Y$  is **pre-exponentiable** in  $\mathbb{D}$  if the lax functor  $- \times Y: \mathbb{D} \rightarrow \mathbb{D}$  has a right adjoint in **LxDbl**, and  $\mathbb{D}$  is **pre-cartesian closed** if every object is pre-exponentiable.

## Theorem

*If  $Y$  is pre-exponentiable in  $\mathbb{D}$ , then  $- \times Y$  is oplax and  $Y$  is exponentiable in  $\mathbb{D}_0$ . The converse holds, if  $\mathbb{D}$  is a glueing category.*

## Proof.

By the Lemma,  $Y$  is pre-exp iff  $- \times Y$  is oplax and  $Y, \text{id}_Y^\bullet$  are exp in  $\mathbb{D}_0, \mathbb{D}_1$ , resp. But,  $\text{id}_Y^\bullet \mapsto (Y \times 2 \rightarrow 2)$  via  $\mathbb{D}_1 \simeq \mathbb{D}_0/2$ , which is exp in  $\mathbb{D}_1$  when  $Y$  is exp in  $\mathbb{D}_0$ , and so the result follows.  $\square$

## Note

For Proposition and Theorem, horizontal invariance of  $\mathbb{D}$  is used to show compatibility with  $s, t$  required in the Lemma.

## Exponentiability: Examples

From [N, 2012; TAC]:  $- \times Y$  is pseudo, if  $Y$  is exponentiable in  $\mathbb{D}_0$ , for  $\mathbb{D} = \mathbf{Cat}, \mathbf{Pos}, \mathbf{Top}, \mathbf{Loc}, \mathbf{Topos}$ , and so for these  $\mathbb{D}$ :

### Corollary

*$Y$  is pre-exponentiable in  $\mathbb{D} \iff Y$  is exponentiable in  $\mathbb{D}_0$ .  
In particular,  $\mathbf{Cat}$  and  $\mathbf{Pos}$  are pre-cartesian closed.*

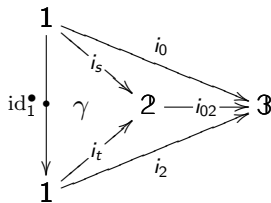
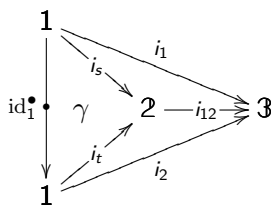
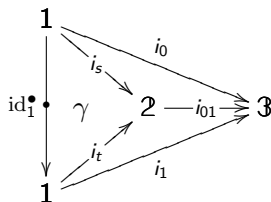
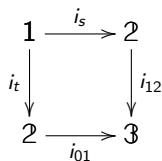
### Note

In [N 2012; TAC], we assumed more, i.e.,  $\mathbb{D}$  is fibrant.

What can we add to (G1) - (G4) so that  $- \times Y$  will be oplax for all glueing categories? How can we deal with  $\odot$ ?

# Exponentiability: Examples, cont.

Suppose  $\mathbb{D}_0$  has pushouts and consider the pushout  $\mathfrak{3}$



where  $i_{02}$  is induced by vertically pasting along  $i_1 = i_{12}i_s = i_{01}i_t$ .

## Exponentiability: Examples, cont.

The diagram below induces a morphism  $j$  s.t.  $(\star)$  is commutative.

$$\begin{array}{ccc} X_s & \xrightarrow{\quad} & \Gamma v \\ \downarrow v & \searrow \gamma_v & \uparrow \\ X_t & \xrightarrow{\quad} & \Gamma w \sqcup_{X_t} \Gamma v \\ \downarrow w & \searrow \gamma_w & \uparrow \\ X_u & \xrightarrow{\quad} & \Gamma w \end{array}$$

$$\begin{array}{ccc} \Gamma(w \odot v) & \xrightarrow{j} & \Gamma w \sqcup_{X_t} \Gamma v \\ \downarrow & (\star) & \downarrow \\ 2 & \xrightarrow{i_{02}} & 3 \end{array}$$

### Definition

We say  $\mathbb{D}$  has the **02-pullback condition** if  $\mathbb{D}_0$  has pushouts and  $(\star)$  is a pullback, for all  $X_s \xrightarrow{v} X_t \xrightarrow{w} X_u$ .

### Note

Cat, Pos, Top, Loc, and Topos satisfy the 02-pullback condition.

# Exponentiability: Examples, cont.

## Corollary

Suppose  $\mathbb{D}$  is a glueing category with the 02-pullback condition.

$Y$  is pre-exponentiable in  $\mathbb{D} \iff Y$  is exponentiable in  $\mathbb{D}_0$

**Proof.** (Sketch)

It suffice to show  $\Gamma\varphi$  is iso, for  $(w \times Y) \odot (v \times Y) \xrightarrow{\varphi} (w \odot v) \times Y$ .

$$\begin{array}{ccc}
 \Gamma((w \times Y) \odot (v \times Y)) & \rightarrow & \Gamma(w \times Y) \sqcup_{X_t \times Y} \Gamma(v \times Y) \\
 \Gamma\varphi \downarrow & & \downarrow \cong \text{ } Y_{\text{exp in } \mathbb{D}_0} \\
 \Gamma((w \odot v) \times Y) & \text{pb} & \\
 \cong \downarrow & & \downarrow \\
 \Gamma(w \odot v) \times Y & \longrightarrow & (\Gamma w \sqcup_{X_t} \Gamma v) \times Y \\
 \downarrow & \text{pb} & \downarrow \\
 2 & \xrightarrow{i_{02}} & 3
 \end{array}$$





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