Graphical abelian logic

David I. Spivak* and Brendan Fong

July 11, 2019

Outline

Abelian categories

Definition

A category $\mathcal{A}$ is *abelian* if

- it has a zero object 0;
- every pair of objects has a product and a coproduct;
- every morphism has a kernel and a cokernel; and
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Biggest math application: homological algebra can be done in $\mathcal{A}$.

- A *chain complex* in $\mathcal{A}$ is a sequence of maps, s.t. wherever you look

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

you have $\mathrm{im}(f) \subseteq \ker(g)$. Then the *homology* there is $\ker(g)/\mathrm{im}(f)$.

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Theorem (Fong-S.)

Abelian categories are reflective in the 2-category of abelian calculi,

$$\mathbb{A}\mathrm{bCalc} \underset{\mathrm{Prd}}{\overset{\mathrm{Syn}}{\rightleftarrows}} \mathbb{A}\mathrm{bCat}. \quad (\Rightarrow)$$

In particular for $\mathcal{A} \in \mathbb{A}\mathrm{bCat}$*, the unit* $\mathcal{A} \xrightarrow{\cong} \mathrm{SynPrd}(\mathcal{A})$ *is an equivalence.*

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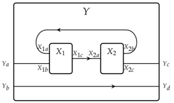
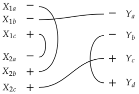
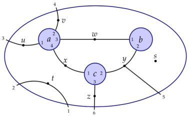
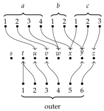

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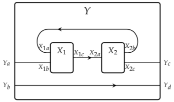

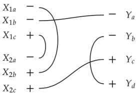

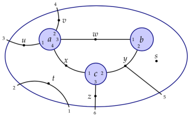

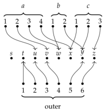

- Brendan talked about how to get regular categories this way.
- Today: abelian categories this way.

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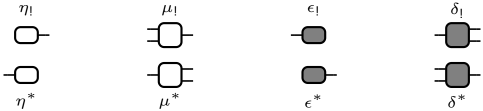

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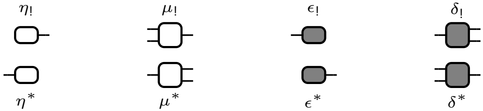

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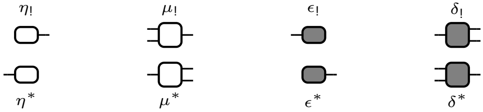

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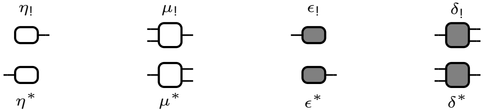

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- Example:  projects onto a coordinate plane,  intersects with it.

Some terms in the graphical language

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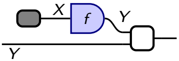

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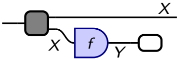

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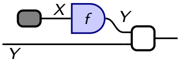

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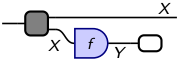

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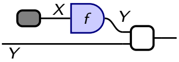

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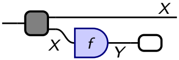

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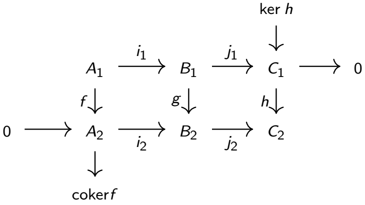

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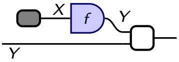

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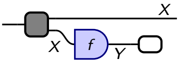

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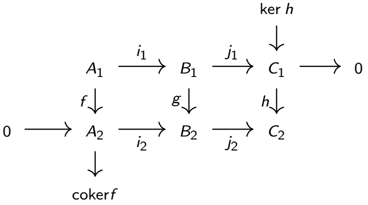

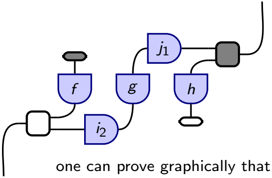

one can prove graphically that this relation is a left adjoint

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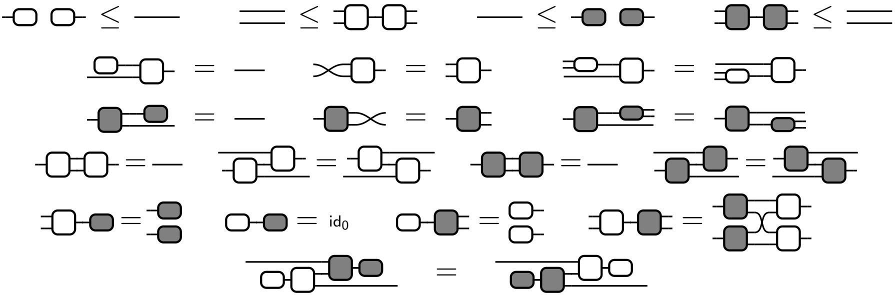

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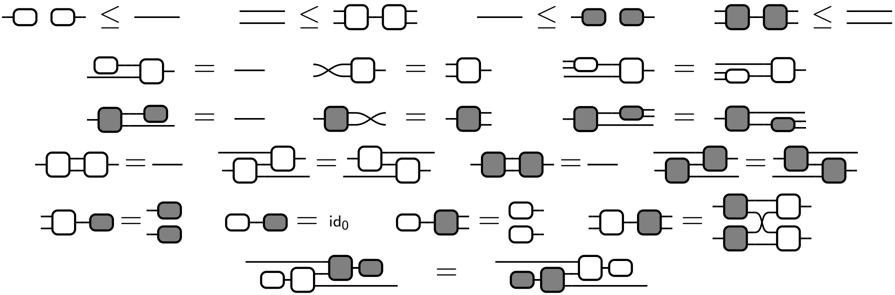

Better characterization?

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- objects are partially ordered sets $(S, \leq)$,
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Let's see one in action.

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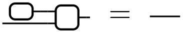
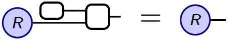
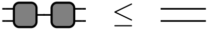
  - All equations in $\mathbb{A}$ are preserved.  translates to 

, i.e. $\{x + y \mid x \in R \text{ and } y = 0\} = R$.
  - Inequalities are preserved.  translates to $\{(x, y, x', y') \mid (x, y) \in R \wedge (x = y = x' = y')\} \subseteq R$

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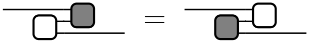
- $\mathbb{A}$ has a "negation involution"  .

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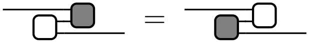
- $\mathbb{A}$ has a "negation involution" 
- $\mathrm{Sub}(\mathbb{Z}^-)$ applied to the negation involution sends $R \mapsto \{x \mid -x \in R\}$.
- Subspaces of $\mathbb{Z}^n$ are closed under negation, so this is identity.
- $\mathrm{Sub}(\mathbb{Z}^-)$ applied to negation involution in $\mathbb{A}$ is identity in $\mathbb{P}\mathrm{oset}$.

The syntactic category of $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$

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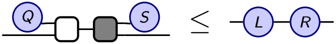

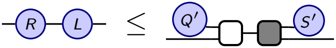

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One can prove that the result $\mathrm{Syn}(P)$ is a category and that it's abelian.

Aside: a sequence being a complex is its homology

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The homology at B *is* the assertion that the sequence is a complex at B.

Outline

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Examples:

- If $\mathcal{C}$ has finite products, it has a supply of comonoids.

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Definition

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$$\begin{array}{ccc} c^{\otimes m} \otimes d^{\otimes m} & \xrightarrow{s_c(\mu) \otimes s_d(\mu)} & c^{\otimes n} \otimes d^{\otimes n} \\ \cong\downarrow & & \downarrow\cong \\ (c \otimes d)^{\otimes m} & \xrightarrow[s_{c \otimes d}(\mu)]{} & (c \otimes d)^{\otimes n} \end{array} \qquad \begin{array}{ccc} I & = & I \\ \cong\downarrow & & \downarrow\cong \\ I^{\otimes m} & \xrightarrow[s_I(\mu)]{} & I^{\otimes n} \end{array}$$

Same definition for SM po-categories and po-props.

Examples:

- If $\mathcal{C}$ has finite products, it has a supply of comonoids.
- There is always a canonical supply of $\mathbb{P}$ in $\mathbb{P}$.

The notion of supply

"Every object is compatibly equipped with algebraic structure from $\mathbb{P}$."

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Examples:

- If $\mathcal{C}$ has finite products, it has a supply of comonoids.
- There is always a canonical supply of $\mathbb{P}$ in $\mathbb{P}$.
- If $\mathcal{A}$ is abelian, its relations po-cat $\mathbb{R}\mathrm{el}_{\mathcal{A}}$ supplies abelian relations $\mathbb{A}$.

Abelian calculi

Definition

An *abelian calculus* is a pair $(\mathcal{C}, P)$, where $\mathcal{C}$ supplies abelian relations and $P\colon \mathcal{C} \to \mathbb{P}\mathrm{oset}$ is bi-ajax and preserves involutions.

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Theorem (Fong-S.)

Abelian categories are reflective in the 2-category of abelian calculi.

$$\mathbb{A}\mathrm{bCalc} \underset{\mathrm{Prd}}{\overset{\mathrm{Syn}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\Rightarrow}}}} \mathbb{A}\mathrm{bCat}$$

In particular for $\mathcal{A} \in \mathbb{A}\mathrm{bCat}$, *the unit* $\mathcal{A} \xrightarrow{\cong} \mathrm{SynPrd}(\mathcal{A})$ *is an equivalence.*

The predicates functor

The inclusion "predicates" functor $\mathsf{Prd}\colon \mathbb{A}\mathsf{bCat} \to \mathbb{A}\mathsf{bCalc}$ is given by

$$\mathsf{Prd}(\mathcal{A}) \coloneqq (\mathbb{R}\mathsf{el}_{\mathcal{A}}, \mathbb{R}\mathsf{el}_{\mathcal{A}}(0, -)).$$

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- Since $\mathcal{A}$ is abelian, $\mathbb{R}\mathsf{el}_{\mathcal{A}}$ supplies abelian relations.
- $\mathbb{R}\mathsf{el}_{\mathcal{A}}(0, -))\colon \mathbb{R}\mathsf{el}_{\mathcal{A}} \to \mathbb{P}\mathsf{oset}$ sends $A \mapsto \mathbb{R}\mathsf{el}_{\mathcal{A}}(0, A) = \mathsf{Sub}(A)$.
  - Easy to see that $\mathbb{R}\mathsf{el}_{\mathcal{A}}(0, -)$ is a po-functor; it's represented by 0.
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$\mathrm{Prd}\colon \mathbb{A}\mathrm{bCat} \to \mathbb{A}\mathrm{bCalc}$ is fully faithful and locally fully faithful.

Outline

Summary

Abelian calculi give a "sketch" approach to abelian categories.

- The 2-cat of abelian categories embeds into that of abelian calculi.
- This embedding is full, and it has a reflector $\mathrm{Syn}\colon \mathbb{A}\mathrm{bCalc} \to \mathbb{A}\mathrm{bCat}$.

An abelian calculus, e.g. $P\colon \mathbb{A} \to \mathbb{P}\mathrm{oset}$, is graphical.

- Have access to icons: $\eta_!$ $\mu_!$ η^* μ^* $\epsilon_!$ $\delta_!$ ϵ^* δ^*

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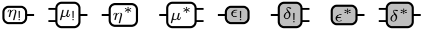
- Have access to icons: 

- These icons and their relations are used in constructing Syn(P)...
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Thanks! Questions and comments welcome!