Quantale-valued dissimilarity

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Frame-valued sets

Let Ω be a frame. An Ω-set is a set X equipped with a map

$$\alpha : X \times X \longrightarrow \Omega$$

such that

- (symmetry) $\alpha(x, y) = \alpha(y, x)$,
- (transitivity) $\alpha(y, z) \wedge \alpha(x, y) \leqslant \alpha(x, z)$

for all $x, y, z \in X$.

$\alpha(x, y)$: the truth-value of x being similar (or equal, or equivalent) to y.

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Guiding example

Let $\mathcal{O}(X)$ be the frame of open sets of a topological space X. Let

$$\mathrm{PC}(X) = \{f \mid f : U \longrightarrow \mathbb{R} \text{ is continuous with open domain } D(f) := U \subseteq X\}.$$

For any $f, g \in \mathrm{PC}(X)$, the assignment

$$\alpha(f, g) := \mathbf{Int}\{x \in D(f) \cap D(g) \mid f(x) = g(x)\}$$

makes $\mathrm{PC}(X)$ an $\mathcal{O}(X)$-set.

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Guiding example

As a dualization of the $\mathcal{O}(X)$-set $(\mathrm{PC}(X), \alpha)$, it is natural to consider the value

$$\beta(f, g) := \mathbf{Int}(X - \mathbf{Int}\{x \in D(f) \cap D(g) \mid f(x) = g(x)\}),$$

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Similarity vs. dissimilarity

In classical logic, with the law of double negation in hand, similarity and dissimilarity are interdefinable via negation.

However, in a non-classical logic, e.g., intuitionistic logic and many-valued logic, the law of double negation may fail, and thus similarity and dissimilarity may not be deduced from each other.

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Apartness relations

Let Ω be a frame. An Ω-valued model of apartness relation consists of a set X and maps

$$E : X \longrightarrow \Omega \quad \text{and} \quad \gamma : X \times X \longrightarrow \Omega,$$

such that

- $\gamma(x, y) \leqslant E(x) \wedge E(y)$,
- $\gamma(x, x) = \bot$,
- $\gamma(x, y) = \gamma(y, x)$,
- $\gamma(x, z) \wedge E(y) \leqslant \gamma(x, y) \vee \gamma(z, y)$

for all $x, y, z \in X$.

$E(x)$: the extent of existence of x.
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Apartness relations

Although apartness relations may be considered as a theory of positive inequalities, it is unfortunate that the assignment

$$\beta(f,g) = \mathbf{Int}(X - \mathbf{Int}\{x \in D(f) \cap D(g) \mid f(x) = g(x)\})$$

cannot be made into an $\mathcal{O}(X)$-valued apartness relation on $\mathrm{PC}(X)$.

Purpose: A positive theory of dissimilarity

Let

$$\mathsf{Q} = (\mathsf{Q}, \&, k, ^\circ)$$

be an involutive quantale, with left and right implications $/$, $\backslash$ satisfying

$$p \,\&\, q \leqslant r \iff p \leqslant r \,/\, q \iff q \leqslant p \,\backslash\, r$$

for all $p, q, r \in \mathsf{Q}$.

Our purpose: establishing a positive theory of Q-valued dissimilarity without the aid of negation.

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Our purpose: establishing a positive theory of Q-valued dissimilarity without the aid of negation.

A Q-valued similarity on a set X is a map

$$\alpha : X \times X \longrightarrow \mathsf{Q}$$

such that

(S1) (strictness) $\alpha(x,y) \leqslant \alpha(x,x) \wedge \alpha(y,y)$,

(S2) (divisibility) $(\alpha(x,y) \,/\, \alpha(x,x)) \,\&\, \alpha(x,x) = \alpha(x,y) = \alpha(y,y) \,\&\, (\alpha(y,y) \setminus \alpha(x,y))$,

(S3) (symmetry) $\alpha(x,y) = \alpha(y,x)^{\circ}$,

(S4) (transitivity) $(\alpha(y,z) \,/\, \alpha(y,y)) \,\&\, \alpha(x,y) = \alpha(y,z) \,\&\, (\alpha(y,y) \setminus \alpha(x,y)) \leqslant \alpha(x,z)$

for all $x,y,z \in X$, and the pair (X,α) is called a Q-valued set.

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Q-valued dissimilarity

A Q-valued dissimilarity on a set X is a map

$$\beta : X \times X \longrightarrow Q$$

such that

(D1) (strictness) $\beta(x,y) \geqslant \beta(x,x) \vee \beta(y,y)$,

(D2) (regularity) $\beta(x,x) \,/\, (\beta(x,y) \setminus \beta(x,x)) = \beta(x,y) = (\beta(y,y) \,/\, \beta(x,y)) \setminus \beta(y,y)$,

(D3) (symmetry) $\beta(x,y) = \beta(y,x)^{\circ}$,

(D4) (contrapositive transitivity) $\beta(x,z) \leqslant \beta(x,y) \,/\, (\beta(y,z) \setminus \beta(y,y)) = (\beta(y,y) \,/\, \beta(x,y)) \setminus \beta(y,z)$

for all $x,y,z \in X$.

Guiding example

Let $\mathcal{O}(X)$ be the frame of open sets of a topological space X. Then

$$\beta(f,g) := \mathbf{Int}(X - \mathbf{Int}\{x \in D(f) \cap D(g) \mid f(x) = g(x)\})$$

defines an $\mathcal{O}(X)$-valued dissimilarity on $\mathrm{PC}(X)$.

The axiom (D1) of strictness

$\beta(x, y)$: the truth-value of the statement that x is dissimilar to y.

$\beta(x, x)$: the extent of x being undefined, since each entity is supposed to be similar to itself unless it is undefined.

$\beta(x, y) \geqslant \beta(x, x) \vee \beta(y, y)$: each entity is less dissimilar to itself than to any other entity.

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The axiom (D2) of regularity

$\beta(x,y) \setminus \beta(x,x)$: the extent that the dissimilarity between x and y forces x to be undefined; in other words, it is the truth value of the contrapositive of the assertion that "if x is defined, then x is similar to y".

$\beta(x,y) = \beta(x,x) \swarrow (\beta(x,y) \setminus \beta(x,x))$: x is dissimilar to y if, and only if, x being similar to y would force x to be undefined.

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The axiom (D3) of symmetry

$\beta(x, y) = \beta(y, x)^\circ$: if x is dissimilar to y, then y is dissimilar to x.

The axiom (D4) of contrapositive transitivity

The inequality

$$\beta(x,z) \leqslant \beta(x,y) \,/\, (\beta(y,z) \setminus \beta(y,y))$$

is equivalent to

$$\beta(x,z) \,\&\, (\beta(y,z) \setminus \beta(y,y)) \leqslant \beta(x,y),$$

which claims that if x is dissimilar to z, and if the dissimilarity between y and z forces y to be undefined, then x is dissimilar to y.

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Q-valued dissimilarities as enriched categories

Fundamental structures are themselves categories.

— F. W. Lawvere

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Q-valued dissimilarities as enriched categories

Each quantale Q induces a quantaloid $\mathbf{B}(\mathsf{Q})$ of back diagonals of Q:

- objects: elements $p, q, r, \ldots$ of Q;
- morphisms: $b \in \mathbf{B}(\mathsf{Q})(p, q)$ if

$$p \swarrow (b \searrow p) = b = (q \swarrow b) \searrow q;$$

- composition: for $b \in \mathbf{B}(\mathsf{Q})(p, q)$ and $c \in \mathbf{B}(\mathsf{Q})(q, r)$,

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- the identity morphism on $q \in \mathsf{Q}$ is q itself;
- each $\mathbf{B}(\mathsf{Q})(p, q)$ is equipped with the reversed order inherited from Q.

L. Shen, Y. Tao and D. Zhang. Chu connections and back diagonals between $\mathcal{Q}$-distributors. *Journal of Pure and Applied Algebra*, 220(5):1858–1901, 2016.

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Q-valued dissimilarities as enriched categories

For each $p, q \in \mathsf{Q}$, let

$$\mathbf{B}_*(\mathsf{Q})(p,q) := \{b \in \mathbf{B}(\mathsf{Q}) \mid p \vee q \leqslant b\}.$$

Then $\mathbf{B}_*(\mathsf{Q})$ is a subquantaloid of $\mathbf{B}(\mathsf{Q})$.

Q-valued dissimilarities as enriched categories

Theorem

A set equipped with a Q*-valued dissimilarity is precisely a symmetric* $\mathbf{B}_*(\mathsf{Q})$*-category.*

Q-valued similarities as enriched categories

Each quantale Q induces a quantaloid $\mathbf{D}(\mathsf{Q})$ of diagonals of Q:

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$$(d \swarrow p) \,\&\, p = d = q \,\&\, (q \searrow d);$$

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- composition: for $d \in \mathbf{D}(\mathrm{Q})(p, q)$ and $e \in \mathbf{D}(\mathrm{Q})(q, r)$,

$$e \diamond d := (e \swarrow q) \mathbin{\&} d = e \mathbin{\&} (q \searrow d);$$

- the identity morphism on $q \in \mathrm{Q}$ is q itself;
- each $\mathbf{D}(\mathrm{Q})(p, q)$ is equipped with the order inherited from Q.

U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.

I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*, 256:95–116, 2014.

Q-valued similarities as enriched categories

Each quantale Q induces a quantaloid $\mathbf{D}(\mathsf{Q})$ of diagonals of Q:

- objects: elements $p, q, r, \dots$ of Q;
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Q-valued similarities as enriched categories

For each $p, q \in \mathsf{Q}$, let

$$\mathbf{D}_*(\mathsf{Q})(p, q) := \{d \in \mathbf{D}(\mathsf{Q})(p, q) \mid d \leqslant p \wedge q\}.$$

Then $\mathbf{D}_*(\mathsf{Q})$ is a subquantaloid of $\mathbf{D}(\mathsf{Q})$, and

Theorem

A set equipped with a Q*-valued similarity is precisely a symmetric* $\mathbf{D}_*(\mathsf{Q})$*-category.*

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When Q is a Girard quantale

A Girard quantale Q is a quantale equipped with a cyclic dualizing element m; that is,

$$m \,/\, q = q \setminus m \quad \text{and} \quad (m \,/\, q) \setminus m = q = m \,/\, (q \setminus m)$$

for all $q \in \mathsf{Q}$.

In this case, the linear negation of $q \in \mathsf{Q}$ is defined as

$$q^{\perp} := m \,/\, q = q \setminus m,$$

which clearly satisfies

$$q^{\perp\perp} = q.$$

Hence, a Girard quantale may be considered as a table of truth-values in which the law of double negation is satisfied.

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When Q is a Girard quantale

Theorem

If Q is a Girard quantale, then there are isomorphisms

$$\mathbf{D}(\mathsf{Q}) \cong \mathbf{B}(\mathsf{Q}) \quad \textit{and} \quad \mathbf{D}_*(\mathsf{Q}) \cong \mathbf{B}_*(\mathsf{Q})$$

of quantaloids, and consequently, Q-valued similarities and Q-valued dissimilarities are interdefinable by the aid of linear negation.

When Q is a Girard quantale

Conversely:

Theorem

Let Q be a commutative quantale. Then there is an isomorphism

$$\mathbf{D}(\mathrm{Q}) \cong \mathbf{B}(\mathrm{Q})$$

of quantaloids if, and only if, Q is a Girard quantale.

Related papers

This talk is based on:

- L. Shen, H. Lai, Y. Tao and D. Zhang. Quantale-valued dissimilarity. *arXiv:1904.05565*.

References:

- M. P. Fourman and D. S. Scott. Sheaves and Logic. *Lecture Notes in Mathematics*, 753:302–401, 1979.
- D. S. Scott. Identity and existence in intuitionistic logic. *Lecture Notes in Mathematics*, 753:660–696, 1979.
- U. Höhle and T. Kubiak. A non-commutative and non-idempotent theory of quantale sets. *Fuzzy Sets and Systems*, 166:1–43, 2011.
- I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets and Systems*, 256:95–116, 2014.
- L. Shen, Y. Tao and D. Zhang. Chu connections and back diagonals between $\mathcal{Q}$-distributors. *Journal of Pure and Applied Algebra*, 220(5):1858–1901, 2016.