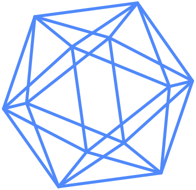
An Axiomatic Approach to Algebraic Topology: A Theory of Elementary $(\infty, 1)$-Toposes

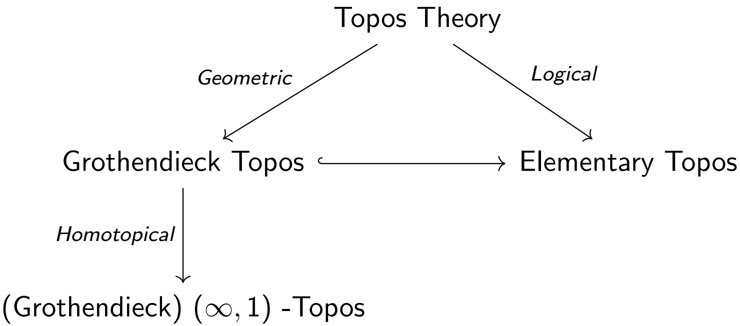
Nima Rasekh

Max-Planck-Institut für Mathematik



July 8th, 2019

Two Paths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Geometric
Logical
Grothendieck Topos
Elementary Topos
Homotopical
(Grothendieck) $(\infty, 1)$ -Topos

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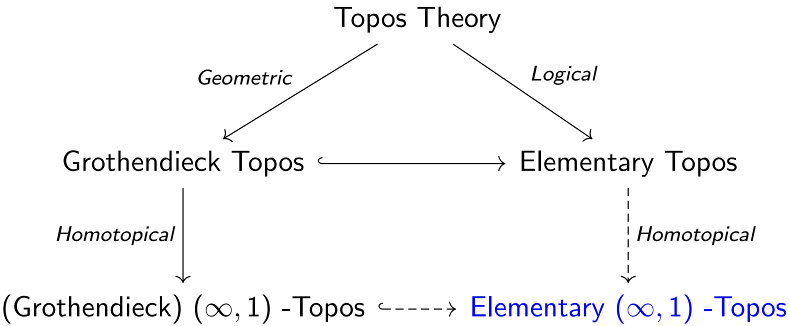
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Why Elementary Toposes?

1. **Category Theory:** It's a fascinating category! It has epi-mono factorization, we can classify left-exact localizations, we can construct finite colimits, ...
2. **Type Theory:** We get models of certain (i.e. higher-order intuitionistic) type theories.
3. **Set Theory:** We can construct models of set theory and so better understand the axioms of set theory.

$(\infty, 1)$-Categories

An $(\infty, 1)$-category $\mathcal{C}$ has following properties:

1. It has **objects** $x, y, z, ...$
2. For any two objects x, y there is a **mapping space** (Kan complex) $map_{\mathcal{C}}(x, y)$ with a notion of composition that holds only "up to homotopy".
3. This is a direct generalization of classical categories and all categorical notions (limits, adjunction, ...) generalize to this setting.

Elementary $(\infty, 1)$-Topos

Definition

An *elementary* $(\infty, 1)$*-topos* is an $(\infty, 1)$-category $\mathcal{E}$ that satisfies following conditions:

1. $\mathcal{E}$ has finite limits and colimits.
2. $\mathcal{E}$ is locally Cartesian closed.
3. $\mathcal{E}$ has a subobject classifier Ω.
4. $\mathcal{E}$ has sufficient universes $\mathcal{U}$.

Locally Cartesian Closed & Subobject Classifier

Definition

$\mathcal{E}$ is *locally Cartesian closed* if for every $f : x \to y$ the functor

$$\mathcal{E}_{/y} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{E}_{/x}$$

has a right adjoint.

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Definition

A *subobject classifier* Ω in $\mathcal{E}$ represents the functor $Sub : \mathcal{E}^{op} \to \mathcal{S}\text{et}$ that has value

$$Sub(x) = \text{Subobjects of } x = \{i : y \to x \mid i \text{ mono}\}/\cong .$$

Universes

Definition

An object $\mathcal{U}$ in $\mathcal{E}$ is a universe if there exists an embedding of functors

$$i_{\mathcal{U}} : Map_{\mathcal{E}}(-,\mathcal{U}) \hookrightarrow \mathcal{E}_{/-}$$

where $\mathcal{E}_{/-}$ is the functor which takes an object x to the slice $\mathcal{E}_{/x}$.

Definition

$\mathcal{E}$ has sufficient universes if for every morphism $f : y \to x$, there exists a universe $\mathcal{U}$ such that f is in the image of $i_{\mathcal{U}}$.

Meaning of Universes

Informally, by definition every universe comes with a *universal fibration* $\mathcal{U}_* \to \mathcal{U}$ and we have sufficient universes if for every morphism $f : y \to x$ there is a pullback square

$$\begin{array}{ccc} y & \longrightarrow & \mathcal{U}_* \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ x & \longrightarrow & \mathcal{U} \end{array} .$$

Examples

Example

Every Grothendieck $(\infty, 1)$-topos is an elementary $(\infty, 1)$-topos.

Example

In particular, the $(\infty, 1)$-category of spaces $\mathcal{S}$ is an elementary $(\infty, 1)$-topos.

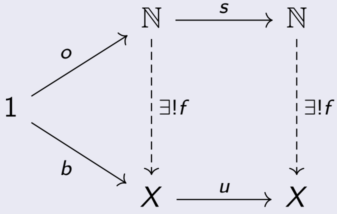
Why Elementary $(\infty, 1)$-Toposes?

1. $(\infty, 1)$-**Category Theory:** It is a fascinating $(\infty, 1)$-category! It has truncations, we should be able to classify all left-exact localizations, we might be able to construct finite colimits, ...
2. **Type Theory:** We should get all models of certain (i.e. homotopy) type theories.
3. **Space Theory:** It should give us models of the homotopy theory of spaces.

Natural Number Objects

Definition

Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos. A *natural number object* is an object $\mathbb{N}$ along with two maps $o : 1 \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $(X, b : 1 \to X, u : X \to X)$



the space of maps f making the diagram commute is contractible.

What does Natural Number Object mean?

Natural number objects allow us to do "infinite constructions" by just using finite limits and colimits.

Theorem (Theorem D5.3.5, Sketches of an Elephant, Johnstone)

Let $\mathcal{E}$ be an **elementary** 1-**topos** *with natural number object. Then we can construct free finitely-presented finitary algebras (monoids, ...).*

But, natural number objects don't always exist (e.g. finite sets).

NNOs in an Elementary $(\infty, 1)$-Topos

However, things are different in elementary $(\infty, 1)$-toposes.

Theorem (R)

Every elementary $(\infty, 1)$-topos $\mathcal{E}$ has a natural number object.

Step I: Algebraic Topology

We use the fact from algebraic topology that $\pi_1(S^1) = \mathbb{Z}$.

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We use the fact from algebraic topology that $\pi_1(S^1) = \mathbb{Z}$.
We can take a coequalizer

$$1 \underset{id}{\overset{id}{\rightrightarrows}} 1 \longrightarrow S^1$$

The object S^1 behaves similar to the circle in spaces. In particular we can take it's loop object.

Step I: Algebraic Topology

$$\begin{array}{ccc} \Omega S^1 & \longrightarrow & 1 \\ \big\downarrow & \ulcorner & \big\downarrow \\ 1 & \longrightarrow & S^1 \end{array}$$

ΩS^1 behaves similar to the classical loop space of the circle: It comes with an automorphism $s : \Omega S^1 \to \Omega S^1$ and a map $o : 1 \to \Omega S^1$. It is also an object in the underlying elementary topos $\tau_{\leq 0}\mathcal{E}$ (the elementary topos of 0-truncated objects).

Step II: Elementary Topos Theory

We can now use results from elementary topos theory to show that the smallest subobject of ΩS^1 closed under s and o is a natural number object in $\tau_{\leq 0}\mathcal{E}$ [Lemma D5.1.1, Sketches of an Elephant, Johnstone].

Step III: Homotopy Type Theory

We need to show that the universal property holds for all objects and not just the ones in the underlying elementary topos. For that we need to be able to use induction arguments in an $(\infty, 1)$-category, which does not simply follow from classical mathematics and needs to be reproven using concepts of homotopy type theory, which is due to Shulman.

Summary

1. Elementary $(\infty, 1)$-toposes sit at the intersection of **elementary topos theory**, $(\infty, 1)$-**categories**, **homotopy type theory** and **algebraic topology**.
2. Some things we know: Natural number objects, some are models of homotopy type theory, some algebraic topology (truncations, Blakers-Massey).
3. Some things we still don't know: classify localizations, free algebras, ...

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Thank You!