

A type theory for cartesian closed bicategories

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Cartesian closed bicategories

Cartesian closed categories 'up to isomorphism'.

Examples:

- Generalised species and cartesian distributors
particularly for applications in higher category theory
(Fiore, Gambino, Hyland, Winskel), (Fiore & Joyal)
- Categorical algebra (operads)
(Gambino & Joyal)
- Game semantics (concurrent games)
(Yamada & Abramsky, Winskel *et al.*, Paquet)

Internal monoids

In a category with finite products:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

Unit law

$$\begin{array}{ccccc} 1 \times M & \xrightarrow{e \times M} & M \times M & \xleftarrow{M \times e} & M \times 1 \\ & \searrow & \downarrow m & \swarrow & \\ & \cong & M & \cong & \end{array}$$

Assoc. law

$$\begin{array}{ccccc} (M \times M) \times M & \xrightarrow{\cong} & M \times (M \times M) & \xrightarrow{M \times m} & M \times M \\ m \times M \downarrow & & & & \downarrow m \\ M \times M & \xrightarrow{\quad m \quad} & & & M \end{array}$$

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In a category with finite products:

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In **Set**: monoids

In **Cat**: **strict** monoidal categories

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Internal pseudomonoids

In Cat:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

Unit 2-cells

$$\begin{array}{ccccc}
 1 \times M & \xrightarrow{e \times M} & M \times M & \xleftarrow{M \times e} & M \times 1 \\
 & \searrow & \downarrow m & \swarrow & \\
 & & M & & \\
 \cong & \nearrow & & \nwarrow & \cong
 \end{array}$$

Diagram illustrating Unit 2-cells. The top row shows the multiplication $m: M \times M \rightarrow M$ and the unit $e: 1 \rightarrow M$. The bottom row shows the multiplication $m: M \times M \rightarrow M$. The 2-cells are λ (left) and ρ (right), both marked with \cong . Blue arrows labeled "data" point from the λ and ρ 2-cells to the m multiplication in the bottom row.

Assoc. 2-cell

$$\begin{array}{ccccc}
 (M \times M) \times M & \xrightarrow{\cong} & M \times (M \times M) & \xrightarrow{M \times m} & M \times M \\
 m \times M \downarrow & & & & \downarrow m \\
 M \times M & \xrightarrow{m} & & & M
 \end{array}$$

Diagram illustrating the Associativity 2-cell. The top row shows the multiplication $m: M \times M \rightarrow M$ and the multiplication $m: M \times M \rightarrow M$. The bottom row shows the multiplication $m: M \times M \rightarrow M$. The 2-cell is α , marked with \cong . A blue arrow labeled "data" points from the α 2-cell to the m multiplication in the bottom row.

Internal pseudomonoids

In Cat:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

Unit 2-cells

$$\begin{array}{ccccc}
 1 \times M & \xrightarrow{e \times M} & M \times M & \xleftarrow{M \times e} & M \times 1 \\
 & & \downarrow m & & \\
 & & M & & \\
 \text{---} & \xrightarrow{\cong} & & \xleftarrow{\cong} & \text{---} \\
 & & & &
 \end{array}$$

λ (red) and ρ (red) are 2-cells from the top row to the bottom row. Blue arrows point from a box labeled "data" to the λ and ρ 2-cells.

Assoc. 2-cell

$$\begin{array}{ccccc}
 (M \times M) \times M & \xrightarrow{\cong} & M \times (M \times M) & \xrightarrow{M \times m} & M \times M \\
 m \times M \downarrow & & & & \downarrow m \\
 M \times M & \xrightarrow{m} & & & M \\
 & & & & \text{---} \\
 & & & & \alpha
 \end{array}$$

α (red) is a 2-cell from the top row to the bottom row. A blue arrow points from the "data" box to the α 2-cell.

+ triangle and pentagon laws

\rightsquigarrow monoidal category

Internal pseudomonoids

In Cat:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

...likewise in any fp-bicategory

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$$\begin{array}{ccccc}
 1 \times M & \xrightarrow{e \times M} & M \times M & \xleftarrow{M \times e} & M \times 1 \\
 & \searrow & \downarrow m & \swarrow & \\
 & & M & & \\
 \simeq & \searrow & & \swarrow & \simeq \\
 & & M & &
 \end{array}$$

Diagram illustrating Unit 2-cells. A box labeled "data" has blue arrows pointing to the 2-cells λ and ρ in the diagram above.

Assoc. 2-cell

$$\begin{array}{ccccc}
 (M \times M) \times M & \xrightarrow{\simeq} & M \times (M \times M) & \xrightarrow{M \times m} & M \times M \\
 m \times M \downarrow & & & & \downarrow m \\
 M \times M & \xrightarrow{m} & & & M \\
 & & \alpha & &
 \end{array}$$

Diagram illustrating Assoc. 2-cell. A box labeled "data" has a blue arrow pointing to the 2-cell α in the diagram above.

+ triangle and pentagon laws


\rightsquigarrow monoidal category

In a CCC every $[X \Rightarrow X]$ becomes a monoid:

$$\left(1 \xrightarrow{\text{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X] \right)$$

? In a cc-bicategory every $[X \Rightarrow X]$ becomes a **pseudomonoid**:

$$\left(1 \xrightarrow{\text{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X] \right)$$


need to check
coherence laws
(i.e. triangle + pentagon)

Programme:

1. Construct a type theory $\Lambda_{\text{ps}}^{\times, \rightarrow}$ for cartesian closed bicategories
(this work),
2. Use NBE to prove the type theory is coherent
bicategorical version of [Fiore2002]
(my thesis),

Coherence

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Application:

Algebraic structure definable in every CCC

\Rightarrow algebraic pseudo-structure definable in every cc-bicategory

A type theory $\Lambda_{\text{ps}}^{\times, \rightarrow}$ that:

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1. Generalises the simply-typed lambda calculus,
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3. Is *sound* and *complete*
i.e. freeness property for the syntactic model.

Bicategories

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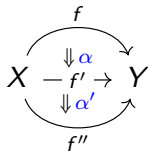
$$\begin{array}{l} \text{1-cells } X \xrightarrow{f} Y \\ \text{2-cells } X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{f'} \end{array} Y \end{array}$$

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- Functors

$$\begin{array}{l} \mathbf{1} \xrightarrow{Id_X} \mathcal{B}(X, X) \\ \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \xrightarrow{\circ_{X, Y, Z}} \mathcal{B}(X, Z) \end{array}$$

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$$X \begin{array}{c} \xrightarrow{f} \\ \Downarrow \alpha \\ \xrightarrow{f'} \end{array} Y \begin{array}{c} \xrightarrow{g} \\ \Downarrow \beta \\ \xrightarrow{g'} \end{array} Z$$

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- Invertible 2-cells

$$(h \circ g) \circ f \xrightarrow{a_{h, g, f}} h \circ (g \circ f)$$

$$Id_X \circ f \xrightarrow{l_f} f$$

$$g \circ Id_X \xrightarrow{r_g} g$$

subject to a triangle law and pentagon law.

Cartesian closed bicategories

Bicategories \mathcal{B} equipped with *biuniversal* 1-cells

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$$(fp) \quad \pi_i : \prod_n(A_1, \dots, A_n) \rightarrow A_i \quad (1 \leq i \leq n)$$

$$(cc) \quad eval : (A \Rightarrow B) \times A \rightarrow B$$

NB: Differ from the ‘cartesian bicategories’ of Carboni and Walters!

Cartesian closed bicategories

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inducing families of equivalences

$$\mathcal{B}(X, \prod_n(A_1, \dots, A_n)) \simeq \prod_{i=1}^n \mathcal{B}(X, A_i)$$

$$\mathcal{B}(X, A \Rightarrow B) \simeq \mathcal{B}(X \times A, B)$$

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inducing families of equivalences

$$\begin{array}{ccc} & \xrightarrow{(\pi_1 \circ -, \dots, \pi_n \circ -)} & \\ \mathcal{B}(X, \prod_n(A_1, \dots, A_n)) & \perp \simeq & \prod_{i=1}^n \mathcal{B}(X, A_i) \\ & \xleftarrow{\langle -, \dots, = \rangle} & \\ & \text{(tupling)} & \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{eval_{A,B} \circ (- \times A)} & \\ \mathcal{B}(X, A \Rightarrow B) & \perp \simeq & \mathcal{B}(X \times A, B) \\ & \xleftarrow{\lambda} & \\ & \text{(currying)} & \end{array}$$

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

Substitution and composition

In any CCC:

$$\llbracket x_k[u_1/x_1, \dots, u_n/x_n] \rrbracket = \llbracket u_k \rrbracket = \pi_k \circ \langle \llbracket u_1 \rrbracket, \dots, \llbracket u_n \rrbracket \rangle$$

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Question: what is **bicategorical** substitution?

An algebraic theory with substitution:

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$$t, (u_1, \dots, u_n) \mapsto t[u_i/x_i]$$

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such that

$$x_k[u_i/x_i] = u_k \quad (1 \leq k \leq n)$$

$$t[x_i/x_i] = t$$

$$t[u_i/x_i][v_j/y_j] = t[u_i[v_j/y_j]/x_i]$$

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such that

$$\begin{aligned} p^{(k)}[u_1, \dots, u_n] &= u_k & (1 \leq k \leq n) \\ t[p^{(1)}, \dots, p^{(n)}] &= t \\ t[u_\bullet][v_\bullet] &= t[v_\bullet[u_\bullet]] \end{aligned}$$

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Note: every clone defines a category

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- Structural isomorphisms

$$\begin{aligned} p^{(k)}[u_1, \dots, u_n] &\xrightarrow{\varrho_{u_\bullet}^{(k)}} u_k & (1 \leq k \leq n) \\ t[p^{(1)}, \dots, p^{(n)}] &\xRightarrow{\iota_t} t \\ t[u_\bullet][v_\bullet] &\xrightarrow{\text{assoc}_{t; u_\bullet; v_\bullet}} t[v_\bullet[u_\bullet]] \end{aligned}$$

subject to a triangle law and pentagon law.

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Vertical composition:
$$\frac{\Gamma \vdash \tau' : t' \Rightarrow t'' : B \quad \Gamma \vdash \tau : t \Rightarrow t' : B}{\Gamma \vdash \tau' \bullet \tau : t \Rightarrow t'' : B}$$

Identities:
$$\frac{\Gamma \vdash t : B}{\Gamma \vdash \text{id}_t : t \Rightarrow t : B}$$

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A *substitution functor*

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A type theory for biclones

A *substitution functor*

$$\begin{aligned} \mathbb{C}(X_1, \dots, X_n; Y) \times \prod_{i=1}^n \mathbb{C}(\Gamma; X_i) &\rightarrow \mathbb{C}(\Gamma; Y) \\ t, (u_1, \dots, u_n) &\mapsto t[u_1, \dots, u_n] \\ \tau, (\sigma_1, \dots, \sigma_n) &\mapsto \tau[\sigma_1, \dots, \sigma_n] \end{aligned}$$

Explicit substitution:

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash t : B \quad (\Delta \vdash u_i : A_i)_{i=1..n}}{\Delta \vdash t \{x_i \mapsto u_i\} : B}$$

$$\frac{x_1 : A_1, \dots, x_n : A_n \vdash \tau : t \Rightarrow t' : B \quad (\Delta \vdash \sigma_i : u_i \Rightarrow u'_i : A_i)_{i=1, \dots, n}}{\Delta \vdash \tau \{x_i \mapsto \sigma_i\} : t \{x_i \mapsto u_i\} \Rightarrow t' \{x_i \mapsto u'_i\} : B}$$

\rightsquigarrow binds the variables x_1, \dots, x_n

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Structural isomorphisms $\varrho^{(k)}, \iota, \text{assoc}$

A type theory for biclones

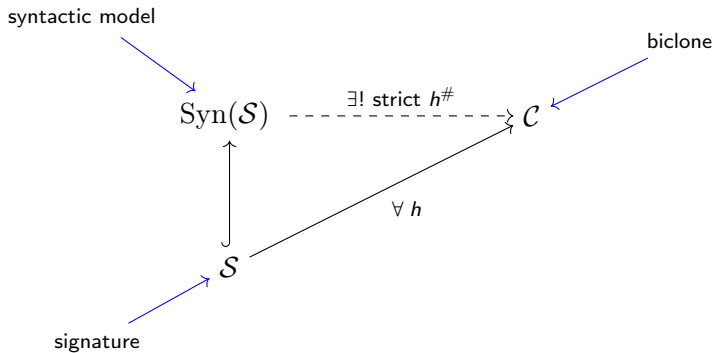
Structural isomorphisms $\rho^{(k)}, \iota, \text{assoc}$

Distinguished invertible rewrites e.g.:

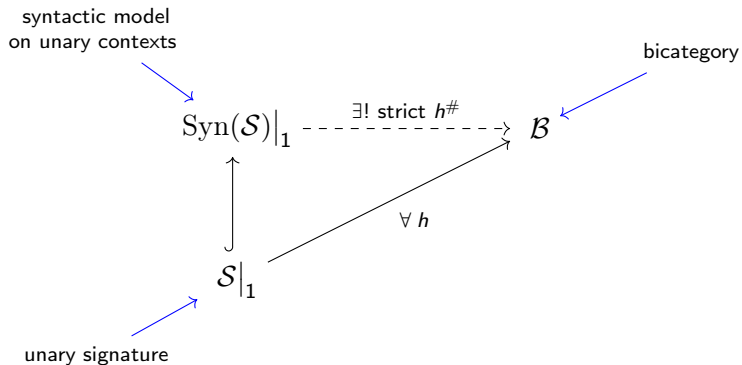
$$\frac{(\Delta \vdash u_i : A_i)_{i=1, \dots, n}}{x_1 : A_1, \dots, x_n : A_n \vdash \rho_{u_\bullet}^{(k)} : x_k \{x_i \mapsto u_i\} \xrightarrow{\cong} u_k : A_k} \quad (1 \leq k \leq n)$$

The syntactic model is free

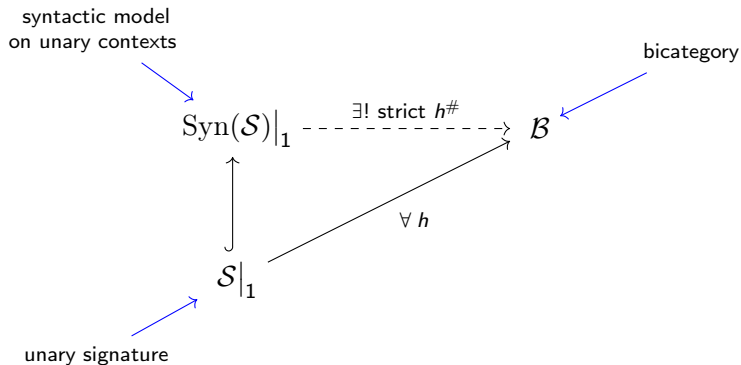
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The syntactic model is free



~> An internal language for bicategories.

1-cells

$$\pi_i : \prod_n(A_1, \dots, A_n) \rightarrow A_i \quad (1 \leq i \leq n)$$

Adjoint equivalences

$$\mathcal{B}(X, \prod_n(A_1, \dots, A_n)) \perp \simeq \prod_{i=1}^n \mathcal{B}(X, A_i)$$

$(\pi_1 \circ -, \dots, \pi_n \circ -)$

$\langle -, \dots, = \rangle$

A type theory for fp-bicategories

1-cells $\pi_i : \prod_n(A_1, \dots, A_n) \rightarrow A_i \quad (1 \leq i \leq n)$

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1-cells $\pi_i : \prod_n(A_1, \dots, A_n) \rightarrow A_i \quad (1 \leq i \leq n)$

Projections $\frac{}{p : \prod_n(A_1, \dots, A_n) \vdash \pi_i(p) : A_i} \quad (1 \leq i \leq n)$

A type theory for fp-bicategories

$$\text{Equivalences } \mathcal{B}(X, \Pi_n(A_1, \dots, A_n)) \perp \simeq \prod_{i=1}^n \mathcal{B}(X, A_i)$$

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A type theory for fp-bicategories

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$\xrightarrow{(\pi_1 \circ -, \dots, \pi_n \circ -)}$
 $\xleftarrow{\langle -, \dots, = \rangle}$

$$\varpi^{(i)} \bullet (\pi_i \circ (-)) \left(\frac{\pi_i \circ u \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \langle t_1, \dots, t_n \rangle : \Pi_n(A_1, \dots, A_n)} \right) \rho^\dagger(-, \dots, =)$$

for a counit $(\varpi^{(i)} : \pi_i \circ \langle t_1, \dots, t_n \rangle \Rightarrow t_i : A_i)_{i=1, \dots, n}$

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$(\pi_1 \circ -, \dots, \pi_n \circ -)$
 $\langle -, \dots, = \rangle$

syntactic sugar

$$\varpi^{(i)} \bullet \pi_i \{(-)\} \left(\frac{\pi_i \{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \text{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)} \right) \rho^\dagger(-, \dots, =)$$

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A type theory for fp-bicategories

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Tupling map

$$\frac{(\Gamma \vdash t_i : A_i)_{i=1, \dots, n}}{\Gamma \vdash \text{tup}(t_1, \dots, t_n) : \prod_n(A_1, \dots, A_n)}$$

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Counit (β -law)
$$\frac{(\Gamma \vdash t_i : A_i)_{i=1, \dots, n}}{\Gamma \vdash \varpi_{t_\bullet}^{(k)} : \pi_k \{\text{tup}(t_1 \dots, t_n)\} \cong t_k : A_k} \quad (1 \leq k \leq n)$$

A type theory for fp-bicategories

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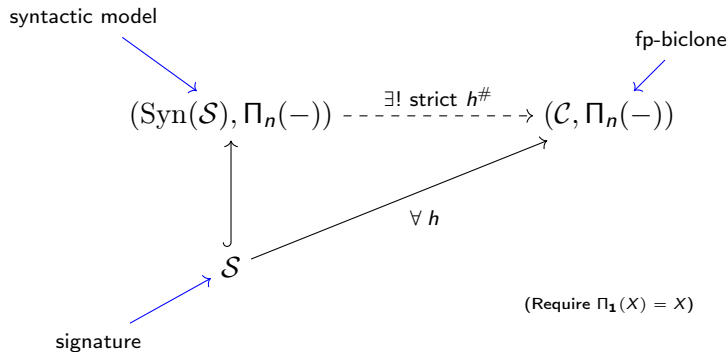
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Mediating 2-cell
$$\frac{(\Gamma \vdash \alpha_i : \pi_i \{u\} \Rightarrow t_i : A_i)_{i=1, \dots, n}}{\Gamma \vdash \text{p}^\dagger(\alpha_1, \dots, \alpha_n) : u \Rightarrow \text{tup}(t_1, \dots, t_n) : \prod_n(A_1, \dots, A_n)}$$

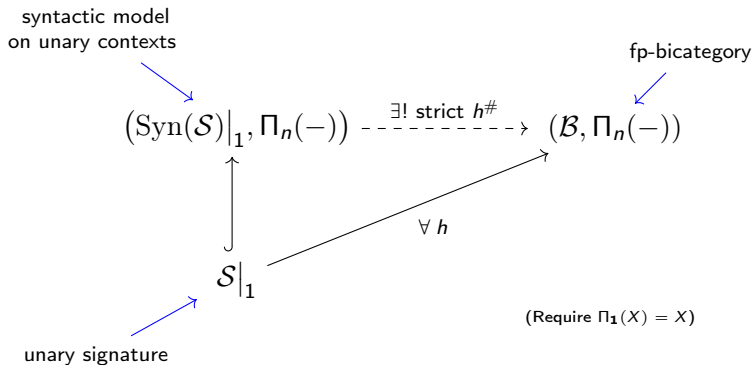
+ three equational rules.

\rightsquigarrow η -law is derivable

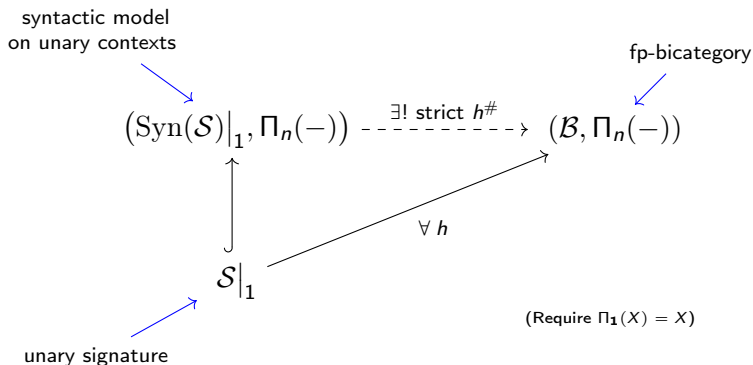
The syntactic model is free



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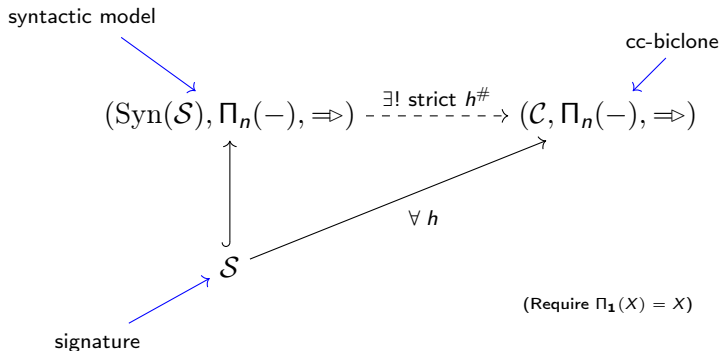


The syntactic model is free



⤿ An internal language for fp-bicategories.
derived from definition of biadjoint

The syntactic model is free



The syntactic model is free

syntactic model
on unary contexts

cc-bicategory with
strict products

$$(\text{Syn}(\mathcal{S})|_1, \Pi_n(-), \Rightarrow) \overset{\exists! \text{ strict } h^\#}{\dashrightarrow} (\mathcal{B}, \Pi_n(-), \Rightarrow)$$

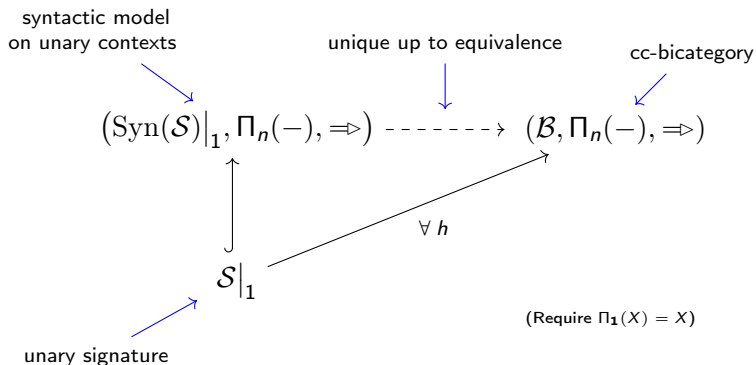
$\mathcal{S}|_1$

$\forall h$

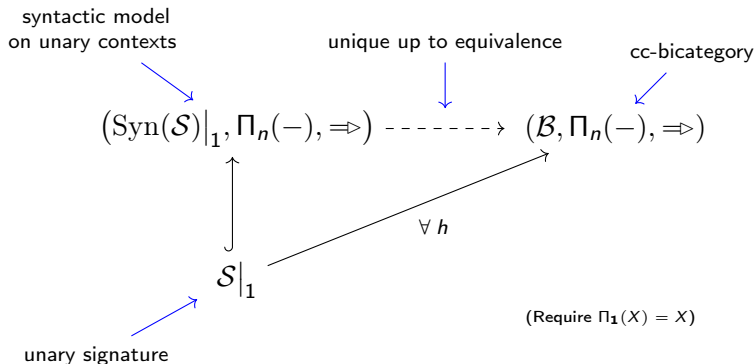
unary signature

(Require $\Pi_1(X) = X$)

The syntactic model is free



The syntactic model is free



\rightsquigarrow An internal language for cartesian closed bicategories.

STLC up to isomorphism

Embedding of STLC-terms to $\Lambda_{\text{ps}}^{\times, \rightarrow}$ -terms:

$$x_k \mapsto x_k$$

$$\pi_k(t) \mapsto \pi_k \{ \langle t \rangle \}$$

$$\langle t_1, \dots, t_n \rangle \mapsto \text{tup}(\langle t_1 \rangle, \dots, \langle t_n \rangle)$$

$$\text{app}(t, u) \mapsto \text{eval} \{ \langle t \rangle, \langle u \rangle \}$$

$$\lambda x. t \mapsto \lambda x. \langle t \rangle$$

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$$(\text{STLC terms } \Gamma \vdash t : B) / \beta\eta \cong (\Lambda_{\text{ps}}^{\times, \rightarrow}\text{-terms } \Gamma \vdash t : B) / \cong_{\Gamma}^B$$

$$t \cong_{\Gamma}^B t' \Leftrightarrow \Gamma \vdash \tau : t \stackrel{\cong}{\Rightarrow} t' : B$$

for some invertible τ

Key properties of $\Lambda_{\text{ps}}^{\times, \rightarrow}$:

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1. Principled development \Rightarrow few rules,
2. An internal language for cc-bicategories,
3. STLC up-to-isomorphism.

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1. Principled development \Rightarrow few rules,
2. An internal language for cc-bicategories,
3. STLC up-to-isomorphism.

A type theory for cartesian closed bicategories (LICS'19):

<https://arxiv.org/abs/1904.06538>

1-cells

$$\text{eval}_{A,B} : (A \Rightarrow B) \times A \rightarrow B$$

Adjoint equivalences

$$\begin{array}{ccc} & \text{eval}_{A,B} \circ (- \times A) & \\ & \curvearrowright & \\ \mathcal{B}(X, A \Rightarrow B) & \perp \simeq & \mathcal{B}(X \times A, B) \\ & \curvearrowleft & \\ & \lambda & \end{array}$$

Rules for exponentials

$$\begin{array}{c}
 \text{explicit weakening by } x \quad \leftarrow \quad (x : A) \quad \leftarrow \quad \text{free variable in context} \\
 \epsilon \bullet \text{eval} \{(-) \{inc_x\}, x\} \left(\frac{\text{eval} \{u \{inc_x\}, x\} \Rightarrow t : B}{u \Rightarrow \lambda x.t : A \Rightarrow B} \right) e^\dagger(x. -)
 \end{array}$$

$$\frac{}{f : A \Rightarrow B, x : A \vdash \text{eval}(f, x) : B} \text{eval} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B} \text{lam}$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma, x : A \vdash \epsilon_t : \text{eval} \{(\lambda x.t) \{inc_x\}, x\} \Rightarrow t : B} \text{\epsilon-intro } (\beta\text{-rule})$$

$$\frac{\Gamma, x : A \vdash t : B \quad \Gamma \vdash u : A \Rightarrow B \quad \Gamma, x : A \vdash \alpha : \text{eval} \{u \{inc_x\}, x\} \Rightarrow t : B}{\Gamma \vdash e^\dagger(x.\alpha) : u \Rightarrow \lambda x.t : A \Rightarrow B} e^\dagger(x.\alpha)\text{-intro}$$

+ three equational rules

\rightsquigarrow η -rule derivable